

## ON THE FILTERED DIFFERENTIAL GROUP

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Some of the information contained in a filtered differential group  $\mathfrak{A}$  may be studied by relative homology groups, by a spectral sequence, or by an exact couple. We give a method for obtaining the complete information available in the form of a category  $\mathfrak{A}^\#$  of groups and homomorphisms. We then represent this by a diagram  $\Delta$ ; more precisely, a function is defined from the groups of the category into the regions of the diagram, such that two groups are represented by the same region if and only if they are canonically isomorphic. The function preserves lattice properties, so that the relations between the groups in the category may be easily read from the diagram. In particular, the diagram offers a graphic and intuitive approach to spectral sequences.

Section 1 describes the category, Section 2 the diagram, and in Section 3 we prove the validity of the representation.

In Section 4 we describe the application to the homology of a fibre space. In this and other applications, a certain subcategory may be described as invariant (independent of the method of calculation, whether it be by sheaves or by singular techniques, say). Section 5 identifies the corresponding invariant part of the diagram. The spectral sequence from the  $E^2$  term onwards is shown to capture all the invariant information except for certain group extensions. We give an invariant basis from which everything may be calculated. Since the basis is itself calculated from the relative homology groups, we justify the use of the latter, instead of the original filtered differential group, in constructing invariants.

In the final section we consider the analysis of the bifiltered differential group. The problem is much harder, and as yet unsolved. The difficulty can be traced to the non-distributivity of a certain lattice. A simple example is given of a free abelian group of rank 5, with a differential and two finite filtrations of lengths 2 and 3 respectively, whose resulting category contains all cyclic groups.

I should like to express my warm appreciation to Saunders MacLane for valuable criticisms and suggestions.

### 1. Algebraic constructions

**DEFINITION.** A *filtered differential group*  $\mathfrak{A}$  of length  $m$  is an abelian group  $A$ , together with an increasing sequence of subgroups  $\{A_p\}$ , where  $p$  takes integer values,  $-\infty < p < \infty$ , such that  $A_p = 0$ ,  $p \leq 0$ , and  $A_p = A$ ,  $p \geq m$ ; and an endomorphism  $d$ , such that  $dd = 0$  and  $dA_p \subset A_p$ . As a convention we allow  $p$  to take the values  $\pm \infty$  and put  $A_{-\infty} = 0$ ,  $A_\infty = A$ . The *homology group*  $H(A, d)$  is defined in the usual way.

**REMARK.** We have restricted ourselves to the above simple case to avoid

complication. In the following discussion, groups may be replaced by vector spaces over a field, modules over a ring, or by algebras. A more serious limitation is the finiteness of the filtration. In the case of an infinite filtration we can draw a similar “infinite” diagram and derive the same theorems. In most applications, however, although there is an infinite filtration, there is a grading as well, which imparts a finite property to the filtration. We discuss this in Section 4.

*Generated categories.* Let  $\mathfrak{G}$  be an arbitrary set of abelian groups and homomorphisms. We shall give a set of building operations, and define the set  $\mathfrak{G}^\#$  of all groups and homomorphisms which can be built from  $\mathfrak{G}$ . We do this by constructing inductively an increasing sequence of sets  $\mathfrak{G}_i$ .

Let  $\mathfrak{G}_0 = \mathfrak{G}$ . Let  $\mathfrak{G}_{i+1}$  comprise  $\mathfrak{G}_i$  together with

- (i) the range and domain of any homomorphism in  $\mathfrak{G}_i$  ;
- (ii) the composite  $fg$  of two composable homomorphisms  $f, g$  in  $\mathfrak{G}_i$  ;
- (iii) the natural <sup>1</sup>injection  $F \rightarrow G$ , if the groups  $F, G$  are in  $\mathfrak{G}_i$  and  $F \subset G$ ;
- (iv) the natural <sup>1</sup>projection  $F/H \rightarrow F/G$ , if the groups  $F, G, H$  are in  $\mathfrak{G}_i$  and  $F \supset G \supset H$ ;
- (v) the kernel, cokernel, image, and co-image of a homomorphism in  $\mathfrak{G}_i$ , together with the induced isomorphism between co-image and image; and
- (vi) the inverse of any isomorphism in  $\mathfrak{G}_i$  .

Let  $\mathfrak{G}^\# = \bigcup_{i=0}^\infty \mathfrak{G}_i$  .

It may be observed that  $\mathfrak{G}^\#$  is in fact a <sup>2</sup>category of groups and homomorphisms, in virtue of (i), (ii) and (iii) (which ensures that the identity homomorphism on any group in  $\mathfrak{G}^\#$  is in  $\mathfrak{G}^\#$ ). We say that  $\mathfrak{G}^\#$  is *the category generated by*  $\mathfrak{G}$ . Notice that  $\mathfrak{G}^{\#\#} = \mathfrak{G}^\#$ . In view of the following lemma,  $\mathfrak{G}^\#$  represents, in a sense, the maximum information obtainable from  $\mathfrak{G}$ .

LEMMA 1.  $\mathfrak{G}^\#$  is closed under the operations of

- (a) forming groups by taking intersections, (group-) unions, or quotients of groups, or kernels, cokernels, images or co-images of homomorphisms;
- (b) forming homomorphisms by injecting subgroups, projecting onto quotient groups, or by composing, decomposing, restricting or inducing homomorphisms;
- (c) forming isomorphisms by inverting isomorphisms, or by using the first or second isomorphism theorems.

PROOF. The proof of most of the lemma is immediate from the building operations; that of the rest is a straight-forward application. We give one example, of proving  $\mathfrak{G}^\#$  closed under intersection, and leave the remainder to the reader.

Given  $F, G \subset H$  in  $\mathfrak{G}^\#$ , we have to show that  $F \cap G \in \mathfrak{G}^\#$ . For some  $i$ ,  $F, G, H \in \mathfrak{G}_i$ . Therefore the injections  $j:F \rightarrow H, j':G \rightarrow H$  are in  $\mathfrak{G}_{i+1}$  by (iii). The quotient  $H/G \in \mathfrak{G}_{i+2}$ , by (v) since it is the cokernel of  $j'$ . The projection  $p:H \rightarrow$

<sup>1</sup> We shall throughout use the terms *injection* and *projection* to mean the natural injection and projection such as occur above.

<sup>2</sup> For the definition of *category*, see [4], Chapter IV.

$H/G$  is in  $\mathfrak{G}_{i+3}$  by (iv). The composite  $pj \in \mathfrak{G}_{i+4}$  by (ii). The required intersection  $F \cap G \in \mathfrak{G}_{i+5}$  by (v), being the kernel of  $pj$ .

*The category  $\mathfrak{A}^\#$ .* The main object of this paper is to examine the category generated by the filtered differential group  $\mathfrak{A} = \{A, A_p, d\}$ . We deduce at once, from the type of building operations to which we have restricted ourselves, that any group of  $\mathfrak{A}^\#$  is a subquotient group of  $A$ , or, more precisely, is obtained from  $A$  in a finite number of steps by successively taking subgroups and quotient groups. To assist the discussion we introduce two subsets  $\mathfrak{L}, \mathfrak{Q}$  of the groups of  $\mathfrak{A}^\#$ . The set of all subgroups of  $A$  form a modular lattice under intersection and group-union, which admits  $d$  and  $d^{-1}$  as unary operators. Let

$$\mathfrak{L} = \{A_p; \cap, +, d, d^{-1}\},$$

namely the smallest sublattice containing the groups  $A_p$  and closed under the four operators indicated. We shall in future always use  $X, Y, Z, \dots$  to denote groups of  $\mathfrak{L}$ .

LEMMA 2.  $\mathfrak{L} \subset \mathfrak{A}^\#$ .

PROOF.  $\{A_p\} \subset \mathfrak{A}^\#$ , and  $\mathfrak{A}^\#$  is closed under  $\cap$  and  $+$ . If  $X \in \mathfrak{L} \cap \mathfrak{A}^\#$ , the restriction of  $d:A \rightarrow A$  to  $d:X \rightarrow A$  is in  $\mathfrak{A}^\#$  by Lemma 1, and hence also the image  $dX$ . Similarly the kernel  $d^{-1}X$  of the composite

$$A \xrightarrow{d} A \xrightarrow{\nu} A/X$$

is in  $\mathfrak{A}^\#$ , where  $\nu$  is the natural projection. Therefore  $\mathfrak{L} \cap \mathfrak{A}^\#$ , being closed under  $d$  and  $d^{-1}$  contains  $\mathfrak{L}$ , and so  $\mathfrak{L} \subset \mathfrak{A}^\#$ .

THEOREM 1. *The lattice  $\mathfrak{L}$  is finite and distributive, and is generated by the two<sup>3</sup> chains*

$$\mathfrak{L}: 0 = A_0 \subset A_1 \subset \dots \subset A_p \subset A_{p+1} \subset \dots \subset A_m = A,$$

$$\mathfrak{L}': 0 = dA_0 \subset dA_1 \subset \dots \subset dA \subset d^{-1}0 \subset d^{-1}A_1 \subset \dots \subset d^{-1}A_m = A.$$

A typical element  $X$  of  $\mathfrak{L}$  may be written

$$X = (T_1 \cap T'_1) + (T_2 \cap Y'_2) + \dots + (T_s \cap T'_s),$$

where  $T_1 \subset T_2 \subset \dots \subset T_s$  in  $\mathfrak{L}$ , and  $T'_1 \supset T'_2 \supset \dots \supset T'_s$  in  $\mathfrak{L}'$ .

We shall reserve the proof of Theorem 1, and likewise of the following Theorem 2, until Section 3. Theorem 1 shows that  $\mathfrak{L}$  admits of a fairly concise description. The category  $\mathfrak{A}^\#$  is not so readily accessible, owing to the duplication of information in the form of canonically isomorphic groups. However, if we work modulo the first and second isomorphism theorems, we are able in Theorem 2 to refer the examination of  $\mathfrak{A}^\#$  back to that of  $\mathfrak{L}$ . Let us first attach a precise meaning to this statement.

*Canonical isomorphisms.* An isomorphism

$$\frac{F/H}{G/H} \rightarrow \frac{F}{G}$$

<sup>3</sup> A chain, in the sense of [1], is a simply ordered subset of a lattice.

where  $F \supset G \supset H$  in  $\mathfrak{A}^\#$ , will be called a 2-isomorphism of  $\mathfrak{A}^\#$ . An isomorphism  $e$  is said to be a 2c-isomorphism if it is equal to the composite  $e_1 e_2 \cdots e_r$ , where, for each  $i$ , either  $e_i$  or  $e_i^{-1}$  is a 2-isomorphism. We say of two groups,  $F = G$  modulo the second isomorphism theorem, if there exists a 2c-isomorphism between them.

Similarly an isomorphism

$$\frac{F}{F \cap G} \rightarrow \frac{F + G}{G}$$

is a 1-isomorphism, and the composite of such and their inverses is called a 1c-isomorphism. A canonical isomorphism, or c-isomorphism, is an arbitrary composite of 1c-isomorphisms and 2c-isomorphisms. Two groups in  $\mathfrak{A}^\#$  are said to be canonically isomorphic, or equal modulo the first and second isomorphism theorems, if there exists in  $\mathfrak{A}^\#$  a canonical isomorphism between them. Clearly this is an equivalence relation between the groups of  $\mathfrak{A}^\#$ . We say of homomorphisms,  $f = g$  modulo the first and second isomorphism theorems, if there are c-isomorphisms  $e, e'$  such that  $f = ege'$ .

*Notation.* Recall that  $X, Y, Z, \dots$  always denote groups of  $\mathfrak{L}$ . Let  $\mathfrak{Q}$  be the set of quotients  $X/Y$ , where  $X \supset Y$ . We make the convention of identifying  $X = X/0$ , so that  $\mathfrak{L} \subset \mathfrak{Q}$ . Let  $\mathfrak{I}$  be the set of injections  $Y/Z \rightarrow X/Z$ , where  $X \supset Y \supset Z$ . Let  $\mathfrak{P}$  be the set of projections  $X/Z \rightarrow X/Y$ , where  $X \supset Y \supset Z$ . If  $X \supset Y \supset d^{-1}0$ , then  $d$  induces an isomorphism  $X/Y \rightarrow dX/dY$ . Let  $\mathfrak{E}$  be the groupoid of isomorphisms between the elements of  $\mathfrak{Q}$  generated by 1-isomorphisms and those induced by  $d$ . Let  $\mathfrak{F}$  be the set of homomorphisms  $f = jep$ ,  $j \in \mathfrak{I}, e \in \mathfrak{E}, p \in \mathfrak{P}$ .

