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The Impact of Mathematics on Meteorology and Weather Prediction

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1. *The role of mathematics in understanding weather*

During the afternoon of 21st October 2012, forecasters at the European Centre for Medium-Range Weather Forecasts (ECMWF), in the quiet leafy suburb of Shinfield Park, near Reading, perused the computer output from the 1200hrs forecast for the next 10 days. In addition to the familiar charts of surface pressure patterns, showing the highs and the lows with their attendant warm and cold fronts, the forecasters turned to an impressive range of diagnostic tools that facilitated rapid, detailed analyses of weather systems around the world. One such diagnostic is the "storm strike probability map", which indicates the risk (a probability of 25% or greater) of a severe weather event: these charts were highlighting something unusual for the New York area on the 29th and 30th October.

Ever since Edward Lorenz epitomised the capricious nature of weather in his famous allegory of the butterfly effect in 1972, chaos has been responsible for what has become almost an anti-theory of weather prediction: we have focussed on reasons why weather is inherently unpredictable. But in the week leading up to the landfall of "Superstorm Sandy" in New Jersey, which wreaked havoc in New York, the forecasters at ECMWF were able to help their American colleagues with a series of predictions that turned out to be remarkably accurate (Roulstone and Norbury 2013). And on the afternoon of 21st October, the tropical storm that was to gain so much notoriety, hadn't even formed.

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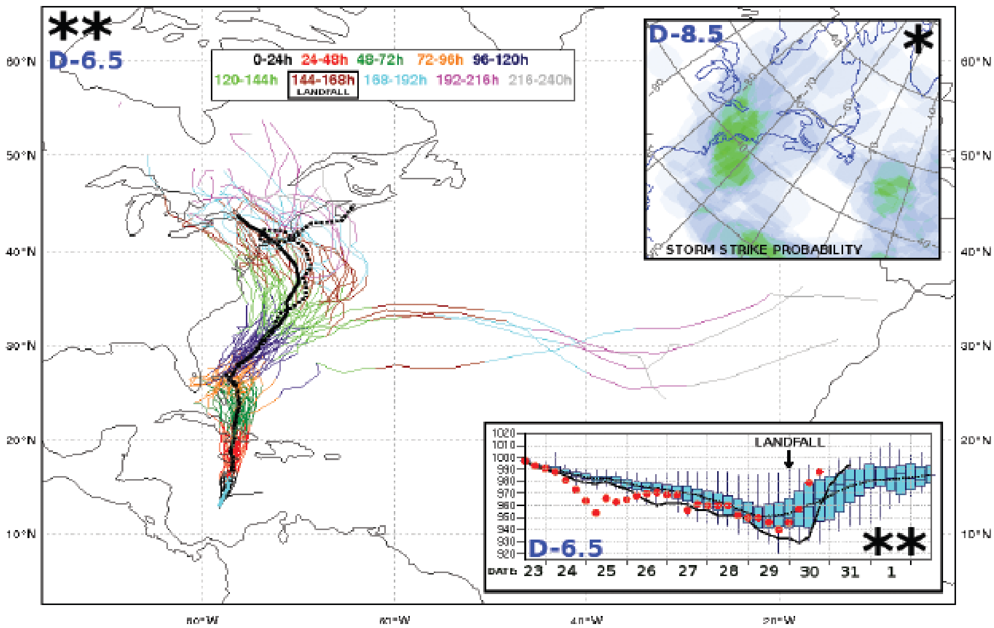
FIGURE 1

The equations governing weather are highly nonlinear. *This means that the 'cause and effect' relationships between the basic variables can become ferociously complex. To deal with the potential loss of predictability, forecasters study not one, but many forecasts, each one started from slightly different initial states, which reflect our ignorance of exactly how a weather system such as Hurricane Sandy formed. If the forecasts predict similar outcomes, we can be reasonably confident, but if they produce very different scenarios, then the situation is more problematic.*

The figure below shows the ensemble of forecasts for Sandy from the European Centre for Medium-Range Weather Forecasts covering the 10 days from the formation of the cyclone; the solid line is the track the forecasters decided to focus on. The ensemble illustrates the high probability of the 'left turn' and the most probable landfall – information that helped save lives.

The inset at the top right is the storm strike probability chart from 1200hrs on 21st October – 36 hours before the tropical storm even formed.



