by (13). We subtract suitable multiples of L_n from $L_1, \ldots, L_{n-1}$ in such a way as to remove the terms in u_n, and write the results as

$$L_i - \lambda_i L_n = \sum_{j=1}^{n-1} \alpha_{ij} u_j \qquad (i = 1, \ldots, n-1). \tag{23}$$

Suppose that we can ensure, by a preliminary unimodular substitution on $u_1, \ldots, u_n$, that the coefficients α_{ij} satisfy

$$\alpha_{ii} = M + o(M), \quad \alpha_{ij} = o(M) \quad \text{for } i \neq j, \tag{24}$$

where M is arbitrarily small. We then define $\theta_1, \ldots, \theta_{n-1}$ by

$$L_n = \alpha(\theta_1 u_1 + \ldots + \theta_{n-1} u_{n-1} + u_n),$$

where $\alpha \sim M^{-n+1} \Delta$ by comparison of determinants. The desired inequality for $C'(F)$ now follows from (13) by straightforward arguments, on using the same continuity property of F as was used in §4.

It remains to be proved that we can satisfy the conditions (24). Let the expressions for $u_1, \ldots, u_n$ in terms of $L_1, \ldots, L_n$ be

$$u_i = \sum_{j=1}^{n} \beta_{ij} L_j \quad (i = 1, \ldots, n).$$

Then, arguing as in the Corollary to Theorem 2, we can ensure that the elements in the first $n-1$ *rows* of the matrix β_{ij} are approximately N times the corresponding elements of the unit matrix. Now the matrix in (23) is the reciprocal of the matrix β_{ij} $(i, j = 1, \ldots, n-1)$. Hence (24) holds, with $M = N^{-1}$. This gives the desired result.

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ON DIRECT SUMS OF FREE CYCLES

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Introduction.

All the groups considered in this paper are abelian. If A, B are two groups, we denote by $A \pitchfork B$ the group of homomorphisms of A into B. This is usually written Hom (A, B), but the shorter notation brings out more clearly the dual relationship it bears to the tensor-product, $\otimes$. A strong case for this duality is suggested by Eilenberg and Steenrod ([1],

† Received 12 April, 1954; read 22 April, 1954.

Ch. V), and the advantages of the notation are apparent in Theorem 2 below.

Let Z be a free cyclic group. Let F and G be respectively the weak and strong† direct sums of a set of free cyclic groups indexed by an arbitrary indexing set, Γ. Thus $F \subset G$, and F is free abelian but G is not free (unless Γ is finite)‡. An element $g \varepsilon G$ is uniquely determined by its integer coordinates $\{g_i\}_{i \varepsilon \Gamma}$. The condition for g to be in F is that only a finite number of its coordinates are non-zero.

The main object of this paper is to establish

THEOREM 1. *There are natural isomorphisms*

$$\text{(i) } F \pitchfork Z \cong G, \qquad \text{(ii) } G \pitchfork Z \cong F.$$

The first half is well known§. The second half is obvious when Γ is finite, and has been proved when Γ is countable by Specker [3]||. In §1 we extend his result to the general case by means of transfinite induction. For this we need to assume

I. *The Axiom of Choice,*

and

II. *The Axiom of Accessibility of Ordinals*¶.

These have been shown to be consistent with the usual axioms for set theory by Gödel and Shepherdson [2] respectively. We do not need to assume the continuum hypothesis. The main step in the proof is Lemma 1.6, which represents, for example, the jump from $\aleph_0$ to $\mathfrak{c}$.

In the rest of the paper we discuss the "associativity" of the symbols $\otimes$ and $\pitchfork$; and prove

THEOREM 2. *If A, B, C are abelian groups, there are natural isomorphisms*

(i) $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$;

(ii) $A \pitchfork (B \pitchfork C) \cong (A \otimes B) \pitchfork C$;

(iii) $A \otimes (B \pitchfork C) \cong (A \pitchfork B) \pitchfork C$, *provided both A and C are free*;

(iv) $A \pitchfork (B \otimes C) \cong (A \pitchfork B) \otimes C$, *provided A or C is free and A or C is finitely-generated.*

---

† Some authors use "restricted" and "unrestricted" instead of "weak" and "strong".

‡ An immediate consequence of Theorem 1 (ii).

§ [1], p. 133.

|| My attention was drawn to Specker's paper by G. Higman.

¶ See Lemma 1.7 below. My attention was drawn to Shepherdson's work by M. H. A. Newman.

Part (i) merely states the associativity of the tensor-product†. The proof of (ii) is straightforward‡. The proof of (iii) is in essence the same as that of Theorem 1(ii), and the necessary modifications are indicated in §2. We also give examples to show the necessity of the conditions; and ask the question: is $F \otimes G \simeq F \pitchfork F$?

Section 3 is concerned with proving Theorem 2 (iv), and giving examples to show the necessity of the conditions.

The motivation behind the two theorems is homology theory in algebraic topology. For instance, let F and G represent the groups of finite integral chains and infinite integral cochains respectively, and Theorem 1 expresses the duality between them. Theorem 2 is of interest in dihomology§, where A, B represent integral chain groups and C a coefficient group.

I should like to acknowledge the fact that this paper grew out of discussions with P. J. Hilton.

Notation. $\alpha, \beta, \gamma, \ldots$ denote subsets of Γ.

If $g \in G$, $g|\beta$ denotes the element of G given by

$$(g|\beta)_i = \begin{cases} g_i, & i \in \beta, \\ 0, & \text{otherwise.} \end{cases}$$

A *unit*, e^i of G (or F) is the element which has its i-th coordinate equal to unity and the remaining coordinates zero. The *standard* element, e, of G is the element with all its coordinates unity.

$j, k, l, r, s, \omega_r, \ldots$ denote ordinals.

O_1, O_2 denote the classes of ordinals of the first and second kinds (*i.e.* with or without a predecessor).

$\lambda, \lambda', \aleph_r, \ldots$ denote cardinals.

$\overline{\overline{\Gamma}}, \overline{\overline{k}}, \ldots$ denote the cardinals of the set Γ, and the ordinal k, etc.

1. *Proof of Theorem* 1 (ii).

There is a natural embedding|| $\theta: F \to G \pitchfork Z$, uniquely determined by its effect upon the units of F

$$\theta e^i(g) = g_i, \quad e^i \in F, \quad \theta e^i : G \to Z.$$

θ is clearly $1-1$, so that to prove the theorem we have to show that it is onto. We shall prove (Corollary 1.11) that given $\phi : G \to Z$, then ϕ maps all but a finite number of units to zero. Therefore there exists $f \in F$, such

† N. Bourbaki, *Algèbre*, Livre II, Ch. III.