This notation enables us to state, and in Section 3 will facilitate the proof of:

**THEOREM 2.**  $\mathfrak{A}^\#$  is finite, and  $\mathfrak{A}^\# = \mathfrak{Q} \cup \mathfrak{F}$ , modulo the second isomorphism theorem.

We conclude this section by describing two well known significant subsets of  $\mathfrak{A}^\#$ .

**DEFINITION.** A graded group  $D$  is a set  $\{D_p; -\infty < p < \infty\}$  of groups  $D_p$  suffixed by the integer  $p$ . The graded group associated with the filtered group is written

$$\text{Gr } A = \{A_p/A_{p-1}; -\infty < p < \infty\}.$$

A differential of degree  $-r$  on  $D$  is a set  $d^r = \{d_p^r; -\infty < p < \infty\}$  of homomorphisms  $d_p^r: D_p \rightarrow D_{p-r}$ , such that  $d_p^r d_{p+r}^r = 0$ . The homology group  $H(D, d^r)$  is defined in the usual way and is likewise graded.

**REMARK.** It is customary to write a graded differential group as the direct sum of its homogeneous components. The value of this is purely notational, and Steenrod has pointed out that it is mistaken from the point of view of considering the category of graded groups, since the image of a graded group under the functor  $\text{Hom}(\ , G)$  is then no longer a graded group. The reason for our preference for the above definition in the present context, is that the building operations

do not include direct summation. We claim that this is not a limitation of the building operations, because the inclusion of such would not echo any useful application, and arbitrary summation, rather than enlarging the information available, would tend to confuse the issue.

*The relative homology groups.* Let  $\mathfrak{S}$  be the set of relative homology groups

$$H(A_p, A_q), \quad -\infty < q \leq p < \infty,$$

and homomorphisms

$$\begin{aligned} \bar{j}: H(A_p, A_q) &\rightarrow H(A_{p'}, A_{q'}), & p \leq p', q \leq q', & \text{induced by injection, and} \\ d: H(A_p, A_q) &\rightarrow H(A_q, A_s), & p \geq q \geq s, & \text{induced by } d. \end{aligned}$$

Let  $D_p$  be the image of the homomorphism  $\bar{j}: H(A_p) \rightarrow H(A)$  in  $\mathfrak{S}$ . The groups  $D_p$  filter  $H(A)$ ; denote by  $\text{Gr } H(A)$  the associated graded group.

*The spectral sequence.* For  $-\infty < p < \infty$  and  $0 \leq r \leq \infty$ , let

$$\begin{aligned} C_p^r &= A_p \cap d^{-1}A_{p-r}, & B_p^r &= A_p \cap dA_{p+r}, \\ E_p^r &= C_p^r / (C_{p-1}^{r-1} + B_p^{r-1}), & E^r &= \{E_p^r; -\infty < p < \infty\}. \end{aligned}$$

If  $r = \infty$ ,  $d$  trivially induces  $d_\infty = 0: E_\infty \rightarrow E_\infty$ . If  $0 \leq r < \infty$ , it is easy to show

$$d(C_p^r) \subset C_{p-r}^r, \quad d(C_{p-1}^{r-1} + B_p^{r-1}) \subset (C_{p-r-1}^{r-1} + B_{p-r}^{r-1}).$$

Hence  $d$  induces  $d_p^r: E_p^r \rightarrow E_{p-r}^r$ . The collection  $\{d_p^r; -\infty < p < \infty\}$  defines a differential  $d^r$  on  $E^r$ . Let

$$\mathfrak{S}^s = \{(E^r, d^r); s \leq r \leq \infty\},$$

be the sequence of differential graded groups.  $\mathfrak{S}^0$  is defined to be the *spectral sequence of  $\mathfrak{A}$* . We prove in Section 2 the elementary properties:

**THEOREM 3.** (Leray [5]).

- (i)  $E_p^r = 0$  unless  $1 \leq p \leq m$ .
- (ii)  $E^m = E^{m+1} = \dots = E^\infty$  (i.e.  $d^r = 0$  for  $m \leq r \leq \infty$ ).
- (iii)  $E^{r+1} \cong H(E^r, d^r)$ ,  $0 \leq r < \infty$ .
- (iv)  $E^\infty \cong \text{Gr } H(A)$ .

### 2. The diagram $\Delta$

In the euclidean plane let  $[\pi, \rho]$  be the unit square given by  $\pi - 1 < x \leq \pi$ ,  $\rho - 1 < y \leq \rho$ , where  $\pi, \rho$  are integers. Let

$$\Delta = \cup \{[\pi, \rho]; 1 \leq \pi \leq m, \pi - m - 1 \leq \rho \leq \pi\}.$$

Thus  $\Delta$  is the union of a set of unit squares (see Figure 1), which we shall denote by  $a, b, c, \dots$  etc. Write  $a \leq b$  if  $a = [\pi, \rho]$ ,  $b = [\pi', \rho']$ , with  $\pi \leq \pi', \rho \leq \rho'$ ; and write  $a < b$  if, further,  $a \neq b$ . The decomposition of  $\Delta$  into a partially ordered set of disjoint pieces is, essentially, the only property of  $\Delta$  we shall use.

The union  $M$  of any subset of the squares of  $\Delta$  we shall call a *region* of  $\Delta$ .  $\mathcal{M}$

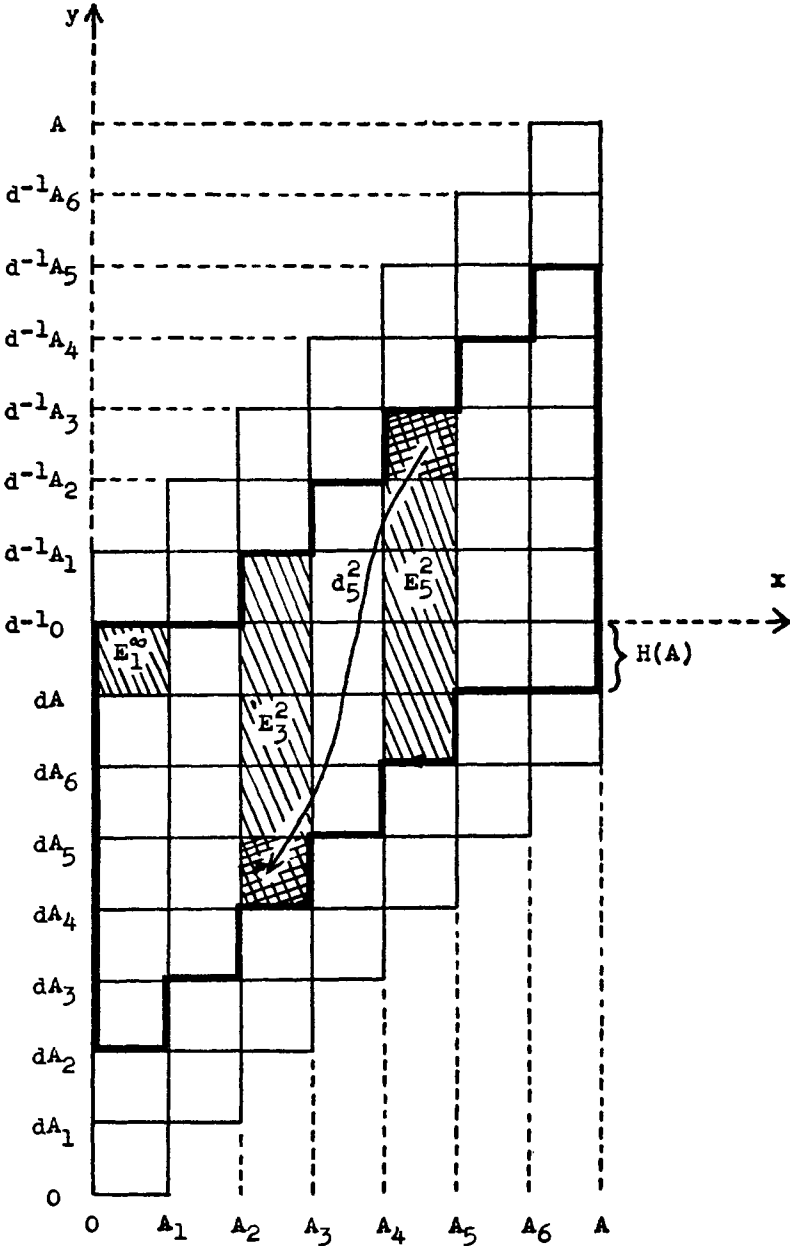


FIGURE 1. The diagram  $\Delta$ . ( $m = 7$ ).

may be uniquely determined by a set  $\Phi$  of formulae governing the range of integer values of  $\pi, \rho$  for which  $[\pi, \rho] \subset M$ ; we write  $M = \Delta(\Phi)$ . In particular let

$$\Delta^+ = \Delta(\rho > 0), \quad \Delta^0 = \Delta(\rho = 0), \quad \Delta^- = \Delta(\rho < 0).$$

We define a function  $\delta$  (corresponding to  $d$ ) from the regions of  $\Delta^+$  to those of  $\Delta^-$ , by

$$\begin{aligned} \delta[\pi, \rho] &= [\rho, \pi - m - 1], \quad \text{for } [\pi, \rho] \subset \Delta^+, \text{ and} \\ \delta M &= \cup \{ \delta a; a \subset M \}, \quad \text{for } M \subset \Delta^+. \end{aligned}$$

$M$  and  $\delta M$  are called  $\delta$ -paired regions. In particular  $\Delta^+, \Delta^-$  are  $\delta$ -paired.

We shall define a function  $\lambda$  from the groups of  $\mathfrak{A}^\#$  into the regions of  $\Delta$ . We run into the technical difficulty, however, of confusing the actual groups of  $\mathfrak{A}^\#$  with the formulae for them; in a particular example of  $\mathfrak{A}$ , two different formulae may give rise to the same group, whereas in general they represent different groups. Two solutions to this difficulty would be either to regard the objects of the category  $\mathfrak{A}^\#$  not as groups but as equivalence classes of formulae, or else to permit  $\lambda$  to be many valued in special cases (as in the examples of Section 3). However, in order to keep the exposition free from logical niceties and as intuitive as possible, we use the following device.

*Definition.* The filtered differential group  $\mathfrak{A}$  of length  $m$  is *sufficiently general* if any two formulae defining isomorphic groups of  $\mathfrak{A}^\#$  also define isomorphic groups in an arbitrary filtered differential group of length  $m$ .

A sufficiently general  $\mathfrak{A}$  does exist, for it will transpire that, since  $\mathfrak{A}^\#$  is finite, one may be constructed by combining suitable examples. From now on we shall assume that  $\mathfrak{A}$  is sufficiently general unless otherwise stated.

DEFINITION OF  $\lambda$ . Let

$$\begin{aligned} \lambda(A_p) &= \Delta(\pi \leq p), \\ \lambda(dA_p) &= \Delta(\rho \leq p - m - 1), \\ \lambda(d^{-1}A_p) &= \Delta(\rho \leq p). \end{aligned}$$

In Figure 1,  $\lambda(A_p)$  is the region to the left of the vertical line marked  $A_p$ , and  $\lambda(dA_p), \lambda(d^{-1}A_p)$  are the regions below the horizontal lines. Extend  $\lambda$  to  $\mathfrak{F}$  by Theorem 1 and the lattice-homomorphism properties of Theorem 4 (i) below. Extend  $\lambda$  to  $\mathfrak{Q}$  by Property (ii) below. Theorem 2 and Property (iii) are sufficient to extend  $\lambda$  to all the groups of  $\mathfrak{A}^\#$ .

THEOREM 4. (the representation theorem). Let  $\mathfrak{A}$  be sufficiently general. Let  $F, G, H$  denote groups in  $\mathfrak{A}^\#$ .

(i) If  $F, G \subset H$ , then  $\lambda(F \cap G) = \lambda F \cap \lambda G$ , and

$$\lambda(F + G) = \lambda F \cup \lambda G.$$

(ii) If  $F \supset G$ , then  $\lambda(F/G) = \lambda F - \lambda G$ .

(iii)  $\lambda F = \lambda G$  if and only if  $F$  and  $G$  are canonically isomorphic.

(iv)  $\delta(\lambda F) = \lambda G$  if and only if  $d$  induces an isomorphism  $F \rightarrow G$ , modulo the first and second isomorphism theorems.

(v) If  $\lambda F, \lambda G$  are neither equal nor  $\delta$ -paired, then  $F \not\cong G$ .

