DIHOMOLOGY
I. RELATIONS BETWEEN HOMOLOGY THEORIES

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1. Introduction

DIHOMOLOGY is a homology theory based on the use of pairs of cells instead of single cells as in classical homology. The pairs considered have some specified geometrical or topological relationship, which is called a *facing relation.* The resulting double complex is handled by means of spectral sequences. The technique is dual in a certain sense to the taking of tensor products over a ring. For if K, L are Λ-complexes we feed the Λ-structure into $K \otimes L$ by forming the quotient complex $K \otimes_\Lambda L$. So in dihomology we feed the facing relation into $K \otimes L$ by forming a subcomplex. There is no reason for restricting ourselves to pairs of cells, and the theory extends quite happily to three or more. In fact in the proof of the last theorem in the second paper (**12**) we have to use a quintuple facing relation.

Dihomology lends itself to a variety of topological situations. The three applications explored so far form the subjects of three papers of which this is the first:

I. In this paper we introduce the notions of dihomology, and use it to give short proofs of known equivalences between various homology theories, between for example Čech and Vietoris homology, Čech and Alexander cohomology, simplicial and singular-simplicial (co)homology, and simplicial and singular-cubical (co)homology. Another application is Dowker's Theorem on the homology groups of a relation (**4**). The equivalences are natural and do not depend on any choice of chain map, because they are induced by augmentation.

An example of the type of facing relation used is the Čech-singular facing relation, in which a Čech simplex is related to a singular simplex (or cube) if the support of the former contains the image of the latter. This particular facing relation leads to the natural transformation Υ from singular homology to Čech homology, and Υ^* from Čech cohomology to singular cohomology. It also leads to the spectral ring relating Čech and singular cohomology, which is examined in detail in the last section.

II. In the second paper (**12**) we use dihomology to introduce the simplicial spectral theory of a simplicial map, and the Čech spectral theory of a continuous map. These are the simplicial and Čech analogues of Leray's sheaf theory (**9**), and Serre's singular theory for fibre maps (**10**). On a simplicial map all four theories are proved to be isomorphic. On a continuous map the Leray and Čech theories are proved to be isomorphic, and Υ^* is generalized to a natural homomorphism from Leray's spectral ring to Serre's spectral ring. To illustrate the type of facing relation used, suppose we are given a simplicial map from K to L: then a simplex of K is related to a simplex of L if the image of the star of the former meets the star of the latter.

III. In the third paper (**13**) we define a spectral sequence on a space which is a topological invariant but not an invariant of homotopy type. It is a mixed functor involving homology and cohomology together, and relates local and global structure; on a manifold the sequence collapses to the Poincaré Duality isomorphism. As a functor it is a functor on a category of maps (or carriers) and the spectral sequence on a space is that associated with the identity map. To compare this spectral sequence with the Čech-singular spectral ring which is described in this paper, one might say that the latter measures how far a topological space falls short of being a polyhedron, while the former measures how far a polyhedron falls short of being a manifold. To illustrate the type of facing relation used, suppose we are given a simplicial complex: then a simplex is related to another if it faces the other. This particularly simple example suggested the term 'facing relation'.

The idea of dihomology and the subject matter of III were contained in a thesis submitted for a doctorate at Cambridge in 1954, and were announced in (**11**). I studied under Shaun Wylie, to whom I should like to express my gratitude for his unfailing fount of intuition and encouragement.

This paper is divided into six sections:

1. Introduction.
2. Facing relations.
3. Isomorphisms between homology groups.
4. Equivalences between homology and cohomology theories.
5. The spectral sequence of a facing relation.
6. The Čech-singular spectral ring of a space.