‡ [1], p. 160, ex. 1.

§ Dihomology is a homology theory based on pairs of simplexes; see a forthcoming paper by the author.

|| In effect the Kronecker index between F and G.

that $\phi - \theta f$ maps all the units to zero. We shall then show (Lemmas 1.2-1.7) that this condition is sufficient to ensure $\phi - \theta f = 0$ in $G \pitchfork Z$. In other words if Δ, $\subset G \pitchfork Z$, is the annihilator of F, $\subset G$, then $\Delta = 0$.

LEMMA† 1.1. *Given $g \varepsilon G$, $\phi \varepsilon G \pitchfork Z$, and a set $\{\alpha_r\}$ of disjoint subsets of Γ, then there exist at most a finite number of r such that $\phi(g \mid \alpha_r) \neq 0$.*

Proof. Suppose not. Then there exists at least a countable set $\{\alpha_n\}$, $n = 1, 2, \ldots$ such that $\phi(g \mid \alpha_n) = a_n \neq 0$, a_n an integer. Construct recursively a strictly increasing sequence of integers $\{k_n\}$, by choosing $k_1 = 1$, $k_n =$ the minimum integer satisfying

$$2^{k_n} > \sum_{m=1}^{n-1} 2^{k_m} |a_m| + n.$$

Define $h \varepsilon G$ by

$$h_i = \begin{cases} 2^{k_n} g_i, & i \varepsilon \alpha_n, \quad n = 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\phi h = a$, and choose $n > |a|$. Then

$$h - \sum_{1}^{n-1} 2^{k_m} (g \mid \alpha_m) = 2^{k_n} h',$$

some $h' \varepsilon G$. Take the ϕ-image:

$$a - \sum_{1}^{n-1} 2^{k_m} a_m = 2^{k_n} \phi h'.$$

$$2^{k_n} |\phi h'| = \left| a - \sum_{1}^{n-1} 2^{k_m} a_m \right| < 2^{k_n},$$

by construction. Therefore $\phi h' = 0$ and

$$a = \sum_{1}^{n-1} 2^{k_m} a_m = \sum_{1}^{n} 2^{k_m} a_m,$$

similarly; so that $a_n = 0$, contradicting our hypothesis.

COROLLARY 1.11. *Any $\phi \varepsilon G \pitchfork Z$ maps all but a finite number of units to zero.*

Proof. Choose $g = e$, the standard element, $\{\alpha_r\}$ to be the set of individual elements of Γ, and apply the lemma.

Consider now the following two statements concerning a cardinal $\lambda \leqslant \bar{\bar{\Gamma}}$.

$\Psi(\lambda)$: For all $g \varepsilon G$, $\phi \varepsilon \Delta$, $\beta \subset \Gamma$, $\bar{\bar{\beta}} = \lambda$, we have $\phi(g \mid \beta) = 0$.

$\chi(\lambda)$: If $g \varepsilon G$, $\phi \varepsilon \Delta$‡, k is an ordinal of cardinal $\leqslant \lambda$, and we are given

† After Specker [3], Satz III.

‡ We could allow $\phi \varepsilon G \pitchfork Z$ for the second statement, $\chi(\lambda)$, but this is not necessary for the proof.

any decreasing transfinite sequence $\{\beta_j\}$ of subsets of Γ, suffixed by j, $1 \leqslant j < k$, such that for each j, $\phi(g|\beta_j) = a$, then $\phi(g|\bigcap_{j<k} \beta_j) = a$.

Clearly $\Psi(\overline{\overline{\Gamma}})$ is equivalent to the theorem, for it implies $\phi g = \phi(g|\Gamma) = 0$, all $\phi \varepsilon \Delta$, $g \varepsilon G$. Conversely if $\theta\colon F \cong G \pitchfork Z$, and $\phi \varepsilon \Delta$, then $\phi = 0$; so that in particular $\phi(g|\beta) = 0$, all β, $\overline{\overline{\beta}} = \overline{\overline{\Gamma}}$.

LEMMA 1.2†. $\Psi(\aleph_0)$.

Proof. Given $g \varepsilon G$, $\phi \varepsilon \Delta$, $\beta \subset \Gamma$, β countable; identify β with the positive integers. Define $h \varepsilon G$ by

$$h_i = \begin{cases} 2^{2^n} g_n, & i = n, \quad n \varepsilon \beta, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\phi h = \phi\left(h - \sum_{m=1}^{n-1} 2^{2^m} g_m e^m\right), \text{ since } \phi e^m = 0, \text{ each } m,$$
$$= 2^{2^n} a, \text{ some integer } a.$$

Therefore 2^{2^n} divides ϕh, for arbitrary n, and so $\phi h = 0$. Similarly $\phi h' = 0$, where $h' \varepsilon G$ is given by

$$h_i' = \begin{cases} (2^{2^n} - 1) g_n, & i = n, \quad n \varepsilon \beta, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $\phi(g|\beta) = \phi(h - h') = 0$, as required.

This is sufficient to establish the theorem when Γ is countable, and also yields the beginning of our inductive proof of the general case.

LEMMA 1.3. *If* $\lambda < \lambda' \leqslant \overline{\overline{\Gamma}}$, *then* $\Psi(\lambda')$ *implies* $\Psi(\lambda)$.

Proof. Given $g \varepsilon G$, $\phi \varepsilon \Delta$, $\beta \subset \Gamma$, $\overline{\overline{\beta}} = \lambda$; then embed $\beta \subset$ some β', $\overline{\overline{\beta'}} = \lambda'$, and define $g' = g|\beta$. Therefore $\phi(g|\beta) = \phi(g'|\beta) = \phi(g'|\beta') = 0$, by $\Psi(\lambda')$.

COROLLARY 1.31. *If* $\overline{\overline{\Gamma}} < \overline{\overline{\Gamma'}}$ *and the theorem is true for* G', *then it is also true for* G.

Proof. Embed $\Gamma \subset \Gamma'$, and hence $G \subset G'$, and apply the lemma.

For the next lemma we need to recall two definitions:

> The *limit* of a transfinite sequence of ordinals, $\lim_{j<k} r_j$ is defined to be the least ordinal $\geqslant$ all r_j, $j < k$.
>
> The ordinal ω_s is said to be *singular* if $\omega_s = \lim_{j<k} r_j$, the limit of some strictly increasing sequence‡, where $k < \omega_s$, $k \varepsilon O_2$.