Summarizing, we may say that  $\lambda$  offers accuracy and maximum economy.

*Interpretation of the regions of  $\Delta$ .* If  $M$  is a region of  $\Delta$ , we say that  $M$  is *permissible* if  $M = \lambda G$ , some  $G \in \mathfrak{A}^\#$ . If  $X \in \mathfrak{X}$  we may write  $X$  in normal form as in Theorem 1. Therefore by Theorem 4(i) the region corresponding to a typical element of  $\mathfrak{X}$  is that below a zig-zag line running from top-left to bottom-right, as shown in Figure 2. Any permissible region is the difference between two such. Permissible regions are characterised by:

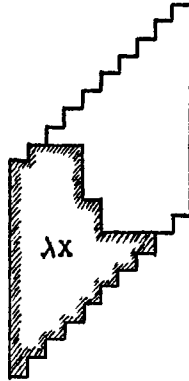


FIG. 2

LEMMA 3.  $M$  is permissible if and only if  $a < b < c$ ,  $a, c \subset M$  implies  $b \subset M$ .

PROOF. If  $M$  is permissible,

$$\begin{aligned} M &= \lambda G = \lambda(X/Y), \quad \text{some } X, Y \text{ in } \mathfrak{X} \text{ by Theorem 2,} \\ &= \lambda X - \lambda Y, \quad \text{by definition of } \lambda. \end{aligned}$$

Suppose  $a < b < c$  and  $a, c \subset M$ . Then from the shape of  $\lambda Y$  (in Figure 2),  $a \not\subset \lambda Y$  implies  $\{e; e > a\} \subset \Delta - \lambda Y$ ; and similarly  $c \subset \lambda X$  implies

$$\{e; e < c\} \subset \lambda X.$$

Hence  $b \subset \lambda X \cap (\Delta - \lambda Y) = M$ .

Conversely suppose  $M$  has this property. Let

$$X = \cap \{X'; \lambda X' \supset M\}, \quad Y = \cup \{(X \cap Y'); \lambda Y' \cap M = \emptyset\},$$

where  $X', Y'$  run through  $\mathfrak{X}$ . Then  $\lambda X - \lambda Y \supset M$ . Let  $b \subset \lambda X - \lambda Y$ . If  $X''$  is maximal subject to  $b \not\subset \lambda X''$ , then  $M \not\subset \lambda X''$ , otherwise  $b \subset \lambda X$  by the construction of  $X$ . Hence there exists  $c \subset M \cap (\Delta - \lambda X'')$  with  $c \geq b$ . Similarly if  $Y''$  minimal subject to  $b \subset \lambda Y''$ , then  $\lambda Y'' \cap M \neq \emptyset$ , otherwise  $b \subset \lambda Y$ . Hence there exists  $a \subset M$ ,  $a \leq b$ . By the given property of  $M$ ,  $b \subset M$ . Therefore  $M = \lambda X - \lambda Y = \lambda(X/Y)$ , and is permissible.



COROLLARY 3.1. *All rectangles, including the unit squares, are permissible. (See Figure 5).*

We call the groups corresponding to rectangles *rectangular*. Lemma 5 will shortly show that all the relative homology groups and spectral sequence terms are rectangular, and in Section 5 we shall see that any other group can be constructed from rectangular groups. An arbitrary region of  $\Delta$  may be interpreted as representing a subcategory of  $\mathfrak{A}^\#$ . In Section 5 we shall discuss the subcategory corresponding to the non-permissible region outlined heavily in Figure 1. We leave the reader to prove the following lemma, which is useful in reading the diagram.

LEMMA 4. *Modulo the first and second isomorphism theorems,*

- (i)  *$F$  is a subquotient group of  $G$  if and only if  $\lambda F \subset \lambda G$ .*
- (ii)  *$F$  is a subgroup of  $G$  if and only if  $\lambda F \subset \lambda G$ , and  $a > b$ ,  $a \subset \lambda F$ ,  $b \subset \lambda G$  implies  $b \subset \lambda F$ .*
- (iii)  *$F$  is a quotient group of  $G$  if and only if  $\lambda F \subset \lambda G$ , and  $a < b$ ,  $a \subset \lambda F$ ,  $b \subset \lambda G$  implies  $b \subset \lambda F$ .*
- (iv)  *$a \leq b$  if and only if there is a group  $G \in \mathfrak{A}^\#$ , having a subgroup  $F$  and a quotient  $H$  such that  $\lambda F = a$ ,  $\lambda H = b$ .*

*Representation of the homomorphisms of  $\mathfrak{A}^\#$  on  $\Delta$ .* We have in Theorem 3 and Lemma 4 given necessary and sufficient conditions on the regions  $\lambda F$  and  $\lambda G$  for there to be, modulo the first and second isomorphism theorems, an injection, projection or isomorphism,  $F \rightarrow G$ . By Theorem 2 an arbitrary homomorphism of  $\mathfrak{A}^\#$  is, modulo the first and second isomorphism theorems, the composition of one of each of these, and we may therefore represent it by suitable "moves" on  $\Delta$ . Conversely we may be given two regions  $M, N$  of  $\Delta$ , and a specific homomorphism  $f$  between the corresponding groups. It is useful to be able to decompose  $f$  into the above primitive moves, or, more precisely, to determine the regions representing the kernel and image of  $f$  etc.

THEOREM 5. *Suppose  $f: F \rightarrow G$  is in  $\mathfrak{F}$ , and  $\lambda F = M, \lambda G = N$ . The following table gives formulae in the two cases when  $f$  is induced  $1^\circ$  by injection or  $2^\circ$  by the differential.*

	$1^\circ$	$2^\circ$
$\lambda(\text{kern } f)$	$M - N$	$M - \delta^{-1}(N \cap \Delta^-)$
$\lambda(\text{co-im } f)$	$M \cap N$	$M \cap \delta^{-1}(N \cap \Delta^-)$
$\lambda(\text{im } f)$	$M \cap N$	$N \cap \delta(M \cap \Delta^+)$
$\lambda(\text{cokern } f)$	$N - M$	$N - \delta(M \cap \Delta^+)$ .

Figure 3 displays cases  $1^\circ$  and  $2^\circ$  when  $M, N$  are rectangles; the shaded areas show the image and co-image.

PROOF OF THEOREM 5. ( $1^\circ$ ) Suppose  $f$  is induced by injection. We may write

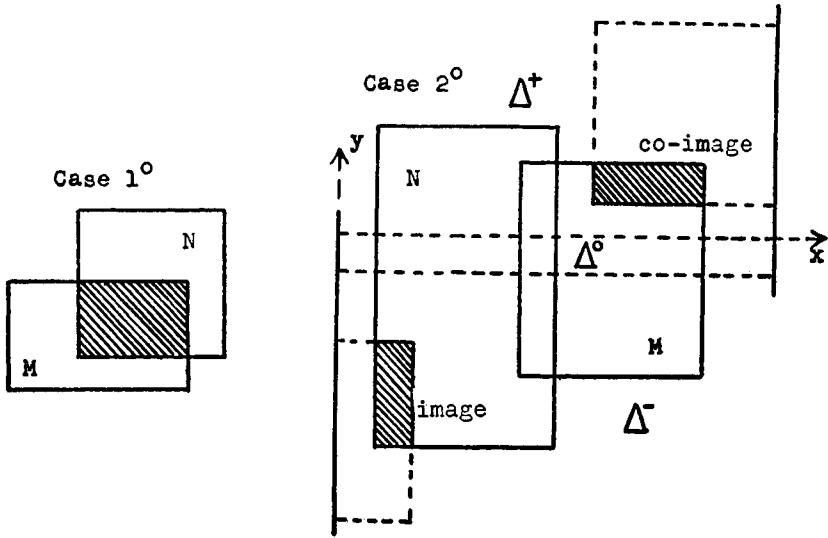


FIG. 3

$F = X/Y, G = X'/Y'$ , where  $Y \subset X, Y' \subset X'$ , and may decompose  $f$

$$\frac{X}{Y} \xrightarrow{p} \frac{X}{X \cap Y'} \xrightarrow{e} \frac{X + Y'}{Y'} \xrightarrow{j} \frac{X'}{Y'}$$

Therefore

$$\begin{aligned} \lambda(\text{im } f) &= (\lambda X \cup \lambda Y') - \lambda Y' = \lambda X - \lambda Y' \\ &= (\lambda X - \lambda Y) - \lambda Y', \text{ since } \lambda Y \subset \lambda Y', \\ &= M - \lambda Y' = M \cap (\lambda X' - \lambda Y'), \text{ since } M \cap \lambda Y' \subset \lambda X', \\ &= M \cap N. \end{aligned}$$

The other formulae may be deduced from Theorem 4(ii).

(2°) If  $f: X/Y \rightarrow X'/Y'$  is induced by  $d$ , then  $dX \subset X', dY \subset Y'$  and  $\text{im } f = (dX + Y')/Y'$ . Therefore

$$\begin{aligned} \lambda(\text{im } f) &= \lambda(dX) - \lambda Y', \text{ as above,} \\ &= [\lambda(dX) - \lambda(dY)] \cap N, \text{ as above,} \\ &= \lambda(dX/dY) \cap N \\ &= \delta\lambda[(X + d^{-1}0)/(Y + d^{-1}0)] \cap N \\ &= \delta(M \cap \Delta^+) \cap N. \end{aligned}$$

The other formulae follow similarly.

REMARK. For general  $f \in \mathfrak{F}$  the situation is not quite so simple. For example if  $f = jep$ , and  $e$  was a 1c-isomorphism (rather than a 1-isomorphism as in case 1° above) then  $\lambda(\text{im } f)$  could be the union of any number of the components of  $M \cap N$ , as shown in Figure 4.

We are now in a position to identify on the diagram the relative homology groups and spectral sequence.

LEMMA 5.

$$\lambda H(A_p, A_q) = \Delta(q < \pi \leq p, \min(0, p - m) \leq \rho \leq \max(0, q)),$$

$$-\infty < q \leq p < \infty.$$

$$\lambda E_p^r = \Delta(\pi = p, \min(0, p - m + r - 1) \leq \rho \leq \max(0, p - r)),$$

$$-\infty < p < \infty, 0 \leq r \leq \infty.$$

The proof is immediate from the definitions of groups concerned and the properties of  $\lambda$ . It is shown diagrammatically in Figure 5 for  $\lambda E_p^r$ .

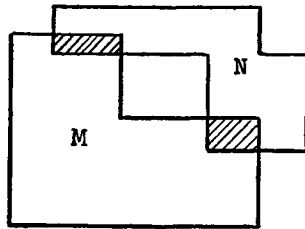


FIG. 4

Description of Lemma 5. The  $\lambda H(A_p, A_q)$  are characterised by being rectangles with one corner on  $x = y$  and the opposite corner on  $x = y + m + 1$ . In particular  $\lambda H(A) = \Delta^0$  and  $\lambda D_p = \Delta^0(\pi \leq p)$ .

The equation  $E^0 = \text{Gr } A$  is echoed on the diagram by decomposing  $\Delta$  into columns: in other words  $\lambda E_p^0$  is the column  $\Delta(\pi = p)$ . As  $r$  increases the columns  $\lambda E_p^r$  become shorter. In fact  $\lambda E_p^{r+1}$  is obtained from  $\lambda E_p^r$  by omitting the top square if it is in  $\Delta^+$ , and the bottom square if it is in  $\Delta^-$ . After at most  $m$  steps we are left with a single square in  $\Delta^0$ , namely  $\lambda E_p^m = \lambda E_p^\infty = [p, 0]$ . Figure 1 gives examples.

Proof of Theorem 3. We use heavily Part (iii) of the representation theorem, which says that if  $\lambda F = \lambda G$  then  $F \cong G$ . In particular if  $\lambda F = \emptyset$  then  $F = 0$ . Parts (i) and (ii) of Theorem 3 are immediate corollaries.