In the current era of big data and mind-boggling digital technology, it would be natural to assume that the accuracy of the forecast that helped save lives and livelihoods in the wake of Hurricane Sandy was attributable to the sheer capacity of modern supercomputers to digest the petabytes of data that stream from satellites and weather radar. Without doubt, technologies such as these have played a huge role in the quest to attain the present level of accuracy we have come to expect from our forecasts. But even with these gargantuan amounts of data, the output of the supercomputer simulations would, more often than not, be virtually useless if it were not for the careful application of novel mathematics.

While the world watched the progress of the superstorm from its birth in the Caribbean, past the Bahamas, along the eastern seaboard of the United States, until it finally swung to the west and made landfall in New Jersey, the choreography of the unfolding drama depended crucially another weather system that originated in the northern Pacific Ocean, thousands of kilometres away from the path of Hurricane Sandy. The salient point is that the accuracy of the forecasts for Sandy relied on capturing the bigger picture — the weather systems that would steer and finally interact with Sandy, causing the highly unusual "left turn" towards New York. While a hurricane is a complex physical system described by nonlinear fluid mechanics and themodynamics, many large-scale weather patterns can be described by much simpler mathematical models. These models reveal the mechanisms that govern large-scale weather patterns, and they indicate that such patterns have much more predictability than is commonly accepted. Such models draw on a wealth of mathematical ideas, from asymptotic analysis and Hamiltonian dynamics, to optimal transport and modern differential geometry.

Mathematics has been responsible for so much in terms of our understanding and prediction of weather and climate. This contribution is perhaps most obvious in the context of the technology behind the observations and the simulations: Bayesian statistics, numerical techniques, stochastic processes, probabilistic methods and optimisation theory, to name but a few, are essential prerequisites for any graduate student entering this challenging field. But while mathematics has helped us decide how to predict, it has also played a perhaps more subtle role in helping us to decide *what* to predict: a problem that has preoccupied forecasters since the advent of numerical weather prediction in the late 1940's.

Weather and climate models are based on the Navier–Stokes equations and moist thermodynamics, but substantial insights into the processes and mechanisms behind ubiquitous features of weather were obtained long before computing power reached the levels we take for granted today. These insights were obtained by applying systematic approximations to the Navier–Stokes equations, while retaining important properties such as conservation laws for energy and potential vorticity. Many decades of careful research have resulted in a deep understanding of what lies behind the choreography of weather, and this understanding is utilized in the design of numerical models. By incorporating this knowledge via elegant numerics, we focus on the features that influence the weather for the week ahead, rather than expending computer power on all the eddying local gusts of wind that play little or no part in the evolution of the big picture.

Many features of the large-scale circulation patterns can be described by solutions of the shallow water equations. These equations are themselves much simpler than the Navier–Stokes equations, and they are used frequently by meteorologists and oceanographers. Their validity as an approximation to the Navier–Stokes equations on large scales is based on the observation that the troposphere is a very thin shell of fluid, of the order of 10km deep, when compared to the radius of the Earth (just over 6000km).

We shall use the shallow water equations as a vehicle to describe the salient mathematical ideas that help meteorologists to quantify some of the large-scale, often highly predictable, features of the atmospheric circulation. For brevity, we ignore thermodynamics in the description that follows, but the model is readily augmented to include the physics of heat and moisture (which are necessary if considering phenomena such as weather fronts), while retaining the key mathematical structures.

2. *A "simple" model of weather*

The shallow water equations on a domain of $\mathbb{R}^2$, with local Cartesian coordinates (x, y), and rotating with constant angular frequency $f/2$, are

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv + g\frac{\partial h}{\partial x} = 0, \quad \frac{\mathrm{D}v}{\mathrm{D}t} + fu + g\frac{\partial h}{\partial y} = 0, \tag{2.1}$$

$$\frac{\mathrm{D}h}{\mathrm{D}t} + h\nabla \cdot \mathbf{u} = 0, \tag{2.2}$$

where t denotes time, g is the acceleration due to gravity (a constant) and $h(x, y, t)$ is the depth of the fluid. The horizontal velocity has two components $\mathbf{u}(x, y, t) = (u, v)$, and the Lagrangian, or material, derivative is

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla. \tag{2.3}$$

The Lagrangian derivative represents the rate of change of a dependent variable as we "follow the flow"; i.e. it is a directional derivative along the trajectory of a fluid particle. The derivative $\mathbf{u} \cdot \nabla$ is often referred to as the advection, or transport, term. If any variable $A(\mathbf{x}, t)$ satisfies $\mathrm{D}A/\mathrm{D}t = 0$, then we say that $A(\mathbf{x}, t)$ is conserved in the Lagrangian sense. Such conservation laws are of fundamental importance in meteorology and oceanography.