2. Facing relations

All complexes in this paper will be *geometric chain complexes* (see (**8**) 108). A geometric chain complex is a combination of geometric and algebraic notions; it is an abstract cell complex ((8) 88), together with the associated chain complex, and is defined as follows:

Definition; algebraic part. A geometric chain complex K is a graded differential group $K = \sum_{-\infty}^{\infty} K_p$, with boundary $\partial : K_p \to K_{p-1}$, such that $\partial^2 = 0$ and $K_p = 0$ if $p < 0$. For each $p \geqslant 0$, K_p is a free Abelian group with a preferred base, whose elements are called the p-cells of K. The dimension, or degree, of a p-cell σ_p^i is p. The 0-cells are called vertices. The augmentation homomorphism $\epsilon : K_0 \to Z$ from K_0 to the integers Z, which is defined by mapping all the vertices to 1, satisfies $\epsilon\partial = 0$.

Geometric part. There is a (transitive and reflexive) partial ordering $\succ$ amongst the cells of K, satisfying the three properties:

(i) If τ_q is a face of σ_p, written $\sigma_p \succ \tau_q$, then $\sigma_p = \tau_q$ or $p > q$.

(ii) The boundary $\partial\sigma_p$ of a p-cell is linearly dependent upon its $(p-1)$-dimensional faces.

(iii) K has at least one vertex, and every cell has at least one vertex as a face.

Remark 1. A geometric chain complex uniquely determines, and is uniquely determined by, its underlying abstract cell complex. The underlying abstract complex consists of the cells (suffixed by dimension), the partial ordering, and the incidence numbers between the cells. The incidence number η^{ij} between σ_p^i and τ_{p-1}^j is given by the coefficient in the boundary formula $\partial\sigma_p^i = \sum_j \eta^{ij}\tau_{p-1}^j$. We prefer the notion of the geometric chain complex to that of the (logically equivalent) underlying abstract cell complex, because the former is algebraically more convenient. Although the topological situation generally presents us with an abstract cell complex, we shall always pass straight to the resulting geometric chain complex.

Remark 2. Occasionally it will be useful to regard K as a category (any partially ordered set may be regarded as a category). An object of the category K is a cell, and a map of the category K from σ to τ is a relation $\sigma \succ \tau$ (i.e. there is either a unique map, or no map, from σ to τ according to whether τ is a face of σ, or not).

A subset M of the cells of K is *closed* if $\sigma \in M$ and $\sigma \succ \tau$ implies $\tau \in M$. A *subcomplex* of K is the graded differential subgroup generated by a closed subset of the cells of K. If M is a subset of the cells of K (not necessarily closed), we denote by $\bar{M}$ the subcomplex generated by $\{\tau;\ \sigma \succ \tau \text{ some } \sigma \in M\}$, which is the smallest subcomplex of K containing M. Of particular interest are $\bar{\sigma}$ and $\overline{\operatorname{st}\sigma}$, where $\sigma \in K$ and $\operatorname{st}\sigma = \{\tau;\ \tau \succ \sigma\}$.

We now give seven examples of geometric chain complexes, all of which we shall use later. In each case the underlying abstract cell complex is well known (see (7) and (8)), and, by the Remark 1 above, determines the geometric chain complex that we are interested in. We use the examples to introduce notation. In Examples iii, iv, v, and vi the underlying complex is semi-simplicial.

EXAMPLE i. *The point complex P.*

The point complex has only one cell, a vertex. Thus $P_0 = H_0(P) = Z$, the integers, and $P_q = H_q(P) = 0$, $q \neq 0$.

EXAMPLE ii. *Oriented simplicial complex K.*

Let K be an oriented Euclidean simplicial complex triangulating the polyhedron $|K|$. We do not necessarily assume K to be finite. Let K also denote the resulting geometric chain complex, whose cells are the oriented simplexes. As usual we write the homology group as $H_*(K) = \sum_p H_p(K)$. If σ is a simplex, let $|\sigma|$ denote the underlying point set of its interior, and if M is a set of simplexes, let $|M| = \bigcup_{\sigma \in M} |\sigma|$. In particular $|\mathrm{st}\,\sigma|$ is an open subset of $|K|$.