(iii) Recall that  $d_p^r$  maps  $E_p^r$  into  $E_{p-r}^r$ . Suppose  $[\pi, \rho] \subset \lambda(\text{co-im } d_p^r)$ . By Theorem 5,  $[\pi, \rho] \subset \lambda E_p^r \cap \Delta^+$ , and  $\delta[\pi, \rho] \subset \lambda E_{p-r}^r$ . Therefore  $\pi = p$  and

$$\rho = p - r > 0.$$

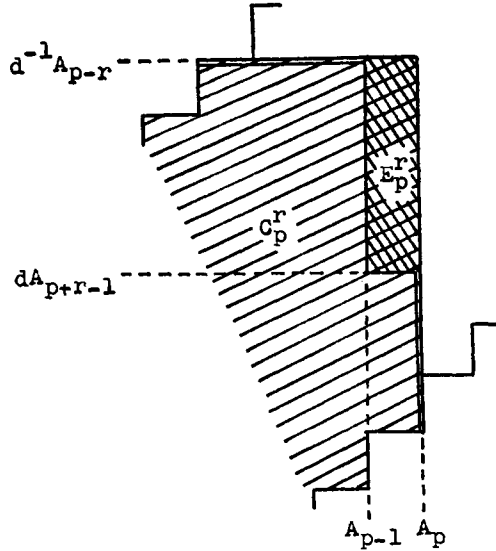


FIG. 5

Hence if  $p > r$ ,  $d_p^r$  “maps” the top square of  $\lambda E_p^r$  onto the bottom square of  $\lambda E_{p-r}^r$  (as shown for  $d_5^2$  in Figure 1). More precisely,

$$\lambda(\text{kern } d_p^r) = \begin{cases} \lambda E_p^r \text{ minus the top square, } r < p, \\ \lambda E_p^r, r \geq p. \end{cases}$$

$$\lambda(\text{im } d_{p+r}^r) = \begin{cases} \text{the bottom square of } \lambda E_p^r, r < m - p, \\ \emptyset, r \geq m - p. \end{cases}$$

Hence

$$\begin{aligned} \lambda H_p(E^r, d^r) &= \lambda(\text{kern } d_p^r) - \lambda(\text{im } d_{p+r}^r), \\ &= \lambda E_p^{r+1}, \end{aligned}$$

by the above description. Therefore  $H_p(E^r, d^r) \cong E_p^{r+1}$ , as desired.

(iv)  $\lambda(D_p/D_{p-1}) = \lambda D_p - \lambda D_{p-1} = [p, 0] = \lambda E_p^\infty.$

Hence

$$\text{Gr } H(A) = \{D_p/D_{p-1}; -\infty < p < \infty\} \cong E^\infty.$$

The proof of Theorem 3 is complete. We should point out that it was somewhat easy because it depended heavily upon Theorems 1, 2 & 4, which have yet to be proved in the following section. On the other hand, once the validity of the

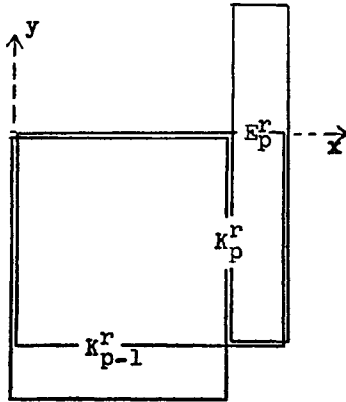


Fig. 6

diagram is established, many other arguments and constructions become equally visual and straightforward. We shall consider two further examples of constructions in this paper, the Massey<sup>4</sup> exact couple below, and the basis for invariants in Section 5. For an example of an argument which is much assisted by using the diagram, the reader is referred to [10].

*The exact couple.* In  $\mathfrak{S}$  we have the exact triple

$$\begin{array}{ccc}
 H(A_{p-1}) & \xrightarrow{J} & H(A_p) \\
 \swarrow \bar{d} & & \searrow \bar{i} \\
 & & H(A_p; A_{p-1})
 \end{array}$$

Observe that  $H(A_p, A_{p-1}) \cong E_p^1$  by Lemma 5. Let  $K_p^1 = H(A_p)$  and  $K^1 = \{K_p^1\}$ . The set of triples as  $p$  varies may be combined to give the exact couple

$$\begin{array}{ccc}
 K^1 & \xrightarrow{\alpha^1} & K^1 \\
 \swarrow \gamma^1 & & \searrow \beta^1 \\
 & & E^1
 \end{array}$$

which we denote by  $\mathfrak{R}^1$ . Let  $\mathfrak{R}^2, \mathfrak{R}^3, \dots$  denote the sequence of derived<sup>4</sup> couples. Then, from Figure 6, it is easy to show that  $\mathfrak{R}^r$  is

$$\begin{array}{ccc}
 K^r & \xrightarrow{\alpha^r} & K^r \\
 \swarrow \gamma^r & & \searrow \beta^r \\
 & & E^r
 \end{array}$$

<sup>4</sup> See Massey [6].

where  $\lambda(K_p^r) = \Delta(\pi \leq p, \min(0, p - m + r - 1) \leq \rho \leq 0), -\infty < p < \infty, 1 \leq r < \infty$ , a rectangle in  $\Delta^- \cup \Delta^0$ ; that  $\alpha^r, \beta^r$  are induced by injection and  $\gamma^r$  by  $d$ , that  $\mathfrak{K}^r = \mathfrak{K}^m, r \geq m$ , and that  $K_p^m = D_p$ , all  $p$ .

### 3. The validity of the diagram

This section is devoted to the proofs of Theorems 1, 2 and 4.

LEMMA 6. (Birkhoff). *The free modular lattice  $\mathfrak{M}$  generated by two finite chains  $\mathfrak{X}, \mathfrak{X}'$  is finite and distributive. A typical element  $X \in \mathfrak{M}$  may be written uniquely in the normal form*

$$X = (T_1 \cap T'_1) \cup (T_2 \cap T'_2) \cup \dots \cup (T_s \cap T'_s),$$

where  $T_1 \subset T_2 \subset \dots \subset T_s$  in  $\mathfrak{X}$ , and  $T'_1 \supset T'_2 \supset \dots \supset T'_s$  in  $\mathfrak{X}'$ .

PROOF. See [1], Theorem 5, p. 72. This theorem is a stroke of good fortune for the filtered differential group. The analogous result for three chains is not true, so that the analysis of the bifiltered group presents greater difficulties: some of them are pointed out in Section 6.

COROLLARY 6.1. *The lattice  $\mathfrak{N}$  of subgroups of  $A$  generated by the chains  $\mathfrak{X}, \mathfrak{X}'$  in Theorem 1 is finite and distributive.*

PROOF.  $\mathfrak{N}$  is modular, although not free due to the relations  $dA_p \subset A_p \subset d^{-1}A_p$ . Therefore there is a lattice-epimorphism  $\mathfrak{M} \rightarrow \mathfrak{N}$ , giving the result.

LEMMA 7.  $\mathfrak{X} = \mathfrak{N}$ .

PROOF. It is sufficient to show  $\mathfrak{N}$  admits the unary operators  $d$  and  $d^{-1}$ , for then

$$\mathfrak{X} = \{A_p; \cap, +, d, d^{-1}\} \subset \{\mathfrak{N}; \cap, +, d, d^{-1}\} = \mathfrak{N} \subset \mathfrak{X}.$$

Let  $X \in \mathfrak{N}$ . We may put  $X$  into normal form (not uniquely in general) by choosing the normal form of some element in  $\mathfrak{M}$  which maps onto  $X$ . Then

$$dX = d(T_1 \cap T'_1) + d(T_2 \cap T'_2) + \dots + d(T_s \cap T'_s).$$

Therefore  $dX \in \mathfrak{N}$  provided  $d(T \cap T') \in \mathfrak{N}$  for each  $T \in \mathfrak{X}, T' \in \mathfrak{X}'$ . If  $T' \subset d^{-1}0$  then  $d(T \cap T') \subset d(d^{-1}0) = 0$ . There remains the case when  $T = A_p, T' = d^{-1}A_q$ . We leave the reader to verify that

$$d(A_p \cap d^{-1}A_q) = dA_p \cap A_q.$$

The proof of the closure of  $\mathfrak{N}$  with respect to  $d^{-1}$  follows dually, interchanging  $d$  with  $d^{-1}$ ,  $\cap$  with  $+$ , and using the dual normal form.

Lemmas 6 & 7 and Corollary 6.1 prove Theorem 1.

For the proof of Theorem 2 we use the notation  $\mathfrak{X}, \mathfrak{Q}, \mathfrak{S}, \mathfrak{P}, \mathfrak{E}, \mathfrak{F}$  introduced in Section 1. Also let  $\mathfrak{Q}_0 = \mathfrak{X}, \mathfrak{Q}_{i+1}$  be the set of quotients of  $\mathfrak{Q}_i$ , and  $\mathfrak{Q} = \cup \mathfrak{Q}_i$ .

LEMMA 8. *Given  $\bar{G} \in \mathfrak{Q}$ , there is a unique 2c-isomorphism  $\tau(\bar{G}): \bar{G} \rightarrow G$  such that  $G \in \mathfrak{Q}$ .*

PROOF. The proof is straightforward.  $\mathfrak{D}_1 = \mathfrak{D}$ . Given  $\bar{G} \in \mathfrak{D}_i, i > 1$ , then  $\bar{G} = F_1/G_1$ , some  $F_1 \supset G_1$  in  $\mathfrak{D}_{i-1}$ ;  $F_1 = F_2/H_2, G_1 = G_2/H_2$ , some  $F_2 \supset G_2 \supset H_2$  in  $\mathfrak{D}_{i-2}$ ; and so on. There is a sequence of 2-isomorphisms

$$\bar{G} = \frac{F_1}{G_1} \rightarrow \frac{F_2}{G_2} \rightarrow \dots \rightarrow \frac{F_i}{G_i} = G,$$

say. The uniqueness follows from observing that  $\bar{G}$  is a set of sets of  $\dots$  sets of elements of  $A$ . The total set of elements of  $A$  involved in  $\bar{G}$ , or in any half-way stage of a 2c-isomorphism, must be uniquely  $F_i$ . Similarly  $G_i$  is unique, and the 2c-isomorphism is the unique natural one.

COROLLARY 8.1.  $\bar{\mathfrak{D}} = \mathfrak{D}$ , modulo the second isomorphism theorem.

COROLLARY 8.2. Any 2c-isomorphism between two elements of  $\mathfrak{D}$  must be an identity.

COROLLARY 8.3. The isomorphisms  $\tau$  give a one-one correspondence between the subgroups (quotient groups) of  $\bar{G}$  in  $\bar{\mathfrak{D}}$  and those of  $G$  in  $\mathfrak{D}$ . The corresponding injections (projections) commute with the  $\tau$ 's.

PROOF. The proof follows from the consideration of successive commutative diagrams such as

$$\begin{array}{ccc} \frac{F'_1/H_1}{G_1/H_1} & \xrightarrow{j} & \frac{F_1/H_1}{G_1/H_1} \\ \downarrow \tau & & \downarrow \tau \\ \frac{F'_1}{G_1} & \xrightarrow{j} & \frac{F_1}{G_1} \end{array}$$

where  $F_1 \supset F'_1 \supset G_1 \supset H_1$ .

COROLLARY 8.4.  $\mathfrak{D}$  is finite.

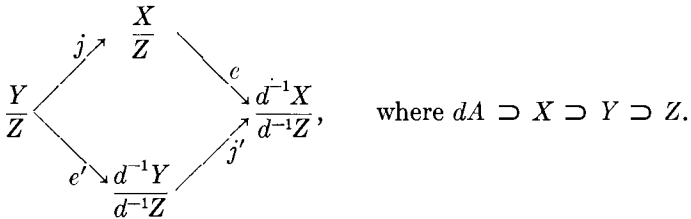
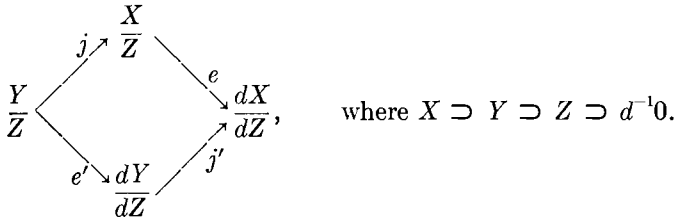
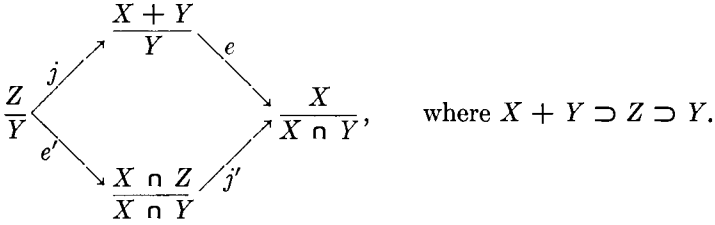
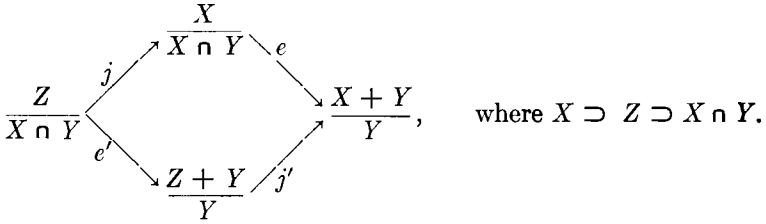
PROOF. Suppose  $\bar{G} \in \mathfrak{D}_i - \mathfrak{D}_{i-1}$ . This implies there is a strictly increasing chain in  $\mathfrak{X}$  of length  $i + 1$ . Since  $\mathfrak{X}$  is finite, there is a maximum length  $k$ , of strictly increasing chains, and so  $i + 1 \leq k$ . Therefore  $\mathfrak{D}_k = \mathfrak{D}_{k+1} = \dots = \bar{\mathfrak{D}}$ . Since  $\mathfrak{D}_0 = \mathfrak{X}$  is finite, we deduce inductively that  $\mathfrak{D}_k = \bar{\mathfrak{D}}$  is finite.