† After Specker [3], Satz III.

‡ Recall that $k \varepsilon O_2$ means that k has no predecessor.

We observe† that if ω_s is singular then $s \varepsilon O_2$, but the converse statement needs some accessibility axiom, and we discuss this in Lemma 1.7 below.

LEMMA 1.4. *If ω_s is singular, and $\Psi(\lambda)$ for all $\lambda < \aleph_s$, then $\Psi(\aleph_s)$.*

Proof. Given $g \varepsilon G$, $\phi \varepsilon \Delta$, $\beta \subset \Gamma$, $\overline{\overline{\beta}} = \aleph_s$; identify β with the section of ordinals defined by ω_s. Thus $i \varepsilon \beta$ implies $1 \leqslant i < \omega_s$. By hypothesis $\omega_s = \lim_{1 \leqslant j < k} r_j$, where $k < \omega_s$, $k \varepsilon O_2$. We now use this accessibility of ω_s to decompose β into subsets for which the result holds. Denote by γ the set $\{j, 1 \leqslant j < k\}$. Then $\overline{\overline{\gamma}} < \aleph_s$, and $\Psi(\overline{\overline{\gamma}})$ by hypothesis. Decompose β into disjoint subsets α_j, $\beta = \bigcup_{j \varepsilon \gamma} \alpha_j$, where

$$\alpha_1 = \{i,\ 1 \leqslant i < r_1\},$$

$$\alpha_j = \left\{i,\ \lim_{j' < j} r_{j'} \leqslant i < r_j\right\},\quad 1 < j < k.$$

Embed the set γ in Γ, which is possible since $\overline{\overline{\gamma}} < \aleph_s \leqslant \overline{\overline{\Gamma}}$, and define an endomorphism ξ: $G \to G$ by

$$(\xi h)_i = \begin{cases} h_j g_i, & i \varepsilon \alpha_j,\quad j \varepsilon \gamma, \\ 0, & i \notin \beta. \end{cases}$$

This is indeed a homomorphism, being linear in the coordinates of h. We deduce that $\phi\xi \varepsilon \Delta$, for

if $j \notin \gamma$, $e^j \overset{\xi}{\to} 0 \overset{\phi}{\to} 0$;

if $j \varepsilon \gamma_j$, $e^j \overset{\xi}{\to} g \mid \alpha_j \overset{\phi}{\to} 0$, since $\overline{\overline{\alpha}}_j \leqslant \overline{\overline{r}}_j < \aleph_s$, and so $\Psi(\overline{\overline{\alpha}}_j)$ by hypothesis.

Therefore $\phi\xi(e \mid \gamma) = 0$, since $\Psi(\overline{\overline{\gamma}})$; but $\xi(e \mid \gamma) = g \mid \beta$, so that $\phi(g \mid \beta) = 0$ as required.

LEMMA 1.5. *$\Psi(\lambda)$ implies $\chi(\lambda)$.*

Proof. The proof follows closely the pattern of that of Lemma 1.4. Suppose we are given the hypothesis of $\chi(\lambda)$, then we have to show

$$\phi\left(g \,\middle|\, \bigcap_{j<k} \beta_j\right) = a.$$

If $k \varepsilon O_1$, then
$$\bigcap_{j<k} \beta_j = \beta_{j-1},$$

and the result is trivial; so assume $k \varepsilon O_2$. Further we may assume the result true for all k', $k' < k$; otherwise replace k by the first such k', for which it does not hold, and we shall achieve a contradiction. Let γ be

† W. Sierpinski, *Leçons sur les nombres transfini* (Paris, 1928), p. 225.

as in Lemma 1.4, and so, by hypothesis and Lemma 1.3, $\Psi(\bar{\bar{\gamma}})$. Let

$$\beta = \beta_1 - \bigcap_{j<k} \beta_j,$$

and decompose β into disjoint subsets α_j, $\beta = \bigcup_{j \varepsilon \gamma} \alpha_j$, where $\alpha_1 =$ empty, and

$$\alpha_j = \bigcap_{j'<j} \beta_{j'} - \beta_j, \quad 1 < j < k.$$

Embed γ in Γ and define ξ as before. Then $\phi\xi \varepsilon \Delta$ for if $j \varepsilon \gamma$,

$$\phi(g \,|\, \alpha_j) = \phi\left(g \,\middle|\, \bigcap_{j'<j} \beta_{j'}\right) - \phi(g \,|\, \beta_j) = 0,$$

by the assumption above. Again $\phi(g \,|\, \beta) = 0$, so that

$$\phi\left(g \,\middle|\, \bigcap_{j<k} \beta_j\right) = \phi(g \,|\, \beta_1) = a,$$

as required.

Remark. The essential difference between Lemmas 1.4 and 1.5 lies in the fact that in the latter case we do not know whether $\Psi(\bar{\bar{\alpha}}_j)$ is true. This is in effect the main obstacle in the transfinite induction and is solved by the following lemma.

LEMMA 1.6. $\chi(\lambda)$ *implies* $\Psi(2^\lambda)$.

Proof. Given $g \varepsilon G$, $\phi \varepsilon \Delta$, $\beta \subset \Gamma$, $\bar{\bar{\beta}} = 2^\lambda$, $\phi(g \,|\, \beta) = a$, we have to show $a = 0$. Suppose not.

Assuming the Axiom of Choice, choose an ordinal k, $\bar{\bar{k}} = \lambda$. (We may choose $k \varepsilon O_2$, but this is not necessary.) Now identify β with the set of all transfinite† sequences of 0's and 1's, of type k; this is possible since both these sets have the same cardinal. An element of β is therefore uniquely determined by its coordinates, 0 or 1, at each l, $1 \leqslant l < k$.

Let us define a decreasing sequence $\{\beta_j^1\}$ of subsets of β, suffixed by j, $1 \leqslant j < k$, as follows:

Let $\beta_j^1 =$ the set of all sequences with 0 at each l, $l < j$. In particular $\beta_1^1 = \beta$, so $\phi(g \,|\, \beta_1^1) = a$.