EXAMPLE iii. *The total complex $N(K)$ of a simplicial complex K.*

Let K be a Euclidean simplicial complex. The total complex is defined using *ordered* rather than oriented simplexes (see ((8) 100)). A p-cell σ of $N(K)$ is an ordered set of $p+1$ vertices of K (possibly with repetitions) which span a simplex of K; and $N(K)$ is the geometric chain complex generated by all such. A face of σ is a subset, with the induced ordering. In particular, if K is a cone with vertex x, we call $N(K)$ a *total cone* with vertex x, and it is easy to show that $N(K)$ is acyclic. More generally, if K is oriented as in the previous example, there is a natural chain equivalence $N(K) \to K$, inducing an isomorphism of homology groups (see ((8) Theorem 3.5.4); we shall also show this isomorphism in Corollary 1.3). As in the previous example, let $|\sigma|$ denote the underlying point set of the interior of the simplex spanned by σ, and let $|M| = \bigcup_{\sigma \in M} |\sigma|$.

EXAMPLE iv. *The nerve $N(\alpha)$ of a covering α.*

We generalize the previous example for Čech theory. Let α be an *open* covering of a space X. A p-cell σ of the nerve $N(\alpha)$ is a Čech simplex, namely an ordered set of $p+1$ sets of α (possibly with repetitions) having non-empty intersection; the intersection is called the support of σ, and is denoted by $\sup \sigma$. Define the nerve $N(\alpha)$ to be the geometric chain complex generated by all such cells. We write $\check{H}_*(X, \alpha)$ for the homology group of $N(\alpha)$; and taking inverse limits over α we obtain the Čech homology group $\check{H}_*(X)$ of X.

In the case when α is the star covering of a Euclidean complex K, Example iv reduces to Example iii; $N(\alpha) = N(K)$, and $\sup \sigma = |\mathrm{st}\,\sigma|$, $\sigma \in K$.

We emphasize once and for all that throughout

(a) *all coverings will be open*;

(b) for the nerve we use *ordered* simplexes (in order that the cup-product be functorial);

(c) for Čech theory we use the set of *all* open coverings; and

(d) for Čech and Alexander cohomology we shall use cochains with *arbitrary supports*.

Example v. *The Vietoris complex $V(X)$ of a space X.*

A p-cell of $V(X)$ is a Vietoris simplex, namely an ordered set of $p+1$ points of X (possibly with repetitions). Define $V(X)$ to be the geometric chain complex generated by all such cells. If $x \in X$, then $V(X)$ is a total cone with vertex x, and so $V(X)$ is acyclic. Of course the topology of X is lost in $V(X)$, but we may reinsert it by means of coverings. If α is an open covering of X, a Vietoris simplex is said to be α-small if it is contained within some set of α. Define the *α-small Vietoris complex* $V(X, \alpha)$ of X to be the subcomplex of $V(X)$ generated by all α-small Vietoris simplexes.

Example vi. *The singular complex $S(X)$ of a space X.*

A p-cell of $S(X)$ is a singular p-simplex of X. Define $S(X)$ to be the geometric chain complex generated by all such (see ((**8**) 315)). The singular homology group of X is defined to be the homology group of $S(X)$, which we write as ${}^sH_*(X)$. If σ is a singular simplex, let $\mathrm{im}\,\sigma$ denote its point-set image. If α is an open covering of X, call σ α-small if $\mathrm{im}\,\sigma$ is contained within some set of α. Define the *α-small singular complex* $S(X, \alpha)$ of X to be the subcomplex generated by all α-small singular simplexes. We write ${}^sH_*(X, \alpha)$ for the resulting homology group. The inclusion $S(X, \alpha) \subset S(X)$ is a chain equivalence, inducing an isomorphism of homology groups ${}^sH_*(X, \alpha) \xrightarrow{\cong} {}^sH_*(X)$ (see ((**7**) 197)).

