

# Proposal for LMS/EPSRC Short Instructional Course

## Derived categories and applications

Centre for Mathematical Science, City University  
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Organizers: J. Chuang and A. Cox

### 1. Introduction

Originally invented and developed as a technical tool for homological algebra, derived categories are now pervasive in many areas of pure mathematics. A recurring motif is the enhancement of existing theorems or conjectures to equivalences of derived categories. Examples include mirror symmetry, Broué's Abelian defect group conjecture, the Deligne-Lusztig theory, the Riemann-Hilbert correspondence, intersection cohomology, Fourier-Mukai transforms, the McKay correspondence, Seiberg duality. So derived categories are indisputably an important tool in modern mathematics. Moreover for the deepest understanding of these areas it is natural and necessary to study the derived categories themselves.

While there are a number of good textbooks on derived categories, the traditional pedagogical approach is to develop the abstract theory to the detriment of motivation. Consequently students can be discouraged by the technical aspects and further put off by the undeserved reputation of the subject for being difficult and overly abstract. An LMS short course offers the chance of exposing students to down-to-earth examples of derived categories in a variety settings, e.g. representation theory, homotopy theory, geometry.

### 2. Objectives of the course

The proposed course aims to introduce students to derived categories and some of their applications, to provide them with the basic theory and diverse examples and to convince them that derived categories are a natural, necessary and approachable part of modern mathematics. We hope to appeal to a relatively broad spectrum of students in algebra, geometry and possibly mathematical physics.

### 3. Meeting the objectives

The course will be organised around three lecture courses and two special guest lectures. Each lecture course will consist of five one-hour lectures and examples classes, which will be run with the help of two assistants. The main lecturers are widely regarded as excellent expositors, able and willing to tailor their presentations to an inexperienced audience. Markus Linckelmann offered a series of lectures in the LMS short course on Fusion Systems held in Birmingham in 2007, and Iain Gordon and Raphaël Rouquier have given instructional courses of a similar nature.

Bernhard Keller (Paris) would present one of the special guest lectures, on the topic of cluster algebras and derived categories. We wish to leave the choice of the second guest lecturer to a later stage, in order to best respond to the most recent developments in the subject. There will certainly be no shortage of attractive candidates to fill the role. The following names come to mind: Catarina Stroppel (Bonn), Volodymyr Mazorchuk (Uppsala), Tom Bridgeland (Sheffield), Jeremy Rickard (Bristol), Alastair King (Bath), Dave Benson (Aberdeen), Will Turner (Aberdeen), ...

#### 3.1 Introduction to derived categories

(M. Linckelmann, Aberdeen University)

The purpose of this course is to provide a gentle introduction to the derived category of an abelian category, taking care to balance the unavoidable technical aspects with heaps of motivation from homological algebra, algebraic topology and elsewhere.

Complexes (over an abelian category) and their role in algebraic topology and homological algebra will be reviewed, and numerous motivating examples given. The notion of homology and quasi-isomorphism will be introduced, leading to a definition of the derived category as a solution to a universal problem. Morphisms will be described using the homotopy category and the calculus of fractions. Examples of objects and morphisms in the derived category will be provided, and then the derived category shown to be a triangulated category. Derived functors will be reinterpreted in terms of derived categories.

Other important examples of triangulated categories will be introduced, in particular the derived categories of differential graded algebras and stable module categories.

### **3.2 Modular representations of finite groups**

(R. Rouquier, University of Oxford)

The modular representation theory of a finite group  $G$  over a field of characteristic  $p$  is organised around the idea that representations of  $G$  should be closely related to representations of certain special subgroups, called  $p$ -local subgroups. This principal originally took a numerical form, as in the conjectures of MacKay and of Alperin. The goal of this course is to explain how derived categories are used to formulate deeper structural counterparts to these conjectures.

The course will begin with the basic material of the ordinary representation theory of finite groups, followed by an introduction to modular representation theory. Some small examples will be worked out in detail. Alperin's weight conjecture and Broué's Abelian defect group conjecture will be stated. Once the motivation for equivalences of derived categories in this context is made clear, Rickard's general tilting theory will be covered. Finally the connection with Deligne-Lusztig theory for finite Chevalley groups will be discussed.

### **3.3 McKay correspondence and symplectic reflection algebras**

(Iain Gordon, University of Edinburgh)

The McKay correspondence relates the representations of a finite subgroup  $G$  of  $SL_2(\mathbb{C})$  with the cohomology of a minimal resolution of the quotient space  $\mathbb{C}/G$ . This course will introduce Bridgeland, King and Reid's reinterpretation of the McKay's correspondence as an equivalence of derived categories, taking it as a starting point to explore an intricate web of ideas involving algebra, geometry and combinatorics.

Among the important areas to be covered: categories of sheaves, resolutions of singularities, Fourier-Mukai transforms, the  $n!$  conjecture, symplectic reflection algebras.

## Provisional Timetable

	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday
09:00-10:00		Course 1	Course 1	Course 1	Course 1	Course 2
10:00-11:00		Course 1	Course 2	Course 2	Course 2	Course 3
11:00-11:30		Coffee	Coffee	Coffee	Coffee	Coffee
11:30-12:30		Course 2	Course 3	Course 3	Course 3	Course 3
12:30-14:00		Lunch	Lunch	Ex Class/Lunch	Lunch	Lunch
14:00-15:00		Ex Class	Ex Class	Free Afternoon	Lecture 1	
15:00-15:30	Arrival Day	Tea	Tea		Tea	
15:30-16:30		Ex Class	Ex Class		Lecture 2	
19:30-21:00					Workshop dinner	

## Budget

			<b>Total</b>
<b>Subsistence</b>			
Bed and breakfast	3 Lecturers	3x5x60	£900.00
	2 tutors	2x5x60	£600.00
	25 students	25x5x55	£6,875.00
	2 guest lecturers	2x1x60	£120.00
Tea and Coffee		32x8x1.5	£384.00
Workshop dinner		34x30	£1,020.00
			<b>£9,899.00</b>
<b>Honoraria</b>			
Lecturers		3x600	£1,800.00
Organisers		1x700	£700.00
Tutors		2x150	£300.00
Guest lecturers		2x100	£200.00
			<b>£3,000.00</b>
<b>Travel</b>			<b>£1,000.00</b>
<b>Room hire</b>	Standard Classroom	5x200	<b>£1,000.00</b>
<b>Total expenditure</b>			<b>£14,899.00</b>
<b>Income</b>	Registration fees	25x100	<b>£2,500.00</b>
<b>Net Cost</b>			<b>£12,399.00</b>