The so-called *semi-geostrophic* equations amount to following approximation to the shallow water equations

$$\frac{\mathrm{D}u_g}{\mathrm{D}t} - fv + g\frac{\partial h}{\partial x} = 0, \quad \frac{\mathrm{D}v_g}{\mathrm{D}t} + fu + g\frac{\partial h}{\partial y} = 0, \tag{2.4}$$

$$\frac{\mathrm{D}h}{\mathrm{D}t} + h\nabla \cdot \mathbf{u} = 0, \tag{2.5}$$

where u_g and v_g are the two components of the *geostrophic wind*

$$g\nabla h = (fv_g, \; -fu_g) \quad .$$

The geostrophic flow is parallel to contours of constant h (in the context of the Navier–Stokes equations on a rotating domain, the geostrophic flow is parallel to the isobars of constant pressure).

The difference between the shallow water equations and the semi-geostrophic equations is the replacement of the fluid velocity (u, v) with the geostrophic wind (u_g, v_g) in the Lagrangian derivatives of u and v, while leaving the derivative operator (2.3) and the continuity equation (2.5) unchanged. This is known as the *geostrophic momentum approximation* (Hoskins 1975), and it applies to flows in which the rate of change of momentum is much smaller than the Coriolis force. The advecting velocity (that is, the velocity appearing in the Lagrangian derivative $\mathbf{u} \cdot \nabla$) is not approximated in semi-geostrophic theory. The incorporation of two velocity fields reflects the fact that many atmospheric flows, such as jet streams and fronts, have two distinct length scales (for example, a weather front is a relatively sharp discontinuity between air masses, but fronts extend to distances of the order of 1000km along the interface between air masses). In essence, the geostrophic momentum approximation tells us that it is important to represent the advecting velocity accurately, while the quantity being advected (e.g. the geostrophic wind) can be approximated. The geostrophic momentum approximation is now well understood from the point of view of Hamiltonian mechanics: the canonical momentum, $\mathbf{p}$, is approximated, while the velocity $\dot{\mathbf{q}}$ is unchanged (McIntyre and Roulstone 2002).

Equations (2.4) and (2.5) conserve energy and the following form of *potential vorticity*

$$q = \frac{1}{h}\left(f + \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} + \frac{1}{f}\frac{\partial(u_g, v_g)}{\partial(x, y)}\right). \tag{2.6}$$

In atmospheric dynamics (with thermodynamics included), potential vorticity is proportional to the vector dot product of vorticity and stratification that, following the flow, can only be changed by diabatic or frictional processes. Potential vorticity is a fundamental concept for understanding the generation of vorticity in cyclogenesis (the birth and development of a cyclone), especially along the polar front, and in analyzing flow in the ocean. The use of potential vorticity played a key role as a diagnostic in understanding the evolution of Hurricane Sandy.

The semi-geostrophic equations have played a major role in understanding the formation of fronts (frontogenesis), and the properties of the equations that facilitate such studies are revealed through a transformation of coordinates. Defining new coordinates (sometimes called geostrophic momentum coordinates)

$$\mathbf{X} \equiv (X,\, Y) \equiv \left(x + \frac{v_g}{f},\, y - \frac{u_g}{f}\right), \tag{2.7}$$

we find that (2.4) may be replaced by

$$\frac{\mathrm{D}\mathbf{X}}{\mathrm{D}t} = \mathbf{u}_g \equiv (u_g,\, v_g)\ . \tag{2.8}$$

Hence the motion in these transformed coordinates is exactly geostrophic. The Jacobian is proportional to the potential vorticity (cf. (2.6))

$$\frac{\partial\,(X,\, Y)}{\partial\,(x,\, y)} = \frac{hq}{f}. \tag{2.9}$$

The vector $\mathbf{X}$ may be expressed as the gradient of a scalar function $P(\mathbf{x})$,

$$\mathbf{X} = \left(\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}\right), \tag{2.10}$$

which, to within an arbitrary additive constant, is uniquely defined by

$$P(\mathbf{x}, t) = \tfrac{1}{2}\left(x^2 + y^2\right) + \frac{gh(\mathbf{x}, t)}{f^2}. \tag{2.11}$$