Example vii. *The normalized singular cubical complex $Q^N(X)$ of a space X.*

We define $Q^N(X)$ to be the quotient complex of the singular cubical complex $Q(X)$ by the subcomplex $Q^D(X)$ of degenerate cubes (see ((**8**) 321), and (**10**)). It matters not for this paper whether degeneracy is defined at the 'front' or the 'back' or on every coordinate of the cubes (see (**6**)), whereas in (**12**) we shall specifically require degeneracy at the 'front'. It is well known (**6**) that there are chain equivalences between $S(X)$ and $Q^N(X)$. Therefore, where no confusion can arise, we shall use the same symbol ${}^sH_*(X)$ to denote the resulting homology group. In particular when X is contractible then $S(X)$ and $Q^N(X)$ are acyclic.

In order to make $Q^N(X)$ into a geometric chain complex, we have yet to define the cells. Now $Q(X)$ is generated by all singular cubes of X, and $Q^D(X)$ by the subset of degenerate cubes; therefore the natural epimorphism $Q(X) \to Q^N(X)$ maps the complementary set of all non-degenerate cubes into a set of generators for $Q^N(X)$, which we call the cells of $Q^N(X)$. The partial ordering on $Q(X)$, given by the faces of the singular cubes, induces the required partial ordering on the cells of $Q^N(X)$.

If σ is a cell of $Q^N(X)$, then σ is a coset containing a unique non-degenerate singular cube, and we define $\operatorname{im}\sigma$ to be the point-set image of the latter. Given an open covering α of X, we can define, in a manner similar to the previous example, the *α-small normalized singular cubical complex* $Q^N(X,\alpha)$ of X. We can prove that the inclusion $Q^N(X,\alpha) \subset Q^N(X)$ is a chain equivalence, either directly, or by transferring the result from the singular simplicial theory. The direct proof uses a subdivision of the n-dimensional cube into 3^n small cubes, analogous to the regular subdivision of a simplex; the explicit formulae will be given in ((**12**) § 7).

The category $\mathfrak{C}$

A *chain map* between two chain complexes is a homomorphism that preserves the grading and commutes with the boundary. A *geometric chain map* between two geometric chain complexes is a chain map that maps cells to cells. Let $\mathfrak{C}$ denote the category of geometric chain complexes and geometric chain maps.

Notice that in Example ii a simplicial map between oriented simplicial complexes induces a chain map which is not in general geometric, because not all cells are mapped to cells (a simplex that is mapped degenerately has algebraic image zero). However, in Example iii a simplicial map between simplicial complexes does induce a geometric chain map between the corresponding total complexes. Similarly in Example iv. In Examples v, vi, and vii a continuous map between spaces does induce a geometric chain map between the respective geometric chain complexes. In fact Examples iii, ..., vii turn out to be natural from the functorial point of view, particularly when we come to the multiplicative structure in § 4.

The augmentation

If we define ϵ to be zero on K_p $(p > 0)$ we extend the augmentation from a homomorphism $K_0 \to Z$ to a chain map $\epsilon_K : K \to P$, where P is the point complex. We denote by the same symbol ϵ_K any homomorphism induced by the augmentation, and in particular the epimorphism $\epsilon_K : H_*(K) \to Z$. As usual we define K to be *connected* if $\epsilon_K : H_0(K) \xrightarrow{\cong} Z$, and *acyclic* if $\epsilon_K : H_*(K) \xrightarrow{\cong} Z$. If f is a geometric chain map then $\epsilon f = \epsilon$, and so the augmentation may be regarded as a functor from the category $\mathfrak{C}$ to the subcategory $\mathfrak{P}$ comprised of one object P and the identity map. We emphasize this seemingly trivial remark because upon it rests the naturality of the many homomorphisms and isomorphisms that we shall establish between various homology and spectral theories.