Now we cannot have $\phi(g \,|\, \beta_j^1) = a$, for all j, $1 \leqslant j < k$; otherwise

$$\phi\left(g \,\middle|\, \bigcap_{j<k} \beta_j^1\right) = a \neq 0,$$

by hypothesis $\chi(\lambda)$. But $\bigcap_{j<k} \beta_j^1$ contains only one element, namely the sequence of all zeros, and, since $\phi \varepsilon \Delta$, we have a contradiction. Therefore there exists an ordinal j_1 (and we may choose the first such), with the

† Since all sequences in this proof are transfinite, we shall in general write "sequence" for "transfinite sequence".

properties $1<j_1<k$, and $\phi(g|\beta^1_{j_1})=a-a_1$, $a_1\neq 0$. Also $j_1 \varepsilon O_1$, otherwise $\beta^1_{j_1}=\bigcap_{j<j_1}\beta_j{}^1$ and $\phi(g|\beta_j{}^1)=a$, $j<j_1$, so that applying $\chi(\lambda)$ yields a contradiction. Let $\gamma_1=\beta^1_{j_1-1}-\beta^1_{j_1}$, namely the set of sequences with coordinates

$$\begin{cases} 0 \text{ at } l, & 1\leqslant l<j_1-1,\\ 1 \text{ at } j_1-1, & \\ \text{free at } l, & l\geqslant j_1. \end{cases}$$

Then $\phi(g|\gamma_1)=a-(a-a_1)=a_1\neq 0$.

Thus we have constructed:

j_1, an ordinal, $1<j_1<k$;

γ_1, a subset of β, namely a set of sequences with coordinates fixed $l<j_1$, free $l\geqslant j_1$;

a_1, an integer, such that $\phi(g|\gamma_1)=a_1\neq 0$.

We now propose to construct a (transfinite) sequence† of triples $\{j_r, \gamma_r, a_r\}$. Let us denote by $P(s)$ the fact that triples have been defined for all r, $1\leqslant r\leqslant s$, where s is an ordinal, $1\leqslant s\leqslant k$, with the following properties:

(i) j_r is an ordinal, $1<j_r\leqslant k$, such that if $r'<r\leqslant s$, $j_{r'}<j_r$;

(ii) γ_r is a subset of β, namely a set of sequences with coordinates fixed $l<j_r$, free $l\geqslant j_r$; such that if $r'<r\leqslant s$, then $\gamma_{r'}\supset\gamma_r$, and if $r \varepsilon O_2$, then $\gamma_r=\bigcap_{r'<r}\gamma_{r'}$;

(iii) a_r is an integer, such that $\phi(g|\gamma_r)=a_r\neq 0$.

We deduce at once that if $P(s)$ is true, then $j_s<k$. Otherwise $j_s=k$, and, by (ii), γ_s is a set of sequences with coordinates fixed for $l<k$; in other words γ_s is comprised of a single element. Therefore, since $\phi \varepsilon \Delta$, $a_s=\phi(g|\gamma_s)=0$, contradicting (iii). From this we see that we cannot have $P(k)$ true, for otherwise, since $j_1>1$ and $j_r>j_{r'}$ for $r>r'$, then $j_k\geqslant k$, contradicting the previous statement.

Therefore, at some point, our construction of a sequence of triples to satisfy the above properties must come to a halt. On the other hand we will show that it can always be continued. The resulting contradiction is the crux of the proof of the lemma, and will show that our original assumption that $a\neq 0$ was invalid.

Let s be the least ordinal such that we can define triples for all r, $1\leqslant r<s$, so that $P(r)$ is true for all r, $1\leqslant r<s$, but such that we cannot have $P(s)$ true. From the foregoing it is clear that s exists and $1<s\leqslant k$.

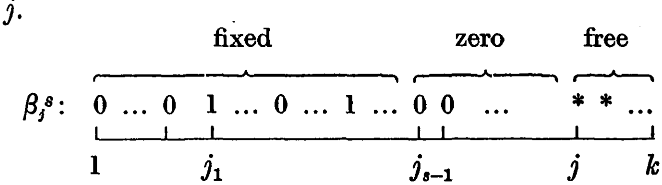
† We can construct the sequence uniquely, but this is not necessary to the proof.

We now construct $\{j_s, \gamma_s, a_s\}$ so as to satisfy $P(s)$, in each of the two cases $s \varepsilon O_1$, or $s \varepsilon O_2$, thereby achieving the contradiction and proving the lemma.

Construction of $\{j_s, \gamma_s, a_s\}$, *for* $s \varepsilon O_1$. Define a decreasing sequence $\{\beta_j^s\}$ of subsets of β, suffixed by j, $1 \leqslant j < k$, as follows:

if $j \leqslant j_{s-1}$, let $\beta_j^s = \gamma_{s-1}$;

if $j > j_{s-1}$, let $\beta_j^s =$ the subset of γ_{s-1}, of sequences with 0 at each l, $j_{s-1} \leqslant l < j$.



Then (as in the above case, $s = 1$),

$$\phi\left(g \,\middle|\, \bigcap_{j<k} \beta_j^s\right) = 0, \quad \phi(g \,|\, \beta^s_{j_{s-1}}) = a_{s-1} \neq 0.$$

Therefore there exists an ordinal j_s (and we may choose the first) such that $j_{s-1} < j_s < k$, $\phi(g \,|\, \beta^s_{j_s}) = a_{s-1} - a_s$, $a_s \neq 0$. Then $j_s \varepsilon O_1$ (as above); so define $\gamma_s = \beta^s_{j_s-1} - \beta^s_{j_s}$†. Thus $\gamma_s =$ a set of sequences with coordinates fixed $l < j_s$, free $l \geqslant j_s$, and $\gamma_s \subset \gamma_{s-1}$. Further

$$\phi(g \,|\, \gamma_s) = a_{s-1} - (a_{s-1} - a_s) = a_s \neq 0;$$

and so we have $P(s)$ true as desired.

Construction of $\{j_s, \gamma_s, a_s\}$, *for* $s \varepsilon O_2$. Define $j_s = \lim\limits_{r<s} j_r$. Thus $j_s \varepsilon O_2$, being the limit of a strictly increasing transfinite sequence of type s, $s \varepsilon O_2$; and $j_s > j_r$, each $r < s$. Further $j_s \leqslant k$, since $j_r < k$, each $r < s$, by the remark above. Condition (i) of $P(s)$ is therefore satisfied.

Define $\gamma_s = \bigcap\limits_{r<s} \gamma_r =$ a set of sequences with coordinates at l fixed if $l <$ some j_r, $r < s$; *i.e.* if $l < j_s$, since $j_s \varepsilon O_2$; and free if $l \geqslant$ each j_r, $r < s$; *i.e.* if $l \geqslant j_s$. Further $\gamma_s \subset$ each γ_r, $r < s$, and so satisfies property (ii). Define $a_s = \phi(g \,|\, \gamma_s)$. To complete the proof we must show $a_s \neq 0$, which requires the use of Lemma 1.1.