LEMMA 9. (a) If  $e \in \mathfrak{E}$  and  $j \in \mathfrak{F}$  are composable, then  $ej = j'e'$  some  $j' \in \mathfrak{F}, e' \in \mathfrak{E}$ .

If  $p \in \mathfrak{P}$  and  $e \in \mathfrak{E}$  are composable, then  $pe = e'p'$  some  $e' \in \mathfrak{E}, p' \in \mathfrak{P}$ .

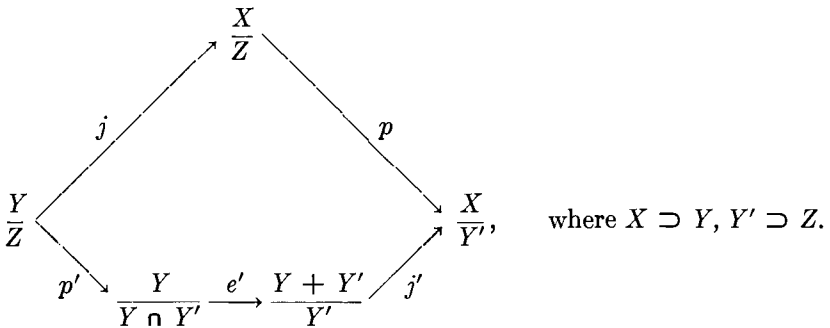
(b) If  $p \in \mathfrak{P}$  and  $j \in \mathfrak{F}$  are composable, then  $pj = j'e'p'$  some  $j' \in \mathfrak{F}, p' \in \mathfrak{P}$ , and some 1-isomorphism  $e' \in \mathfrak{E}$ .

PROOF. (a) We consider a commutative diagram in the four cases when  $e$  or  $e^{-1}$  is a 1-isomorphism or is induced by  $d$ , and we deduce the general case by composition. In each diagram the top half is given and the bottom half is deduced.



The proof for projections is similar.

(b) We are given the top half of the following commutative diagram, and deduce the bottom half.





PROOF OF THEOREM 2. Let  $\tilde{\mathfrak{F}}$  be the set of all homomorphisms  $\tilde{f} = \tau_1^{-1}f\tau_2$  between the elements of  $\tilde{\mathfrak{Q}}$ , where  $f \in \mathfrak{F}$  and  $\tau_1, \tau_2$  are the unique 2c-isomorphisms. Using Corollary 8.1 we deduce that  $\tilde{\mathfrak{Q}} \cup \tilde{\mathfrak{F}} = \mathfrak{Q} \cup \mathfrak{F}$ , modulo the second isomorphism theorem.  $\mathfrak{Q}$  is finite by Theorem 1, and so we may deduce in turn the finiteness of  $\mathfrak{F}, \mathfrak{P}, \mathfrak{E}, \mathfrak{I}$ , and, by Corollary 8.4, of  $\tilde{\mathfrak{Q}}$  and  $\tilde{\mathfrak{F}}$ . Using Lemmas 1 & 2 we observe that

$$\mathfrak{A} \subset \mathfrak{Q} \cup \mathfrak{F} \subset \tilde{\mathfrak{Q}} \cup \tilde{\mathfrak{F}} \subset \mathfrak{A}^\#.$$

To prove Theorem 2 it is sufficient to show that  $\tilde{\mathfrak{Q}} \cup \tilde{\mathfrak{F}}$  is closed with respect to the building operations, and is therefore equal to  $\mathfrak{A}^\#$ . We consider each of the six operations in turn.

(i) The range and domain of any  $\tilde{f} \in \tilde{\mathfrak{F}}$  is in  $\tilde{\mathfrak{Q}}$ .

(ii) Given  $\tilde{f}_1 = \tau_1^{-1}f_1\tau_2$  and  $\tilde{f}_2 = \tau_3^{-1}f_2\tau_4$ , composable, we have to show  $\tilde{f}_1\tilde{f}_2 \in \tilde{\mathfrak{F}}$ . Since the range of  $f_2$  and the domain of  $f_1$  are in  $\mathfrak{Q}$ ,  $\tau_2\tau_3^{-1}$  is an identity by Corollary 8.2. Therefore

$$\tilde{f}_1\tilde{f}_2 = \tau_1^{-1}f_1f_2\tau_4,$$

and it is sufficient to verify that  $f_1f_2 \in \mathfrak{F}$ . With the convention that  $j \in \mathfrak{F}, e_i \in \mathfrak{E}, p_i \in \mathfrak{P}$ , all  $i$ , we can write  $f_1 = \text{some } j_1e_1p_1$  and  $f_2 = \text{some } j_2e_2p_2$ . Therefore

$$\begin{aligned} f_1f_2 &= j_1e_1p_1j_2e_2p_2 \\ &= j_1e_1j_3e_3p_3e_2p_2, & \text{some } j_3, e_3, p_3, & \text{by Lemma 9b,} \\ &= j_1j_4e_4e_3e_5p_5p_2, & \text{some } j_4, e_4, e_5, p_5, & \text{by Lemma 9a,} \\ &= j_6e_6p_6, \end{aligned}$$

since each of the three sets  $\mathfrak{F}, \mathfrak{E}, \mathfrak{P}$ , is closed with respect to composition. Thus  $f_1f_2 \in \mathfrak{F}$  as required.

Parts (iii) and (iv) follow at once from Corollary 8.3 and the fact that  $\mathfrak{F}$  contains all injections and projections between the elements of  $\mathfrak{Q}$ .

(v) If  $f \in \mathfrak{F}$ , the decomposition  $f = jep$  shows that  $\mathfrak{Q}$  contains the kernel, cokernel, image and co-image of  $f$ , and that  $\mathfrak{F}$  contains the induced isomorphism  $e$ . If  $f: \bar{F} \rightarrow \bar{G}$  is in  $\tilde{\mathfrak{F}}$  we decompose

$$\begin{array}{ccc} \bar{F} & \xrightarrow{\tilde{f}} & \bar{G} \\ \downarrow \tau(\bar{F}) & & \downarrow \tau(\bar{G}) \\ Y & \xrightarrow{j'} X & \xrightarrow{f} X' & \xrightarrow{p'} Y' & \xrightarrow{j} X' & \xrightarrow{p'} Y' \\ Z & \xrightarrow{p} Z & \xrightarrow{e} Y & \xrightarrow{e} Z' & \xrightarrow{j} Z' & \xrightarrow{p'} Y' \end{array}$$

for unique  $X \supset Y \supset Z, X' \supset Y' \supset Z'$ . The kernel of  $\tilde{f}$  is obtained by Corollary 8.3 as the unique subgroup of  $\tilde{F}$  corresponding to  $Y/Z \subset X/Z$ ; the cokernel, image and co-image occur similarly. The induced isomorphism

$$\bar{e} = \tau(\text{co-im } \tilde{f})^{-1} e \tau(\text{im } \tilde{f}).$$

(vi) An isomorphism of  $\tilde{\mathfrak{G}}$ , is of the form  $\bar{e} = \tau_1^{-1} e \tau_2$ , where  $e \in \mathfrak{E}$ . Since  $\mathfrak{E}$  is a groupoid,  $e^{-1} \in \mathfrak{E}$ , and  $\bar{e}^{-1} = \tau_2^{-1} e^{-1} \tau_1 \in \tilde{\mathfrak{G}}$ . The proof of Theorem 2 is complete.

The first step in proving the representation theorem, Theorem 4, is to establish that in general  $\lambda$  is one-one on  $\mathfrak{Q}$ . Given a particular filtered differential group  $\mathfrak{A}$ , we say that  $\mathfrak{A}$  attaches  $G$  to  $M$  if  $G \in \mathfrak{A}^\#$  and  $\lambda G = M$ . It will transpire that all groups attached to  $M$  by  $\mathfrak{A}$  are (canonically) isomorphic, but we have not shown this yet. Let  $J$  denote a free cyclic group, and  $J_2$  a cyclic group of order 2.

EXAMPLE 1. Given a unit square  $b \subset \Delta^0$ , there exists an  $\mathfrak{A}$  attaching  $J$  to  $b$  and 0 to all other squares. For suppose  $b = [q, 0]$ ; let  $A \cong J$ , with a generator of filtration  $q$ , and let  $d = 0$ .

EXAMPLE 2. Given  $a' = \delta a$  (implying  $a \subset \Delta^+, a' \subset \Delta^-$ ), there exists an  $\mathfrak{A}$  attaching  $J$  to  $a$  and  $a'$  and 0 to all other squares. For suppose  $a = [p, p - r]$ ,  $p > r$ . Let  $A$  be free abelian with two generators  $x, y$  of filtrations  $p, p - r$  respectively, and define  $d$  by  $dx = y, dy = 0$ .

LEMMA 10. *If  $\mathfrak{A}$  is sufficiently general then  $\lambda \mid \mathfrak{Q}$  is a lattice-monomorphism.*

PROOF. We may represent (as in [1] Chapter V) the elements of the free modular lattice  $\mathfrak{M}$  generated by  $\mathfrak{X}, \mathfrak{X}'$  by the regions of a rectangle

$$\square = \{[\pi, \rho]; 1 \leq \pi \leq m, -m \leq \rho \leq m\}.$$

In other words there is a lattice-monomorphism  $\tilde{\lambda}$  from  $\mathfrak{M}$  to the lattice of regions of  $\square$ , with definition similar to that of  $\lambda$  in Section 2. Now  $\mathfrak{Q}$  is the quotient of  $\mathfrak{M}$  by an equivalence relationship  $\chi$ , say. We claim that the relations  $dA_p \subset A_p \subset d^{-1}A_p$  are sufficient to generate  $\chi$ . Similarly the lattice of regions of  $\Delta$  is the quotient of that of  $\square$  by the equivalence relationship generated by the relations  $\tilde{\lambda}(dA_p) \subset \tilde{\lambda}(A_p) \subset \tilde{\lambda}(d^{-1}A_p)$ ; this is what we are doing when we lop off anything outside the upper and lower zig-zag boundaries of  $\Delta$ . Hence the lattice-monomorphism  $\tilde{\lambda}$  induces the lattice-monomorphism  $\lambda \mid \mathfrak{Q}$  from  $\mathfrak{Q}$  to the lattice of regions of  $\Delta$ .

Now suppose the above relations were insufficient to generate  $\chi$ : this would mean there was a group  $X \in \mathfrak{Q}$ , arising from distinct elements (formulae) in  $\mathfrak{M}$ , and represented under  $\lambda$  by distinct regions  $M, M'$  of  $\Delta$ . We may suppose  $M \supset M'$ , by suitable renaming, since  $X$  is also represented by  $M \cap M'$ . Therefore  $\mathfrak{A}$  attaches 0 to  $M - M'$ , and to any square  $a \subset M - M'$ . Since  $\mathfrak{A}^\#$  is sufficiently general, this is also true for an arbitrary  $\mathfrak{A}$ , a fact which may be contradicted by Examples 1 & 2. Thus the relations  $dA_p \subset A_p \subset d^{-1}A_p$  are sufficient to generate  $\chi$ , and the lemma is proved.

LEMMA 11. *Let  $a, a', b$  be unit squares of  $\Delta$  such that  $a' = \delta a$ . Then  $a' < a$ , and  $a' < b$  or  $b < a$  (or both).*

PROOF. Since  $a \subset \Delta^+$ ,  $a = \text{some } [\pi, \rho]$  with  $1 \leq \rho \leq \pi \leq m$ . Therefore

$$a' = \delta a = [\rho, \pi - m - 1] < [\pi, 0] < [\pi, \rho].$$

The second part follows from the fact that the two regions  $\cup \{c; a' < c\}$  and  $\cup \{c; c < a\}$  cover  $\Delta$  (see Figure 1).