Double complexes

Given two geometric chain complexes $K = \Sigma K_p$, $L = \Sigma L_q$, we form in the usual way the *double complex* $K \otimes L$, which is bigraded with skew-commutative differentials $d_K = \partial_K \otimes 1$, $d_L = \omega_K \otimes \partial_L$, where ω_K is the sign-changing automorphism of K given by $\omega_K x = (-)^p x$ for a homogeneous element x of degree p in K. A cell of $K \otimes L$ is the tensor product $\sigma \otimes \tau$ of cells $\sigma \in K$, $\tau \in L$, and we write $\sigma \otimes \tau \succ \sigma' \otimes \tau'$ if $\sigma \succ \sigma'$ and $\tau \succ \tau'$. Associated with the double complex $K \otimes L$ there is a single geometric chain complex, graded by $n = p+q$ and with differential $d = d_K + d_L$. We shall use the symbol $K \otimes L$ to denote ambiguously both the double and the associated single geometric chain complex. Consequently we may identify $K \otimes P = K$ and $P \otimes L = L$. The augmentations induce chain maps $\epsilon_K : K \otimes L \to L$ and $\epsilon_L : K \otimes L \to K$. If D is a subcomplex of the double complex $K \otimes L$ we shall also use D ambiguously to denote both the double and the associated single complex.

Definition of a facing relation

A *facing relation* $\mathfrak{F}$ between two complexes K and L is a set $\{\sigma \otimes \tau\}$ of cells of $K \otimes L$ satisfying the *facing condition*

$$\sigma \otimes \tau \in \mathfrak{F} \quad \text{and} \quad \sigma \otimes \tau \succ \sigma' \otimes \tau' \quad \text{implies} \quad \sigma' \otimes \tau' \in \mathfrak{F}.$$

A facing relation generates a subcomplex D of the double complex $K \otimes L$. Of course the facing relation $\mathfrak{F}$ and the subcomplex D are logically equivalent, but the notion is so basic that it is worth having two symbols to distinguish between the geometric and algebraic aspects. The augmentations restricted to D induce chain maps $\epsilon_K : D \to L$ and $\epsilon_L : D \to K$.

We next introduce the notion of facets. If $\sigma \in K$ the *right facet* $\mathfrak{F}\sigma$ of σ is the subcomplex of L generated by $\{\tau;\, \sigma \otimes \tau \in \mathfrak{F}\}$. It is a subcomplex because of the facing condition. Similarly if $\tau \in L$ the *left facet* $\mathfrak{F}\tau$ of τ is the subcomplex of K generated by $\{\sigma;\, \sigma \otimes \tau \in \mathfrak{F}\}$. The facets will have both geometrical significance and algebraical usefulness, so it is important to think of them both geometrically as sets of cells and algebraically as chain complexes. We say that $\mathfrak{F}$ is *left acyclic* if all the left facets are acyclic, *right acyclic* if all the right facets are acyclic, and *acyclic* if it is both left and right acyclic.

Examples of facing relations

We give as examples the seven facing relations that we shall use in this paper.

EXAMPLE 1. The acyclic *simplicial* : *total-simplicial* facing relation.

Let K be an oriented simplicial complex. Let $L = N(K)$, the total complex of K. Let $\mathfrak{F}$ be the facing relation between K and L given by

$$\mathfrak{F} = \{\sigma\otimes\tau;\ |\mathrm{st}\,\sigma|\cap|\mathrm{st}\,\tau|\neq\emptyset\}.$$

We verify that $\mathfrak{F}$ satisfies the facing condition, for if $\sigma\succ\sigma'$ in K and $\tau\succ\tau'$ in L and $\sigma\otimes\tau\in\mathfrak{F}$, then

$$|\mathrm{st}\,\sigma'|\cap|\mathrm{st}\,\tau'|\supset|\mathrm{st}\,\sigma|\cap|\mathrm{st}\,\tau|\neq\emptyset,$$

and so $\sigma'\otimes\tau'\in\mathfrak{F}$. A more intuitive way of saying this is that stars expand when passing to faces. Given $\tau\in L$, let σ be the oriented simplex of K underlying τ (obtained by omitting any repetitions in τ); then the left facet $\mathfrak{F}\tau$ is $\overline{\mathrm{st}\,\sigma}$, which is a cone, and consequently acyclic. Therefore $\mathfrak{F}$ is left acyclic. Similarly $\mathfrak{F}$ is right acyclic, because, if $\sigma\in K$, then the right facet $\mathfrak{F}\sigma$ is $N(\overline{\mathrm{st}\,\sigma})$, which is a total cone.