Consider the disjoint subsets $\{\alpha_r\}$ of β defined by

$$\alpha_r = \gamma_{r-1} - \gamma_r, \text{ for } r \varepsilon O_1, \quad 1 \leqslant r < s.$$

By Lemma 1.1, $\phi(g \,|\, \alpha_r) \neq 0$ for at most a finite number of r. Since $s \varepsilon O_2$, we may therefore choose t, $t < s$, such that $\phi(g \,|\, \alpha_r) = 0$, all $r \varepsilon O_1$, $t \leqslant r < s$. From this we may deduce that $a_r = a_t$, all r, $t \leqslant r \leqslant s$. Otherwise let r

† Do not confuse $j_s - 1$ with j_{s-1}.

be the first such that $a_r \neq a_t$. If $r \in O_1$, then $r < s$, and

$$0 = \phi(g \mid \alpha_r) = a_{r-1} - a_r = a_t - a_r, \quad \text{a contradiction.}$$

If $r \in O_2$, then $a_{r'} = a_t$ for all r', $t \leqslant r' < r$; therefore

$$a_r = \phi(g \mid \gamma_r) = \phi\Big(g \,\Big|\, \bigcap_{r'<r} \gamma_{r'}\Big), \quad \text{by condition (ii),}$$

$$= a_t, \quad \text{by } \chi(\lambda), \text{ a contradiction.}$$

In particular when $r = s$ we have $a_s = a_t$, and $a_t \neq 0$ by hypothesis $P(t)$. Thus $a_s \neq 0$, and hence all the requirements for $P(s)$ are satisfied.

This completes the proof of Lemma 1.6.

We must now verify that Lemmas 1.2—1.6 are sufficient to achieve a transfinite induction. We need to assume

I. *The Axiom of Choice.*

II. *The Axiom of Accessibility of Ordinals*: if $s \in O_2$, and $2^{\aleph_r} < \aleph_s$ for all r, $r < s$, then ω_s is singular.

This latter axiom is slightly weaker than the usual Accessibility Axiom which says "if $s \in O_2$ then ω_s is singular". It was introduced by Tarski, and enables us to avoid discussing the Continuum Hypothesis. Shepherdson [2] shows that it is consistent with the Zermelo-Fraenkel system of axioms for set theory (provided these are themselves consistent), by constructing a super-complete inner model of the universe, in which the class of ordinals is the section of the ordinals of the universe by the first inaccessible ordinal of the universe, if such exists. It is not known whether a universe with inaccessible ordinals exists.

LEMMA 1.7. *If*

(*a*) $\Psi(\aleph_0)$,

(*b*) $\lambda < \lambda'$ *and* $\Psi(\lambda')$ *implies* $\Psi(\lambda)$,

(*c*) ω_s *singular and* $\Psi(\lambda)$ *all* $\lambda < \aleph_s$ *implies* $\Psi(\aleph_s)$,

(*d*) $\Psi(\lambda)$ *implies* $\Psi(2^\lambda)$,

then $\Psi(\Gamma)$.

Proof. We may assume Γ infinite, otherwise the lemma is trivial:

By I, $\overline{\overline{\Gamma}} = \aleph_t$, some ordinal t. Suppose the lemma is not true, and suppose s is the first ordinal for which $\Psi(\aleph_s)$ does not hold. By (*a*), $s > 0$. If $s \in O_1$, then $\Psi(2^{\aleph_{s-1}})$ by (*d*). (There is a small detail here, that if we do not assume the generalized continuum hypothesis, and $\aleph_{s-1} < \overline{\overline{\Gamma}} < 2^{\aleph_{s-1}}$, then we must embed G in G', $\overline{\overline{\Gamma'}} = 2^{\aleph_{s-1}}$, and use corollary 1.31.) Then $\Psi(\aleph_s)$ by (*b*), since $\aleph_s \leqslant 2^{\aleph_{s-1}}$.

If $s \varepsilon O_2$, either $2^{\aleph_r} \geqslant \aleph_s$, for some r, $r < s$, whence $\Psi(\aleph_s)$ by (*d*) and (*b*), as above; or ω_s is singular by II, and $\Psi(\aleph_s)$ by (*c*). We have the desired contradiction in each case.

This completes the proof of Theorem 1.

Remark 1.8. Ulam† has tackled a similar set-theoretical problem in showing that there must exist non-measurable sets in the unit interval; *i.e.* that there exists no real-valued function f on the subsets of (0, 1) such that

1° there is at least one subset A for which $f(A) > 0$,

2° for each point p, $f(p) = 0$,

3° for each countable disjoint sequence $\{A_n\}$, $f\left(\bigcup_1^{\infty} A_n\right) = \sum_{n=1}^{\infty} f(A_n)$.

However this problem lacks the analogue of Lemma 1.1, which is used in achieving the induction $\lambda \to 2^{\lambda}$ of Lemma 1.6. Consequently he can only obtain (by a different proof) the induction $\aleph_r \to \aleph_{r+1}$; and to reach $\mathfrak{c}$, for instance, he must assume either the continuum hypothesis, $\mathfrak{c} = \aleph_1$, or the weaker hypothesis: "if $s \varepsilon O_2$ and $\omega_s < \mathfrak{c}$, then ω_s is singular". Whereas in the above proof we may reach $\mathfrak{c}$ without any assumptions.

2. *Proof of Theorem* 2 (iii).

The theorem states that if A and C are free then

$$A \otimes (B \pitchfork C) \cong (A \pitchfork B) \pitchfork C, \text{ naturally.}$$

We notice that if $B = C = Z$ this reduces to Theorem 1 (ii); moreover the proof for the general case is only a modification of the above.

We show first (Lemma 2.1) there is a natural embedding

$$\theta: A \otimes (B \pitchfork C) \to (A \pitchfork B) \pitchfork C.$$

Then, as before, the more difficult step is to prove that θ is onto. For this we must choose a fixed arbitrary base $\{e^i\}_{i \varepsilon \Gamma}$ for A, but it is important to notice that the definition of θ is independent of the choice of base, so that the natural character of the isomorphism is preserved. The base enables us to embed $A \otimes B \subset A \pitchfork B$ (Lemma 2.2), and to define Δ, $\subset (A \pitchfork B) \pitchfork C$, the annihilator of $A \otimes B$. Then, given $\phi \varepsilon (A \pitchfork B) \pitchfork C$, we are able (Corollary 2.31) to choose $f \varepsilon A \otimes (B \pitchfork C)$, such that $\phi - \theta f \varepsilon \Delta$.