LEMMA 12. *Let  $\mathfrak{A}$  be sufficiently general, and suppose  $\mathfrak{A}$  attaches isomorphic groups to  $M$  and  $M'$ . Let  $b \subset \Delta^\circ$  and  $a' = \delta a$ . Then*

- (i)  $b \subset M$  if and only if  $b \subset M'$ ;
- (ii)  $a \subset M - M'$  if and only if  $a' \subset M' - M$ ;
- (iii)  $a \subset M - M'$  implies that  $b \not\subset M \cap M'$ .

PROOF. (i) If  $b \subset M - M'$ , then Example 1 attaches  $J$  to  $M$  and  $0$  to  $M'$  contradicting the hypothesis that  $\mathfrak{A}$  is sufficiently general.

(ii) Suppose  $a \subset M - M'$  and  $a' \not\subset M' - M$ . Consider Example 2. If  $a' \subset M'$ , then  $a' \subset M$  also, and  $\mathfrak{A}$  attaches  $J \oplus J$  to  $M$  and  $J$  to  $M'$ . Otherwise  $a' \not\subset M'$ , and  $\mathfrak{A}$  attaches  $J$  or  $J \oplus J$  to  $M$  and  $0$  to  $M'$ . The contradiction is obtained as in (i).

(iii) By (ii)  $a' \subset M' - M$ . Assume that  $b \subset M \cap M'$ . Suppose  $b = [q, 0]$  and  $a = [p, p - r]$ ,  $p > r$ . There are two cases:

1°. If  $p - r \leq q$ , let  $A$  be free abelian with two generators  $x, y$  of filtrations  $p, q$  respectively, and such that  $2y$  is of filtration  $p - r$ . Define  $dx = 2y, dy = 0$ . We have attached  $J \oplus J_2$  to  $M$  and  $J$  to  $M'$ .

2°. If  $p - r > q$ , let  $A \cong J \oplus J_2$ , with generators  $x, y$  of filtrations  $p, p - r$  respectively, and such that  $2x$  is of filtration  $q$  and  $2y = 0$ . Define  $dx = y, dy = 0$ .  $J$  is attached to  $M$  and  $J \oplus J_2$  to  $M'$ . The contradiction is as before,

PROOF OF THEOREM 4.

Property (ii). The function  $\lambda$  was defined for  $\mathfrak{E}$ , and extended successively to  $\mathfrak{Q}_1, \mathfrak{Q}_2, \dots$  by property (ii)

$$\lambda(F/G) = \lambda F - \lambda G,$$

and so to  $\mathfrak{Q}$ , which is the set of all groups of  $\mathfrak{A}^\#$  by Theorem 2. Property (ii) is therefore automatically verified. Moreover, if  $F \supset G \supset H$  in  $\mathfrak{A}^\#$ , then  $\lambda F \supset \lambda G \supset \lambda H$ , and

$$\begin{aligned} \lambda \left( \frac{F/H}{G/H} \right) &= \lambda(F/H) - \lambda(G/H) \\ &= (\lambda F - \lambda H) - (\lambda G - \lambda H) = \lambda F - \lambda G = \lambda(F/G) \end{aligned}$$

implying that 2-isomorphic, and hence 2c-isomorphic, groups are represented by the same region.

Property (i). The lattice-homomorphic property has been established for groups in  $\mathfrak{E}$  by Lemma 10. Suppose  $F, G \subset H$  in  $\mathfrak{A}^\#$ . By Corollary 8.3 and the above, it is sufficient to consider the case when  $F, G, H \in \mathfrak{Q}$ . Suppose  $F = X/Z, G = Y/Z$ , where  $X, Y \supset Z$  in  $\mathfrak{E}$ . Then

$$\begin{aligned}
 \lambda(F + G) &= \lambda\left(\frac{X + Y}{Z}\right) = \lambda(X + Y) - \lambda Z, \text{ by Property (ii),} \\
 &= (\lambda X \cup \lambda Y) - \lambda Z, \text{ by Lemma 10,} \\
 &= (\lambda X - \lambda Z) \cup (\lambda Y - \lambda Z) \\
 &= \lambda(X/Z) \cup \lambda(Y/Z), \text{ by Property (ii),} \\
 &= \lambda F \cup \lambda G.
 \end{aligned}$$

Similarly for intersections.

Property (iii). Given  $F, G \subset H$  in  $\mathfrak{A}^\#$ ,

$$\begin{aligned}
 \lambda\left(\frac{F}{F \cap G}\right) &= \lambda F - (\lambda F \cap \lambda G), \text{ by Properties (i) and (ii),} \\
 &= \lambda F - \lambda G \\
 &= (\lambda F \cup \lambda G) - \lambda G = \lambda\left(\frac{F + G}{G}\right).
 \end{aligned}$$

Therefore 1-isomorphic groups are represented by the same region, and hence 1c-isomorphic and c-isomorphic groups also.

Conversely, suppose  $\lambda F = \lambda G$ , where  $F, G \in \mathfrak{A}^\#$ . By Lemma 8 and (ii) above, there exist 2c-isomorphisms  $F \rightarrow X/Y, G \rightarrow X'/Y'$ , where  $X \supset Y, X' \supset Y'$  in  $\mathfrak{Q}$ , and  $\lambda X - \lambda Y = \lambda F = \lambda G = \lambda X' - \lambda Y'$ . Therefore

$$\lambda(Y \cap Y') = \lambda(X' \cap Y), \quad \lambda(X \cap X' + Y) = \lambda X.$$

By the monomorphic character of  $\lambda$  on  $\mathfrak{Q}$ ,

$$Y \cap Y' = X' \cap Y. \quad X \cap X' + Y = X.$$

There is a 1-isomorphism

$$\frac{X \cap X'}{Y \cap Y'} = \frac{X \cap X'}{X \cap X' \cap Y} \rightarrow \frac{X \cap X' + Y}{Y} = \frac{X}{Y}.$$

By symmetry and composition we may establish the required canonical isomorphism

$$F \rightarrow \frac{X}{Y} \leftarrow \frac{X \cap X'}{Y \cap Y'} \rightarrow \frac{X'}{Y'} \leftarrow G.$$

Property (iv). If  $a \in \Delta^+$ ,  $a$  is the top-right square of some  $\lambda C_p^r$ , and  $\delta a$  is the top-right square of  $\lambda(dC_p^r) = \lambda B_{p-r}^r$ . Hence if  $a \subset \lambda X, X \in \mathfrak{Q}$ , then  $C_p^r \subset X$  by Lemma 4, and so  $B_{p-r}^r \subset dX, \delta a \subset \lambda B_{p-r}^r \subset \lambda(dX)$ . Similarly if  $a \not\subset \lambda Y$  then  $\delta a \not\subset \lambda(dY)$ .

Suppose  $\delta(\lambda F) = \lambda G$ , where  $F, G \in \mathfrak{A}^\#$ . Since  $\lambda F \subset \Delta^+$ , we may, by Lemma 8 and (iii) above, choose a c-isomorphism  $F \rightarrow X/Y, X \supset Y \supset d^{-1}0$  in  $\mathfrak{Q}$ . Then

$$\lambda G = \delta(\lambda F) = \{\delta a; a \subset \lambda X - \lambda Y\} = \lambda(dX) - \lambda(dY) = \lambda(dX/dY).$$

By (iii) there is a  $c$ -isomorphism  $dX/dY \rightarrow G$ . Meanwhile  $d$  induces an isomorphism  $X/Y \rightarrow dX/dY$ , and so, modulo the first and second isomorphism theorems, an isomorphism  $F \rightarrow G$ , as desired.

The converse is immediate.

Property (v). Suppose we are given distinct regions  $N, N'$  to which any  $\mathfrak{A}$  attaches isomorphic groups. We shall show that they must be  $\delta$ -paired. In other words if  $\lambda F, \lambda G$  are neither equal nor  $\delta$ -paired, and  $\mathfrak{A}$  is sufficiently general, then  $F \not\cong G$ .

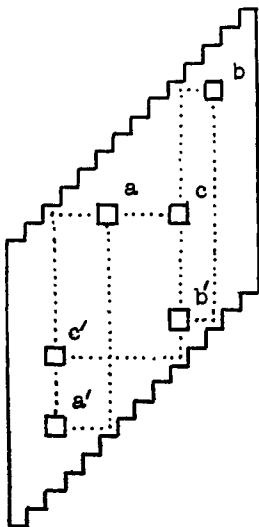


FIG. 7

Since  $N, N'$  are distinct, either  $N - N'$  or  $N' - N \neq \emptyset$ . Suppose  $a \subset N - N'$ . Then  $a \not\subset \Delta^0$ , by Lemma 12 (i). Therefore  $a$  has a  $\delta$ -pair,  $a'$  say; by Lemma 12 (ii)

(1) if  $a \subset N - N'$  then  $a' \subset N' - N$ .

Also

(2)  $N \cap \Delta^0 = \emptyset$ ,

for suppose  $b \subset N \cap \Delta^0$ . Then either  $b \not\subset N'$  contradicting Lemma 12 (i), or  $b \subset N'$  contradicting Lemma 12 (iii).

(3) If  $a \subset N \cap \Delta^+$  and  $b' \subset N \cap \Delta^-$  then  $a \succ b'$ ,

otherwise there exists  $c \subset \Delta^0, a > c > b'$ , which is contained in  $N$  by Lemma 3, contradicting (2). We now deduce the stronger statement

(4)  $N$  cannot meet both  $\Delta^+$  and  $\Delta^-$ .

For let  $a \subset N - N'$ , and its  $\delta$ -pair  $a' \subset N' - N$  by (1). We may suppose without loss of generality that  $a \subset \Delta^+$ , and so  $a' = \delta a$ . Assume the converse of (4) that  $b' \subset N \cap \Delta^-$ . Let  $\delta b = b'$ . From (3)  $a \succ b'$ , and by Lemma 11,  $a < b$ . Therefore  $a' < a < b, a' \subset N'$ , and so  $b \not\subset N'$  by (3). Hence  $b \subset N - N', b' \subset N' - N$  by (1), implying  $b' \subset N'$ . Let  $c$  be the least upper bound of  $a$  and  $b'$ , and let

$c' = \delta c$ . Then  $a < c < b$ , and  $a' < c' < b'$  since  $\delta$  preserves the partial ordering. Since  $a', b' \subset N', c' \subset N'$  also by Lemma 3. Now  $a, c$  have the same  $\rho$ -coordinate by construction, and so  $a', c'$  have the same  $\pi$ -coordinate, implying that  $c' < a$ . But  $a \subset N$ , therefore  $c' \not\subset N$  by (3). Consequently  $c' \subset N' - N$ , and  $c \subset N - N'$  by (1). The existence of  $b', c \subset N$  with  $b' < c$  contradicts (3), thereby establishing (4). (See Figure 7.)

The statements (2) and (4) imply that  $N$  is contained either in  $\Delta^+$  or  $\Delta^-$ . We may suppose without loss of generality that  $N \subset \Delta^+$ . By (1)  $N' - N$  contains the  $\delta$ -pair of some  $a \subset N$ , and so  $N'$  meets, and is therefore contained in,  $\Delta^-$ . Consequently  $N, N'$  are disjoint and composed of  $\delta$ -paired squares; i.e.  $N' = \delta N$ , as desired. The proof of Theorem 4 is complete.

### 4. Application to the homology of a fibre space

In this section we are concerned with the graded filtered differential group, a phenomenon which arises frequently in practice. An example is the singular cubical chain group of a fibre space, [8]. At first sight this appears to be a differential group with infinite grading and infinite filtration. However the filtration is really finite in each homogeneous component, and as we never actually use the direct summation of homogeneous components we may dispense with it. The resulting equivalent algebraic description is as follows:

**DEFINITION.** A *graded filtered differential group*  ${}^*\mathfrak{A}$  is a sequence of groups  $\{{}^nA; -\infty < n < \infty\}$ , each of which possesses a finite filtration, namely an increasing sequence of subgroups  $\{{}^nA_p; -\infty \leq p \leq \infty\}$ , such that  ${}^nA_p = 0, p < 0$ , and  ${}^nA_p = {}^nA, p \geq n$  (implying that  ${}^nA = 0, n < 0$ ); together with a sequence of homomorphisms  ${}^n d: {}^nA \rightarrow {}^{n-1}A$ , such that  ${}^n d^{n+1}d = 0$ , and  ${}^n d({}^nA_p) \subset {}^{n-1}A_p$ .