EXAMPLE 2. The acyclic *simplicial* : *singular* facing relation.

Let K be an oriented simplicial complex triangulating X, and let α be the star covering of K. Let $L = S(X,\alpha)$, the α-small singular complex of X. Let $\mathfrak{F}$ be the facing relation between K and L given by

$$\mathfrak{F} = \{\sigma\otimes\tau;\ |\mathrm{st}\,\sigma|\supset\mathrm{im}\,\tau\}.$$

$\mathfrak{F}$ satisfies the facing condition because stars expand and images shrink when passing to faces; more precisely, if $\sigma\succ\sigma'$ in K and $\tau\succ\tau'$ in L and $\sigma\otimes\tau\in\mathfrak{F}$, then

$$|\mathrm{st}\,\sigma'|\supset|\mathrm{st}\,\sigma|\supset\mathrm{im}\,\tau\supset\mathrm{im}\,\tau'$$

and so $\sigma'\otimes\tau'\in\mathfrak{F}$. Given $\tau\in L$, then the left facet $\mathfrak{F}\tau=\bar{\sigma}$, where σ is the simplex of K spanning those vertices of K whose stars contain $\mathrm{im}\,\tau$ (there is always at least one such vertex because τ is α-small). Therefore the left facets are simplexes, and so $\mathfrak{F}$ is left acyclic. Notice that had we used $S(X)$ instead of $S(X,\alpha)$, then the facet of too large a singular simplex would have been empty, and so $\mathfrak{F}$ would not have been left acyclic, which subsequently would have been technically inconvenient. If $\sigma\in K$, then the right facet $\mathfrak{F}\sigma = S(|\mathrm{st}\,\sigma|)$, which is acyclic because it is the singular complex of a contractible space $|\mathrm{st}\,\sigma|$. Therefore $\mathfrak{F}$ is acyclic on both sides.

EXAMPLE 3. The acyclic *simplicial* : *singular-cubical* facing relation.

In Example 2 replace $L = S(X,\alpha)$ by $L = Q^N(X,\alpha)$, the α-small normalized singular cubical complex of X. Then the same formula for $\mathfrak{F}$ gives a facing relation, which is acyclic for exactly the same reasons.

EXAMPLE 4. The acyclic *total-simplicial* : *singular* facing relation.

Let M be a simplicial complex triangulating X, with star covering α. Let $K = N(M)$, the total complex of M, and let $L = S(X,\alpha)$, the α-small singular complex of X. Then the same formula for $\mathfrak{F}$ as in Example 2 gives a facing relation between K and L, which is acyclic because the left facets are total cones, while the right facets are contractible, as before. The advantage of using the total complex (Example 4 as opposed to Example 2) will become apparent in the functorial approach of § 4.

EXAMPLE 5. The left acyclic *Čech* : *singular* facing relation.