The proof that $\Delta = 0$, and hence of the theorem, follows by Lemmas

† S. Ulam, "Zur Masstheorie", *Fund. Math.*, 16 (1930).

[*Added in proof.* I. Kaplansky has pointed out to me that Lemma 1.6 may also be proved by using a second result of Ulam's concerning *two*-valued measures, for which the induction $\lambda \to 2^{\lambda}$ holds.]

1.2-1.7, with only a few formal changes in notation, *e.g.* replacing G by $A \pitchfork B$.

The notation $g|\beta$, for $g \varepsilon A \pitchfork B$, $\beta \subset \Gamma$, is again dependent upon the base chosen above, being used only as a tool in proving θ onto, and is defined as follows:

$g|\beta$ is the element in $A \pitchfork B$ uniquely given by

$$(g|\beta)\, e^i = \begin{cases} ge^i, & i \varepsilon \beta, \\ 0, & \text{otherwise.} \end{cases}$$

LEMMA 2.1. *If A, B, C, are any abelian groups, there is a natural homomorphism θ: $A \otimes (B \pitchfork C) \to (A \pitchfork B) \pitchfork C$, which is* 1-1 *provided A is free.*

Proof. Define θ by linear extension of the formula

$$[\theta(a \otimes y)](g) = y\big(g(a)\big) \text{ in } C, \text{ for } a \varepsilon A,\ g \varepsilon A \pitchfork B,\ y \varepsilon B \pitchfork C.$$

We leave the reader to verify that θ is indeed one-valued with values in $(A \pitchfork B) \pitchfork C$, and a homomorphism. If A is free, and given $x \varepsilon A \otimes (B \pitchfork C)$ such that $\theta x = 0$, we must show $x = 0$. Let $\{e^i\}$ be any base for A; we may write $x = \sum_{m=1}^{n} e^m \otimes y_m$ (after a suitable relabelling of the e's concerned).

Now given any m, $1 \leqslant m \leqslant n$, and any $b \varepsilon B$, let $g \varepsilon A \pitchfork B$ be the element defined by $ge^m = b$, $ge^i = 0$ for $i \neq m$.

Then $0 = \theta x(g) = \sum_{m=1}^{n} y_m\big(g(e^m)\big) = y_m(b)$. Therefore $y_m = 0$, since this is true for arbitrary $b \varepsilon B$; and so, in turn, $x = 0$.

LEMMA 2.2. *If A is free, then any base of A induces a (non-natural) embedding $A \otimes B \subset A \pitchfork B$, which is an isomorphism onto if and only if A is finitely-generated.*

Proof. Let the base be $\{e^i\}_{i \varepsilon \Gamma}$. Then any element in $A \otimes B$ may be written uniquely as $\Sigma e^i \otimes b_i$, or as $\{b_i\}_{i \varepsilon \Gamma}$ where only a finite number of the b_i are non-zero. Again, an element in $A \pitchfork B$ is uniquely determined by the images $\{b_i\}_{i \varepsilon \Gamma}$ of the base elements e^i. We thus have a representation of $A \otimes B$ and $A \pitchfork B$ as the weak and strong direct sums of a set of groups indexed by Γ, each isomorphic to B. This yields at once the desired embedding and the lemma.

LEMMA† 2.3. *Assume A and C are free. Let $g \varepsilon A \pitchfork B$, $\phi \varepsilon (A \pitchfork B) \pitchfork C$, and let $\{\alpha_r\}$ be a set of disjoint subsets of Γ; then there exist at most a finite number of r such that $\phi(g|\alpha_r) \neq 0$.*

---

† *Cf.* Lemma 1.1.

Proof. Suppose not. Then Γ must be infinite and there exists an infinite set of r such that $\phi(g\,|\,\alpha_r) = c_r \neq 0$ in C. Choose a base $\{d^s\}$ for C. There are two cases:

(i) There is a base element, d say, such that there exists a countable set of r for which the d-coordinate of c_r, an integer a_r say, is non-zero. Let $\zeta : C \rightarrow Z$ be the homomorphism defined by taking the d-coordinate of an element in C. Then $\zeta\phi \,\varepsilon\, (A \pitchfork B) \pitchfork Z$, and there exists a countable set $\{\alpha_n\}$, $n = 1, 2, \ldots$, such that $\zeta\phi(g\,|\,\alpha_n) = a_n \neq 0$.

We may now proceed as in Lemma 1.1. Define $h \,\varepsilon\, A \pitchfork B$ by

$$he^i = \begin{cases} 2^{k_n} g e^i, & i \,\varepsilon\, \alpha_n, \\ 0, & \text{otherwise; etc.} \end{cases}$$

The contradiction is obtained as before.

(ii) There is no such base element.

Then we may pick out countable subsets $\{c_n\}$ of c_r, and $\{d^n\}$ of base elements, such that the d^m-coordinate of c_n, $c_n{}^m$ say†,

$$= \begin{cases} a_n \neq 0, & m = n, \\ 0, & m \neq n. \end{cases}$$

We do this recursively as follows.

Suppose we have chosen c_m and d^m for $m < n$, obeying the above formulae. Certainly $c_1, \ldots, c_{n-1}$ have non-zero coordinates on only a finite subset, D, of base elements, and D clearly contains $d^1, \ldots, d^{n-1}$. By our assumption, there is at most a finite set of c_r which have a non-zero coordinate on at least one $d^s \,\varepsilon\, D$. Choose c_n to be *not* one of these, and so $c_n{}^m = 0$, $m < n$. Further, choose d^n to be any d^s such that $c_n{}^s \neq 0$. Thus $c_n{}^n = a_n$, say, $\neq 0$, and since $d^n \notin D$ by construction of c_n, we have $c_m{}^n = 0$, $m < n$ by definition of D. Hence c_n and d^n are defined, and the formulae satisfied for $m \leqslant n$.

Now let $\zeta : C \rightarrow Z$ be the homomorphism defined by

$$\zeta d^s = \begin{cases} 1, & s = \text{some } n, \\ 0, & \text{otherwise;} \end{cases}$$

so that $\zeta c_n = a_n$, and we reach the same conclusions as in (i).