The infinite category  ${}^*\mathfrak{A}^\#$  generated by  ${}^*\mathfrak{A}$  is seen to be the union of finite categories  ${}^n\mathfrak{A}^\#$ , together with homomorphisms induced by  $d$  between them. For each  $n$ , we may represent  ${}^n\mathfrak{A}^\#$  upon a diagram  ${}^n\Delta$ . The groups of  ${}^n\mathfrak{A}^\#$  are generated (as in Theorems 1 & 2) by the two chains:

$$\begin{aligned} {}^n\mathfrak{T}: 0 \subset {}^nA_0 \subset \dots \subset {}^nA_{p-1} \subset {}^nA_p \subset \dots \subset {}^nA_{n-1} \subset {}^nA, \\ {}^n\mathfrak{T}': 0 \subset {}^{n+1}d({}^{n+1}A_0) \subset \dots \subset {}^{n+1}d({}^{n+1}A) \subset {}^n d^{-1}(0) \subset {}^n d^{-1}({}^{n-1}A_0) \subset \dots \subset {}^nA. \end{aligned}$$

The squares  ${}^n[\pi, \rho]$  in  ${}^n\Delta$  range over the set of integer values

$$0 \leq \pi \leq n, \pi - n - 2 \leq \rho \leq \min(\pi + 1, n).$$

As before

$${}^n\Delta^+ = {}^n\Delta(\rho > 0), \quad {}^n\Delta^- = {}^n\Delta(\rho < 0).$$

The function  ${}^n\delta$ , given by

$${}^n\delta({}^n[\pi, \rho]) = {}^{n-1}[\rho - 1, \pi - n - 1],$$

maps the regions of  ${}^n\Delta^+$  onto those of  ${}^{n-1}\Delta^-$ .

Figure 8 is drawn for  $n = 4$ .

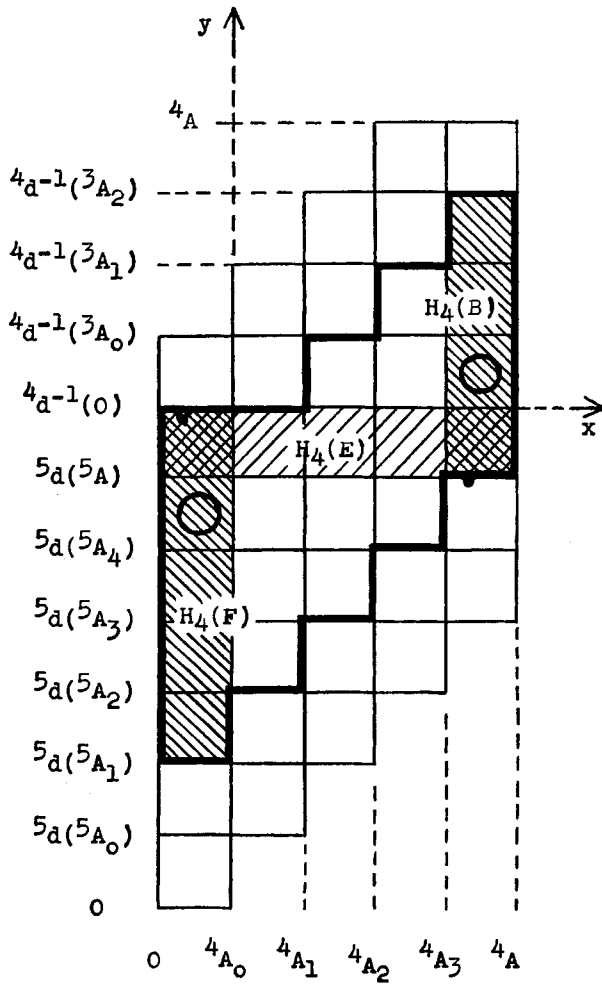


FIGURE 8. The diagram  ${}^4\Delta$ .

The fibre space. Let  $F \xrightarrow{i} E \xrightarrow{h} B$  be a fibre space, with fibre  $F$ , base  $B$  and projection  $h$ . Suppose, for simplicity, that the fundamental group of  $B$  acts trivially on the homology of  $F$ . Then we may identify on  ${}^n\Delta$  the regions which represent the homology groups  $H_n(F)$ ,  $H_n(E)$  and  $H_n(B)$  (as shown in Figure 8 for  $n = 4$ ). The squares  ${}^n[0, 0]$  and  ${}^n[0, n]$  represent the images of  $i_*$  and  $h_*$  respectively, by Theorem 5. The region  ${}^n\Delta^0 (1 \leq \pi \leq n - 1)$  represents a measure of the non-exactness of

$$H_n(F) \xrightarrow{i_*} H_n(E) \xrightarrow{h_*} H_n(B).$$

The squares  ${}^n[n, 1]$  and  ${}^{n-1}[0, -1]$  (marked with circles on Figure 8) are  ${}^n\delta$ -paired,

and represent the transgressive elements of dimension  $n$  in the base, and dimension  $n - 1$  in the fibre, respectively. The fact that they occur in the middle of  $H_n(B)$  and  $H_{n-1}(F)$  indicates, by Lemma 4 that we cannot in general extend the "isomorphism"  $\delta: {}^n[n, 1] \leftrightarrow {}^{n-1}[0, -1]$  either way into a homomorphism between  $H_n(B)$  and  $H_{n-1}(F)$ . However  ${}^n[n, 1]$  does occur at the top of  $\lambda P_n$ , where  $P_n$  is the subgroup of  $H_n(B)$  corresponding to the two lowest squares; and  ${}^{n-1}[0, -1]$  occurs at the bottom of  $\lambda Q_{n-1}$ , where  $Q_{n-1}$  is the quotient group of  $H_{n-1}(F)$  corresponding to the two highest squares. Now we can apply Theorem 5 and obtain the transgression

$$P_n \rightarrow Q_{n-1}$$

induced by  $d$ . Alternatively, if  $H_n(E) = H_{n-1}(E) = 0$ , then  ${}^n[n, 1]$  occurs at the bottom and  ${}^{n-1}[0, -1]$  at the top, enabling us to define the suspension

$$H_n(B) \leftarrow H_{n-1}(F)$$

induced by  $d^{-1}$ .

We denote by  ${}^n\Gamma$  the heavily outlined region of  ${}^n\Delta$  in Figure 8. Its full significance will appear in the next section. Meanwhile we illustrate its use by considering the situation when the fibre is *totally non-homologous to zero*. This is described algebraically by  $E^2 = E^\infty$ , and, on the diagram, by attaching zero groups to all the squares contained in  ${}^n\Gamma \cap ({}^n\Delta^+ \cup {}^n\Delta^-)$ . As a result we see that  $i_*$  becomes a monomorphism, and  $h_*$  an endomorphism.

We conclude this descriptive section by mentioning that similar diagrams can be drawn for cohomology, but that they do not display the multiplicative structure.

### 5. The invariant subcategory

Suppose the three spaces of a fibre space are polyhedra. The terms  $E^r$ ,  $r \geq 2$ , of the associated spectral sequence are "invariant" in the sense that they are independent of the method of calculation, whether it be by singular cubical homology (Serre [8]), by singular simplicial homology, by the use of sheaves (Leray [5]), or by dihomology, [9]. We seek for all such invariant groups in  $\mathfrak{A}^\#$ . In order to preserve simplicity, we return to the discussion of an (ungraded) filtered differential group of length  $m$ , as in Section 1, and manufacture a suitable definition of invariance. The generalization to the graded case presents no difficulties and is left to the reader.

DEFINITION, A *homomorphism*  $f: \mathfrak{A} \rightarrow \mathfrak{A}'$  between two filtered differential groups of length  $m$  is a homomorphism  $f: A \rightarrow A'$  which preserves the structure, that is  $fA_p \subset A'_p$ , each  $p$ , and  $fd = d'f$ .

If  $\psi$  is a formula defining a group of  $\mathfrak{A}^\#$  (in terms of the  $A_p$  and  $d$ ), then  $f$  induces a homomorphism  $\psi(f): \psi(\mathfrak{A}) \rightarrow \psi(\mathfrak{A}')$ . The term  $E_p^r$  of the spectral sequence is an example of such a formula. Define  $\psi$  to be *invariant*<sup>5</sup> if  $\psi(f)$  is an

<sup>5</sup> If we were interested in a spectral sequence from only the  $r^{\text{th}}$  term onwards, we could equally well replace  $E^2$  by  $E^r$  in the definition of invariance, and throughout Section 5.



isomorphism for all homomorphisms  $f$  which induce an isomorphism  $E^2(f)$ . If  $\psi$  is invariant, then the group  $\psi(\mathfrak{A}) \in \mathfrak{A}^\#$  is called *invariant*; and a homomorphism in  $\mathfrak{A}^\#$  is said to be *invariant* if its range and domain are. It follows immediately that the set of all such groups and homomorphisms form a subcategory  $\mathfrak{B}$ , which we call the *invariant subcategory* of  $\mathfrak{A}^\#$ . Glancing at the building operations, we notice that anything we can build from invariants is also invariant, i.e.  $\mathfrak{B}^\# = \mathfrak{B}$ .

If a group is invariant, then its canonical isomorphisms are also invariant, and so, using Theorem 4 (iii), we may call the corresponding region on  $\Delta$  *invariant*. In order to identify the invariant regions on  $\Delta$  we define

$$\Gamma = \bigcup_p \lambda E_p^2 = \Delta(\min(0, \pi - m - 1) \leq \rho \leq \max(0, \pi - 2)).$$

$\Gamma$  is marked with a heavy outline in Figure 1, as is the corresponding region  ${}^n\Gamma$  in Figure 8.

**THEOREM 6.** *A permissible region is invariant if and only if it is contained in  $\Gamma$ .*

**PROOF.** Suppose  $G \in \mathfrak{A}^\#$ , and  $\lambda G \not\subset \Gamma$ . We wish to show  $G$  is not invariant. Choose  $a \subset \lambda G - \Gamma$ . If  $a'$  is the  $\delta$ -pair of  $a$ , then  $a' \not\subset \Gamma$  also. Let  $\psi$  be a formula giving  $G$ . Let  $\mathfrak{A}_1$  be as in Example 2 in Section 3. Let  $\mathfrak{A}_2 = 0$ , and  $f: \mathfrak{A}_1 \rightarrow \mathfrak{A}_2$  be the trivially unique homomorphism. Since  $E^2(\mathfrak{A}_1) = E^2(\mathfrak{A}_2) = 0$ ,  $E^2(f)$  is an isomorphism. But  $\psi(\mathfrak{A}_1) \neq 0$ ,  $\psi(\mathfrak{A}_2) = 0$ , and so  $\psi(f)$  is not an isomorphism. Hence  $\psi$  is not invariant, and neither is  $G$ .

Conversely suppose  $G \in \mathfrak{A}^\#$ ,  $\lambda G \subset \Gamma$ . The filtration

$$\lambda G(\pi \leq p), \quad p = 0, 1, \dots, m$$

of the region  $\lambda G$  imposes a filtration on  $G$ . Let  $\sum_{p=1}^m G_p$  be the associated graded group, where  $\lambda G_p \subset \lambda E_p^r$ . If  $f: \mathfrak{A} \rightarrow \mathfrak{A}'$  induces an isomorphism of each  $G_p$ , then by repeated application of the five lemma,<sup>6</sup> we deduce that it does also on  $G$ . Each  $G_p$  is filtered (vertically) such that  $\text{Gr } G_p$  is the direct sum of groups corresponding to squares in  $\Gamma$ . By the same argument it is sufficient to show that  $f$  induces isomorphisms of the latter. But each square represents the kernel, image or co-image of some  $d_p^r$ ,  $r \geq 2$ . Since  $f$  commutes with  $d$ , and so with each  $d^r$ , by Theorem 3 (iii) it induces an isomorphism on each  $E^r$ ,  $r \geq 2$ , and consequently on each kernel, image or co-image. Thus  $G$  is invariant.

**COROLLARY 6.1.**  $\mathfrak{S}^{2\#} \subset \mathfrak{R}^{2\#} \subset \mathfrak{B}$ ; in other words, the spectral sequence terms and exact couples for  $r \geq 2$ , and anything built therefrom, are invariant.

**COROLLARY 6.2.** Any invariant group may be obtained by group extensions from  $\mathfrak{S}^{2\#}$ , modulo the first and second isomorphism theorems.