We note, using $g\nabla h = (fv_g,\, -fu_g)$ together with (2.7) and (2.11), that (2.9) has the form of a Monge-Ampère equation for P, given $q(x, y, t)$ and suitable boundary conditions

$$q = \frac{f}{h}(P_{xx}P_{yy} - P_{xy}^2). \tag{2.12}$$

When the Jacobian (2.9) is non-singular, we express $\mathbf{x}(\mathbf{X})$ by introducing a scalar function $R(\mathbf{X})$:

$$\mathbf{x} = \left(\frac{\partial R}{\partial X}, \frac{\partial R}{\partial Y}\right)\ , \tag{2.13}$$

where R is given to within an additive constant by

$$R(\mathbf{X}) = \mathbf{x}\cdot\mathbf{X} - P(\mathbf{x}) \tag{2.14}$$

(time is a parameter in this transformation). Equation (2.14) is the expression for the *Legendre transformation* between $R(\mathbf{X})$ and $P(\mathbf{x})$. Local *singularities* of this map can be interpreted as *atmospheric fronts* (Chynoweth and Sewell 1989).

The semi-geostrophic equations can be integrated in time using the conservation of potential vorticity, expressed in geostrophic momentum coordinates. We write the reciprocal of the potential vorticity, q^{-1}, as

$$q(\mathbf{x},t)^{-1} \equiv \rho(\mathbf{X},t) = \frac{h(\mathbf{x},t)}{f}(R_{XX}R_{YY} - R_{XY}^2). \tag{2.15}$$

This may be expressed solely in terms of the geostrophic momentum coordinates by defining $\phi(\mathbf{x},t) = gh(\mathbf{x},t)/f^2$ and $\Phi(\mathbf{X},t) = \frac{1}{2}(X^2+Y^2) - R(\mathbf{X},t)$, then we note

$$\frac{\partial \Phi}{\partial X} = \frac{\partial \phi}{\partial x} = X - x, \quad \frac{\partial \Phi}{\partial Y} = \frac{\partial \phi}{\partial y} = Y - y.$$

Hence (2.15) may be written

$$\rho(\mathbf{X},t) = \frac{f}{g}\left(\Phi - \tfrac{1}{2}(\Phi_X^2 + \Phi_Y^2)\right)|\mathrm{Hess}(\tfrac{1}{2}(X^2+Y^2) - \Phi)|,$$

where Hess($\bullet$) is the hessian matrix of the second derivatives of $\bullet$ with respect to X, Y.

We can show that ρ satisfies

$$\frac{\partial \rho}{\partial t} + \dot{X}\frac{\partial \rho}{\partial X} + \dot{Y}\frac{\partial \rho}{\partial Y} = 0 \tag{2.16}$$

where

$$\dot{X} = f\left(\frac{\partial R}{\partial Y} - Y\right), \ \dot{Y} = -f\left(\frac{\partial R}{\partial X} - X\right). \tag{2.17}$$

The integration begins by solving the dual Monge-Ampère equation (2.15) for R, and then using R in (2.17) to determine the advecting velocities, which are in turn used to update ρ via (2.16). Equations (2.17) can be expressed in Hamiltonian form (McIntyre and Roulstone 2002):

$$\dot{X} = -f\frac{\partial \Phi}{\partial Y}, \quad \dot{Y} = f\frac{\partial \Phi}{\partial X}.$$

The semi-geostrophic equations involve two important types of geometry: symplectic geometry associated with the Hamiltonian structure, and contact geometry associated with the Legendre transformation. These two geometries also play a fundamental role in the theory of the Monge-Ampère equation (Kushner et al. 2007) and, in turn, relate to complex geometries, such as Kähler geometry. The transformation properties of the semi-geostrophic equations can be understood within the context of hyper-Kähler geometry, and this geometry has been exploited in the study of a much broader class of models of cyclones (McIntyre and Roulstone 2002, Delahaies and Roulstone 2010)†.