COROLLARY 2.31. *Given $\phi \,\varepsilon\, (A \pitchfork B) \pitchfork C$, and the base $\{e^i\}$ of A, then there is a finite subset $\beta(\phi)$ of Γ, such that $\phi g = \phi\big(g\,|\,\beta(\phi)\big)$, for any $g \,\varepsilon\, A \pitchfork B$. (Proof as for Corollary 1.11.)*

† In general there will also exist $d^s \neq$ any d^m, such that $c_n{}^s \neq 0$.

Apart from the formal changes in notation in §1, which we leave to the reader, this completes the proof of Theorem 2 (iii).

It is worthwhile noting that the conditions for A and C to be free are both necessary for the above proof, as the following counter examples illustrate.

Example 2.4. *A not free.*

Let A be cyclic of order p: and $B = C = Z$. Then

$$(A \pitchfork B) \pitchfork C = 0, \quad A \otimes (B \pitchfork C) \cong A.$$

Again it is not sufficient merely to have A without torsion, for let A be the additive group of rationals, and the same formulae hold.

Example† 2.5. *C not free.*

Choose Γ infinite, and let $A = F$, $B = Z$, $C = Z_2$.
Then $A \otimes (B \pitchfork C) \cong F \otimes Z_2$, of cardinal $\bar{\bar{\Gamma}}$; but

$$(A \pitchfork B) \pitchfork C \cong G \pitchfork Z_2 \cong (G/2G) \pitchfork Z_2,$$

of cardinal $2^{2^{\bar{\bar{\Gamma}}}}$, since $G/2G$ is a vector space over the field of two elements with cardinal $2^{\bar{\bar{\Gamma}}}$. The two groups cannot be isomorphic for $2^{2^{\bar{\bar{\Gamma}}}} > \bar{\bar{\Gamma}}$.

As before it is not sufficient merely to have C without torsion, for consider:

Example 2.6. *C without torsion, but not free.*

Choose Γ infinite, and let $A = F$, $B = Z$, $C = G$.
Then

$$A \otimes (B \pitchfork C) \cong F \otimes G = K, \text{ say;}$$

$$(A \pitchfork B) \pitchfork C \cong G \pitchfork G, \text{ by Theorem 1 (i)}, = L, \text{ say;}$$

and $\theta : K \to L$ is strictly into, since by the definition of θ, it is clear that the identity in $G \pitchfork G$ cannot be in the image of θ (for it "involves" an infinite number of units of A).

Remark 2.61. Although the natural isomorphism $\theta : K \to L$ is not onto, it is not known whether in fact K and L are non-isomorphic. In this context we observe two simple corollaries.

COROLLARY 2.7. *There are natural isomorphisms*

$$F \pitchfork F \cong G \pitchfork G \ (= L), \quad \textit{and} \quad K \pitchfork Z \cong L, \quad L \pitchfork Z \cong K.$$

† This counter example was pointed out by the referee.

Proof.

$$K \pitchfork Z = (F \otimes G) \pitchfork Z \cong F \pitchfork (G \pitchfork Z), \text{ by Theorem 2 (ii)},$$
$$\cong F \pitchfork F, \text{ by Theorem 1 (ii)},$$
$$\text{similarly } \cong (G \otimes F) \pitchfork Z \cong G \pitchfork (F \pitchfork Z) \cong G \pitchfork G = L.$$
$$L \pitchfork Z \cong (F \pitchfork F) \pitchfork Z \cong F \otimes (F \pitchfork Z) \text{ by Theorem 2 (iii)},$$
$$\cong F \otimes G, \text{ by Theorem 1 (i)} = K.$$

COROLLARY 2.8. *Provided* Γ *is infinite, there exist (non-natural) isomorphisms*

$$F \pitchfork G \cong G, \quad G \pitchfork F \cong F,$$

and inclusions

$$G \subset K \subset G, \; K \not\cong G; \quad G \subset L \subset G, \; L \not\cong G.$$

Proof.

$$F \pitchfork G \cong F \pitchfork (F \pitchfork Z), \text{ by Theorem 1 (i)},$$
$$\cong (F \otimes F) \pitchfork Z, \text{ by Theorem 2 (ii)},$$

$\cong F \pitchfork Z$, since Γ and Γ^2 can be put into 1-1 correspondence (although this is not natural), $\cong G$.

$$G \pitchfork F \cong (F \pitchfork Z) \pitchfork F, \text{ by Theorem 1 (i)},$$
$$\cong F \otimes (Z \pitchfork F), \text{ by Theorem 2 (iii)},$$
$$\cong F \otimes F \cong F \text{ (non-naturally as above)}.$$

For the second part, $Z \subset F \subset G$ implies $F \pitchfork Z \subset F \pitchfork F \subset F \pitchfork G$, *i.e* $G \subset L \subset G$. Again, since Z is contained in F as a direct factor†, $Z \otimes G \subset F \otimes G$, *i.e.* $G \subset K$. Also by example 2.6, $K \subset L$, and by above $L \subset G$, so $G \subset K \subset G$.

Finally we conclude that $K \not\cong G$ (and similarly $L \not\cong G$); otherwise $G \subset L \cong K \pitchfork Z$, by Corollary 2.6, $\cong G \pitchfork Z \cong F$, plainly a contradiction since the cardinal of G is greater than that of F (Γ being infinite).

3. *Proof of Theorem* 2 (iv).

This does not depend upon the transfinite techniques used in §1, and is included primarily for the symmetry of Theorem 2. If A, B and C are any abelian groups, there is a natural homomorphism

$$\eta : (A \pitchfork B) \otimes C \to A \pitchfork (B \otimes C),$$

† [1], p. 142.

defined by $[\eta(y\otimes c)](a) = y(a)\otimes c$, $a \varepsilon A$, $c \varepsilon C$, $y \varepsilon A \pitchfork B$, and η extended linearly. We leave the reader to verify that η is indeed one valued, with values in $A \pitchfork (B\otimes C)$, and a homomorphism. In showing that η is an isomorphism we treat each of the four cases separately.

LEMMA 3.1. *If A is free and finitely-generated, then η is an isomorphism onto.*

Proof.

$$\begin{aligned}(A \pitchfork B)\otimes C &\cong (A\otimes B)\otimes C, \text{ by Lemma 2.2,}\\ &= A\otimes (B\otimes C), \text{ by the associativity of } \otimes,\\ &= A \pitchfork (B\otimes C), \text{ again by Lemma 2.2.}\end{aligned}$$

By inspection we see that these isomorphisms coincide with η (the non-naturality of the isomorphisms of Lemma 2.2 being cancelled out by its double application, once one way and once the other).