Corollary 6.2 may be interpreted by the optimists as saying that apart from certain group extensions, the spectral sequence captures all the information available from a filtered differential group. The pessimists, however, will point out that it is inadequate in not providing these group extensions. The exact couple is better, but still insufficient, for the inclusions in Corollary 6.1 are in

<sup>6</sup> The "five" lemma is given in [4], p. 16.

general both strict. An example of an invariant group which cannot be built from  $\mathbb{R}^2$  is the group attached to the rectangle  $\Delta(3 \leq \pi \leq 4, -2 \leq \rho \leq 1)$  in Figure 1.

We seek therefore for an economical basis or list of invariant groups and homomorphisms from which to generate all invariants. The reason why any such list must necessarily be rather complicated is that the region  $\Gamma$  is not permissible. Therefore, to capture group extensions, we are forced to have the basis regions overlapping one another. If, on the other hand, we are willing to sacrifice group extensions (if we are working with vector spaces over a field, say) then an obvious approach is to decompose  $\Gamma$  as simply as possible into disjoint permissible regions, namely the columns. Thus we are led intuitively to the spectral sequence term  $E^2$ —quite apart from its two fundamental advantages, the properties in Theorem 3 and its relations to other known invariants when used.

*Definition of  $\mathfrak{B}$ .* Given a filtered differential group  $\mathfrak{A}$  of length  $m$ , let  $\mathfrak{B}$  comprise the groups

$$R_{p,q} = \text{the image of } j: H(A_p, A_{q-1}) \rightarrow H(A_{p+1}, A_q), \quad 0 \leq q < p \leq m;$$

and the homomorphisms

$$\bar{j}: R_{p,q} \rightarrow R_{p',q'}, \quad p \leq p', q \leq q', \quad \text{induced by injection, and}$$

$$\bar{d}: R_{p,q} \rightarrow R_{q-1,s}, \quad p > q > s + 1, \quad \text{induced by } d.$$

The regions  $\lambda R_{p,q}$  are characterized by being maximal rectangles contained in  $\Gamma$ .

**THEOREM 7.**  $\mathfrak{B}$  is a basis for the invariant category  $\mathfrak{B}$ ; in other words,  $\mathfrak{B}^\# = \mathfrak{B}$ , modulo the first and second isomorphism theorems.

**REMARK.** In forming a basis for  $\mathfrak{B}$ , we could do with fewer homomorphisms than in  $\mathfrak{B}$ , although we cannot dispense with any of the groups  $R_{p,q}$ . It is easy to write down a minimal set of homomorphisms.

**LEMMA 13.** Any permissible invariant region  $M$  is contained in some invariant rectangle  $R$ .

**PROOF.** Let  $R$  be the smallest rectangle in the plane containing  $M$ , and let  $a, b$  be the top-left and bottom-right unit squares of  $R$  respectively. There must exist  $a_1, a_2 \subset M$  such that  $a_1$  has the same  $\pi$ -coordinate as  $a$ , and  $a_2$  has the same  $\rho$ -coordinate as  $a$ . Hence  $a_1 \leq a \leq a_2$ . Since  $M$  is permissible,  $a \subset M$  by Lemma 3; similarly  $b \subset M$ . Also  $M$  is invariant, and so  $M \subset \Gamma$  by Theorem 6. Therefore  $a, b \subset \Gamma$ , and by observing the shape of  $\Gamma$  we deduce  $R \subset \Gamma$ , proving that  $R$  is invariant.

**PROOF OF THEOREM 7.** Clearly  $\mathfrak{B} \subset \mathfrak{B}$ . To generate  $\mathfrak{B}$  from  $\mathfrak{B}$  we may use the processes mentioned in Lemma 1, of taking images, unions and quotients etc. Since we are working modulo the first and second isomorphism theorems, it is sufficient to generate one group for each permissible invariant region. We first demonstrate this for rectangles.

Suppose we are given the rectangle  $R, \subset \Gamma$ . Let  $a, b$  be the top-right and bottom-left squares of  $R$ . Let  $R_1, R_2$  be maximal invariant rectangles with  $a$  as top-right square, and  $b$  as bottom-left square, respectively. Using Theorem 6 and

Figure 1 we observe that: if  $a \in \Delta^-, R_1$  represents a group  $F_1 \in \mathfrak{B}$ ; if  $a \in \Delta^-, R_1$  represents a group  $F_1$  which is the image of some  $\bar{d} \in \mathfrak{B}$ . Similarly  $R_2 = \lambda_2 F_2$ , where  $F_2$  is either a group in  $\mathfrak{B}$  or the co-image of some  $\bar{d} \in \mathfrak{B}$ . The image of the homomorphism  $F_1 \rightarrow F_2$ , induced by the appropriate  $\bar{j}$  in  $\mathfrak{B}$ , gives a group  $F$  such that  $\lambda F = R$ . Moreover  $F \in \mathfrak{B}^\#$ , as desired, by Lemma 1.

Now let  $M$  be an arbitrary permissible invariant region.  $M$  is contained in some invariant rectangle  $R$ , by Lemma 13. We may choose  $F \in \mathfrak{B}^\#, \lambda F = R$ , as above. Let  $\{H_i\}$  denote the set of all rectangular subgroups of  $F$  such that  $\lambda H_i \cap M = \emptyset$ , and let  $H = \cup_i H_i$ . Let  $\{G_i\}$  denote the set of all rectangular subgroups of  $F$  such that  $\lambda G_i \subset \lambda H \cup M$ , and let  $G = \cup_i G_i$ . Then  $\lambda(G/H) = \lambda G - \lambda H = M$ . By Lemma 1  $G/H \in \mathfrak{B}^\#$ , so that we have generated a group attached to the arbitrary region  $M$ , and the theorem is proved.

**COROLLARY 7.1.**  $\mathfrak{S}^\# \supset \mathfrak{B}$ , modulo the first and second isomorphism theorems.

**PROOF.**  $\mathfrak{B}$  is defined by means of the relative homology groups and homomorphisms  $\mathfrak{S}$ . Therefore  $\mathfrak{B} \subset \mathfrak{S}^\#$ , and  $\mathfrak{B} = \mathfrak{B}^\# \subset \mathfrak{S}^{\#\#} = \mathfrak{S}^\#$ , modulo the first and second isomorphism theorems.

**REMARK.** The significance of Corollary 7.1 is that if we are looking for all the invariant information, and we know the relative groups and homomorphisms  $\mathfrak{S}$ , there is nothing to be gained from going back to the original chain group  $\mathfrak{A}$ . In other words 7.1 justifies Eilenberg's use of  $\mathfrak{S}$  as the fundamental concept in defining the spectral sequence in [3], the Deheuvels constructions from  $\mathfrak{S}$  in [2]. Deheuvels constructed a set of "partial homology groups", which correspond to our rectangular groups, together with their subgroups, and thereby obtained a large part, but not all, of  $\mathfrak{B}$ .

### 6. The bifiltered differential group

The following problem due to Eilenberg is suggested by Massey in [6] (problem number 7). What are the relations between the various spectral sequences arising from a bifiltered differential group? More particularly, suppose to the filtered differential group of length  $m, \mathfrak{A} = \{A, A_p, d\}$ , we add one subgroup  $A'$  of  $A$ , stable under  $d$ . Write  $\mathfrak{D} = \{\mathfrak{A}, A'\}$ . Let  $\mathfrak{A}'$  denote the filtered differential group  $\{A', A' \cap A_p, d|A'\}$ , and  $\mathfrak{A}'' = \{A/A', (A' + A_p)/A', d''\}$ , where  $d''$  is induced by  $d$ . What is the relation between the spectral sequences associated with  $\mathfrak{A}, \mathfrak{A}', \mathfrak{A}''$ ?

The situation arises, for instance, when considering a fibre bundle and the subbundle over a subset of the base (or alternatively a subbundle using only a subset of the fibre).

In our terminology, the second question reduces to the problem of analyzing the category  $\mathfrak{D}^\#$  generated by  $\mathfrak{D}$ . The purpose of the present section is to show why this is so much more complicated than the analysis of  $\mathfrak{A}^\#$ , and why it is yet unsolved. The reason may be traced back to the *non-distributivity* of a certain lattice.

Suppose we try to apply the techniques of the foregoing sections. Let  $\mathfrak{L}^*$  be the lattice  $\{A_p, A'; \cap, +, d, d^{-1}\}$ . Let  $\mathfrak{Q}^*, \mathfrak{F}^*$  be defined as in Section 1. Then

we can show, as in Theorem 2, that  $\mathfrak{D}^\# = \mathfrak{Q}^* \cup \mathfrak{F}^*$ , modulo the second isomorphism theorem, so that the analysis of  $\mathfrak{D}^\#$  is reduced to that of the lattice  $\mathfrak{Q}^*$ . However this is where the difficulty is encountered, in that Lemma 6 no longer holds, and  $\mathfrak{Q}^*$  is not in general distributive. Some of the drastic consequences of this are as follows: in general

1°. We cannot represent  $\mathfrak{D}^\#$  by a diagram (since any lattice of subsets is distributive).

2°. We cannot use the ideas of inclusion, intersection, quotient etc. between equivalence classes of canonically isomorphic groups, as we have, in effect, been able to do in the foregoing (for a group may have distinct canonically isomorphic subgroups).

3°.  $\mathfrak{Q}^*$  is no longer finite (although  $\mathfrak{D}$  is finite).

4°.  $\mathfrak{D}^\#$  contains an infinite number of non-isomorphic groups.

5°. The sequence

$$E_p^r(\mathfrak{A}') \rightarrow E_p^r(\mathfrak{A}) \rightarrow E_p^r(\mathfrak{A}'')$$

is not exact.

We give one simple example, which is sufficient to demonstrate all these points.

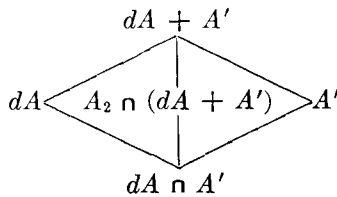
EXAMPLE. Let  $A$  be a free abelian group of rank 5 with generators  $x_1, x_2, x_3, x_4, x_5$ . Let  $A_0 = A$ ;  $A_1$  be the subgroup of rank 2 generated by  $x_1, x_2$ ;  $A_2$  be the subgroup of rank 3 generated by  $x_1, x_2, x_3$ ;  $A_3 = A$ . Let  $d$  be given by  $dx_1 = dx_3 = dx_4 = 0, dx_2 = x_1$  and  $dx_5 = x_4$ . Thus  $\mathfrak{A}$  is a filtered differential group of length 3. Let  $A'$  be the subgroup of rank 2 generated by  $x_1, x_2 + x_3 + x_4$ .

1°.  $\mathfrak{Q}^*$  is not distributive, for

$$A_2 \cap (dA + A') \text{ is of rank 2, but}$$

$$(A_2 \cap dA) + (A_2 \cap A') \text{ is of rank 1.}$$

2°. The diagram



gives a sublattice of  $\mathfrak{Q}^*$ , and so there are 1-isomorphisms

$$F = \frac{dA}{dA \cap A'} \rightarrow \frac{dA + A'}{A_2 \cap (dA + A')} \leftarrow \frac{A'}{dA \cap A'} = G.$$

Consequently  $F$  and  $G$  are c-isomorphic, and of rank 1, but  $F \cap F \not\cong F \cap G = 0$ . Therefore if  $[F]$  represents the equivalence class of groups canonically isomorphic to  $F$ ,  $[F] \cap [F]$  is ambiguous.

3°. Define<sup>7</sup> inductively

$$\begin{aligned} X_1 &= A', \\ Y_q &= [X_q + (A_2 \cap d^{-1}0)] \cap (dA + A_1) \\ X_{q+1} &= [Y_q + (A_2 \cap (dA + A'))] \cap (A_1 + A'). \end{aligned}$$

Let  $Z_q = A_1 \cap (Y_q + dA)$ . Then  $Z_q$  is of rank 2 generated by  $x_1, qx_2$ ; the verification is left to the reader. The  $Z_q$  form an infinite subset of  $\mathfrak{L}^*$ , showing that  $\mathfrak{L}^*$  is not finite.

4°. The group  $J_q = A_1/Z_q$  is cyclic of order  $q$ . Therefore  $\mathfrak{D}^\#$  contains all cyclic groups.

5°. The sequence

$$E_2^\infty(\mathfrak{A}') \rightarrow E_2^\infty(\mathfrak{A}) \rightarrow E_2^\infty(\mathfrak{A}'')$$

is not exact, since the center group is free cyclic, while the other two are zero.

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<sup>7</sup> I am indebted to Bjarni Jónsson for the idea leading to this construction.