The emergence of a complex structure, which at first sight is somewhat surprising given the classical nature of the fluid mechanics, occurs when the Monge-Ampère equation (2.12) is elliptic. This equation is elliptic when $q > 0$. The ellipticity is related to a convexity condition on the energy and to the stability of the flow (Cullen 2006). The total energy, which is conserved by the shallow water equations (2.1), (2.2), is

$$E = \int \left(\tfrac{1}{2}(u^2+v^2) + \tfrac{1}{2}gh\right) h\mathrm{d}x\mathrm{d}y. \tag{2.18}$$

This is a functional of u, v and h, and the conditions for E to be *minimized* corresponds to the stability of a geostrophic flow, viewed as a solution of the unapproximated equations, to perturbations of the form

$$\delta u = f\delta y, \quad \delta v = -f\delta x, \quad \nabla \cdot (\delta x, \delta y) = 0. \tag{2.19}$$

†A recent Clay Mathematics Institute international workshop on *Geometry and Fluids* (http://www.claymath.org/events/geometry-and-fluids) explored the application of complex geometries to the study of Navier–Stokes turbulence.

The second variation of E is greater than zero when the Hessian matrix of the Legendre function P is positive definite (cf. Cullen et al. 1987). This corresponds to the ellipticity of the Monge-Ampère equation (2.12).

Introducing momentum coordinates, (X', Y'), via $(u, v) \equiv f(y - Y', X' - x)$, we can show that the perturbations (2.19) imply $\delta\sigma = 0$, where

$$\sigma = h \frac{\partial(x, y)}{\partial(X', Y')}. \tag{2.20}$$

If we define a distance $d(\mathbf{x}, \mathbf{X}')$ between $\mathbf{x}$ and $\mathbf{X}'$ such that

$$d^2 = f^2((X' - x)^2 + (Y' - y)^2),$$

then the energy functional can be rewritten

$$E = \int \left(\tfrac{1}{2} d^2(\mathbf{x}, \mathbf{X}') + \tfrac{1}{2} gh\right) h \mathrm{d}x \mathrm{d}y. \tag{2.21}$$

The proof that E can be uniquely minimized is established by showing that, given σ as a non-negative function of the momentum coordinates, there is a unique mapping from (X', Y') to (x, y) that minimizes E and satisfies (2.20). This is the starting point for expressing the energy minimization as a *Monge optimal mass transport* problem (Benamou and Brenier 1998; Cullen 2006; Villani 2008). Utilizing theorems on the regularity of solutions of the Monge-Ampère equation, together with optimal mass transport theory, it has been shown that the semi-geostrophic equations can be integrated for large times from suitable initial data. Cullen and Roulstone (1993) showed that a semi-geostrophic finite element numerical simulation of an Eady wave — the basic building block of cyclogenesis — could be performed through multiple life-cycles, and the predictability of the key features of the system were remarkably robust to perturbations of the initial data. Recent work by Visram *et al.* (2014) has demonstrated how semi-gestrophic theory can be used to validate the numerical schemes used in operational weather forecasting (based on the Navier–Stokes equations), and thereby assess the performance of the models in terms of their ability to represent the life-cycles of cyclones.

3. *Summary and Outlook*

Mathematics gives us considerable insight into the dynamics of the atmosphere and oceans, and these insights are incorporated in operational forecast models via the design of integration schemes, sub-grid models, and data assimilation (Bauer *et al.* 2015). The fusion of novel mathematics with computer science and observational meteorology has resulted in the current 3-day forecast being as reliable as the 1-day forecast 20 years ago. And such improvements could not have been achieved by relying solely on the phenomenal increase in the performance of supercomputers. If the numerical methods fail to capture the big picture — the conservation principles that orchestrate the cyclones and fronts — then the forecasts will be unreliable and risk analysis is much more problematic.

Using simpler models such as the semi-geostrophic equations, we are able to pick apart the predictable from the unpredictable amid the complexities of the atmospheric circulation. This enables us to concentrate our resources on observing and forecasting the more predictable large-scale patterns, which organise and transport the local weather features – and occasionally superstorms such as Sandy – that impact on our lives and livelihoods.

Weather and climate prediction are “big data” problems, and the mathematical techniques developed so successfully by forecasters are now helping us address the grand challanges in a variety of different disciplines, from ecology to health care, by enabling us to make sense of the data deluge.

4. *Further reading*

Invisible in the Storm: the role of mathematics in understanding weather, by Ian Roulstone and John Norbury; Princeton University Press 2013
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