LEMMA 3.2. *If C is free and finitely-generated, then η is an isomorphism onto.*

Proof.

$$\begin{aligned}(A \pitchfork B)\otimes C &\cong C \pitchfork (A \pitchfork B), \text{ by Lemma 2.2,}\\ &\cong (C\otimes A) \pitchfork B, \text{ by Theorem 2 (ii),}\\ &\cong A \pitchfork (C \pitchfork B), \text{ similarly,}\\ &\cong A \pitchfork (C\otimes B), \text{ by Lemma 2.2, } \cong A \pitchfork (B\otimes C).\end{aligned}$$

Again by inspection, we see that these coincide with η.

LEMMA 3.3. *If A is free and C finitely generated, then η is an isomorphism onto.*

Proof. To show that η is 1-1 we only need the second condition. Let $c_1, \ldots, c_n$ be a canonical set of generators for C; that is to say the set of linear dependence relations amongst them is generated by the equations $t_m c_m = 0$, $1 \leqslant m \leqslant n$, where t_m are integers, some possibly zero. Suppose $\eta x = 0$, $x \varepsilon (A \pitchfork B)\otimes C$; then we must show $x = 0$. We may write (non-uniquely) $x = \sum_{m=1}^{n} y_m \otimes c_m$, $y_m \varepsilon A \pitchfork B$.

For any $a \varepsilon A$,

$$\begin{aligned}\sum_{1}^{n} y_m(a)\otimes c_m &= \eta x(a), \text{ by definition of } \eta,\\ &= 0 \text{ in } B\otimes C, \text{ since } \eta x = 0.\end{aligned}$$

Therefore, by our choice of c_m, we may deduce if $t_m = 0$, $y_m(a) = 0$, each $a \varepsilon A$, and so $y_m = 0$; if $t_m \neq 0$, t_m divides $y_m(a)$, each $a \varepsilon A$, and so we may define $y_m' = y_m/t_m$ in $A \pitchfork B$.

Hence $x = \sum_{t_m \neq 0} t_m y_m' \otimes c_m = \Sigma\, y_m' \otimes t_m c_m = 0$, as desired.

To show η is onto, we need both conditions.

Let $\{a^i\}_{i \varepsilon \Gamma}$ be a base for A. Given $x' \varepsilon A \pitchfork (B \otimes C)$ we have to find $x \varepsilon (A \pitchfork B) \otimes C$, such that $\eta x = x'$.

x' is uniquely determined by the set $\{x'(a^i)\}_{i \varepsilon \Gamma}$ in $B \otimes C$.

We may write (non-uniquely) $x'(a^i) = \sum_{m=1}^{n} y_m(a^i) \otimes c_m$, where the $y_m(a^i)$ are suitable elements in B.

For fixed m, the set $\{y_m(a^i)\}_{i \varepsilon \Gamma}$ define, by linear extension, $y_m \varepsilon A \pitchfork B$. Let† $x = \sum_{1}^{n} y_m \otimes c_m$, and by definition of η, $\eta x = x'$ as desired. This completes the proof of Lemma 3.3.

LEMMA 3.4. *If A is finitely-generated and C is free, then η is an isomorphism onto.*

Proof. As in the previous lemma, to show that η is 1-1 we only need the second condition. Let $\{c^i\}$ be a base for C. Suppose $\eta x = 0$, $x \varepsilon (A \pitchfork B) \otimes C$. We may write, uniquely, $x = \Sigma y_i \otimes c^i$, a finite sum. Then $\Sigma y_i(a) \otimes c^i = 0$, and so $y_i(a) = 0$ in B, each $a \varepsilon A$. Hence each $y_i = 0$, and in turn $x = 0$, as desired.

To show that η is onto we need both conditions. Given $x' \varepsilon A \pitchfork (B \otimes C)$, since A is finitely-generated, $x'A \subset \sum_I B \otimes c^i$, summed over a finite set I of i. Define $\zeta_i : B \otimes C \to B$ by

$$\zeta_i(b \otimes c^j) = \begin{cases} b, & j = i, \\ 0, & j \neq i, \end{cases}$$

and extending linearly.

Let $x = \sum_I \zeta_i x' \otimes c^i$ in $(A \pitchfork B) \otimes C$.

Then $\eta x(a) = \sum_I \zeta_i x'(a) \otimes c^i = x'(a)$, each $a \varepsilon A$, and so $\eta x = x'$, as desired.

This completes the proof of Lemma 3.4 and Theorem 2 (iv).

We conclude the paper by giving two examples, in which η is not onto, to show the necessity of the conditions, and one to show that η need not be 1-1.

Example 3.5. *Both A and C free but neither finitely-generated.*

Let $A = C = F$, countable free, and $B = Z$. Then

$$(A \pitchfork B) \otimes C \cong G \otimes F \cong K, \quad A \pitchfork (B \otimes C) \cong F \pitchfork F \cong L,$$

and as in Example 2.6, $\eta : K \to L$ is not onto.

† The x, in fact, must be unique, independently of the choice of $y_m(a^i)$ above, but this is not necessary to the proof.

Example 3.6. *Both A and C finitely generated but neither free.*

Let $A = C = Z_2$, $B = Z$.

Then $(A \pitchfork B) \otimes C = 0$, but $A \pitchfork (B \otimes C) \cong Z_2 \pitchfork Z_2 \cong Z_2$.

Example 3.7. *η not $1-1$, C being neither free nor finitely-generated.*

Let $A = F$, countable free,

$B = Z_2 + Z_3 + \ldots$ the weak sum of cyclic groups of all orders,

$C =$ the additive group of rationals.

Let $y \in A \pitchfork B$ be given by mapping a set of base elements of A, one onto each generator of a cyclic group. Then y is of infinite order, and if $c \neq 0$ in C then $y \otimes c \neq 0$. Thus $(A \pitchfork B) \otimes C \neq 0$.

But if $x \in B \otimes C$,

$$x \in (Z_2 + \ldots + Z_p) \otimes C, \text{ for some } p,$$

$$\cong (Z_2 \otimes C) + \ldots + (Z_p \otimes C) = 0.$$

Therefore $B \otimes C = 0$.

Hence $A \pitchfork (B \otimes C) = 0$, so that η cannot be 1-1.

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ON THE CLASSES $C\{M_n\}$ OF INFINITELY DIFFERENTIABLE REAL FUNCTIONS

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1. We consider a sequence of positive numbers:

$$M_n = (M_1, M_2, M_3, \ldots),$$

and a given closed interval:

$$I = [a, b].$$

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