We may generalize the previous example by passing from the simplicial theory on polyhedra to Čech theory on arbitrary spaces. Let X be a topological space and α an open covering of X. Let $K = N(\alpha)$, the nerve of α, and $L = S(X,\alpha)$, the α-small singular complex of X. Let $\mathfrak{F}$ be the facing relation between K and L given by

$$\mathfrak{F} = \{\sigma\otimes\tau;\ \sup\sigma\supset \operatorname{im}\tau\}.$$

$\mathfrak{F}$ satisfies the facing condition because supports expand and images shrink when passing to faces. $\mathfrak{F}$ is left acyclic because any left facet $\mathfrak{F}\tau$ is a total cone with vertex any vertex of K whose support contains $\operatorname{im}\tau$; in particular if only a finite number of sets of α contain $\operatorname{im}\tau$, and if σ is the corresponding simplex of the nerve, then the left facet $\mathfrak{F}\tau = N(\sigma)$. On the other hand, $\mathfrak{F}$ is not in general right acyclic, because the singular homology groups of $\sup\sigma$ may be far from trivial. Indeed the right facets do in a sense capture the local singularities of X, as we shall see in Theorem 4.

EXAMPLE 6. The acyclic *Čech* : *Vietoris* facing relation.

Let α be an open covering of a space X. Let $K = N(\alpha)$, the nerve of α, and $L = V(X,\alpha)$, the α-small Vietoris complex of X. Let $\mathfrak{F}$ be the facing relation between K and L given by

$$\mathfrak{F} = \{\sigma\otimes\tau;\ \sup\sigma\supset\tau\}.$$

$\mathfrak{F}$ satisfies the facing condition because supports expand and Vietoris simplexes shrink when passing to faces. $\mathfrak{F}$ is left acyclic as in the previous example, and is right acyclic because any right facet $\mathfrak{F}\sigma = V(\sup\sigma)$ is a total cone.

EXAMPLE 7. The acyclic facing relation of a *relation* $\mathbf{R}$.

This is a generalization of the previous example. Let $\mathbf{R}$ be a relation between two sets X, Y; in other words $\mathbf{R}$ is a subset of $X \times Y$. Dowker (**4**) observed that $\mathbf{R}$ gives rise to two complexes K, L; a simplex $\sigma \in K$ is a finite ordered set of elements of X (possibly with repetitions) all related to one element of Y, and a simplex $\tau \in L$ is a similar subset of Y. We define the facing relation $\mathfrak{F}$ to be the set of all pairs $\sigma \otimes \tau$ such that each element of σ is related to each element of τ. $\mathfrak{F}$ clearly satisfies the facing condition, and is acyclic because all the facets are total cones. It does not matter if we replace either or both of K and L by a complex formed of oriented simplexes without repetitions, instead of ordered simplexes with repetitions; the same facing relation does the job.

Remark 1. The above examples are all concerned with a single space; on the space a single complex gives a homology group, and a facing relation between two complexes will relate two different homology groups. In the next paper (**12**) we shall start with a continuous map between two spaces; a facing relation between two complexes on the two spaces will give a spectral sequence of the map, while a quadruple facing relation between four complexes will relate two different spectral sequences. In the next paper, therefore, we shall introduce multiple facing relations.

Remark 2. A facing relation resembles a carrier. Assigning to a simplex its facet is indeed a functor $\sigma \to \mathfrak{F}\sigma$ from the category of K to the category of subcomplexes of L, like a carrier, but it is a covariant functor whereas a carrier is contravariant: the facets have the carrier condition reversed, namely if $\sigma \succ \sigma'$ then $\mathfrak{F}\sigma \subset \mathfrak{F}\sigma'$. Facing relations which have closed facets obeying the carrier condition the right way round turn out to be mixed functors involving both homology and cohomology at once; we examine them in the third paper (**13**).

3. Isomorphisms between homology groups

The corollaries in this and the next section are classical. Our purpose is to show how they are unified under the notion of a facing relation and all follow from the one theorem:

THEOREM 1. *Let D be the double complex generated by a facing relation $\mathfrak{F}$ between two complexes K and L. If $\mathfrak{F}$ is left acyclic, then the augmentation of K induces an isomorphism* $\epsilon_K : H_*(D) \xrightarrow{\cong} H_*(L)$.

Proof. The proof follows a standard spectral sequence argument. Consider the spectral sequences obtained from D and $P \otimes L = L$, respectively, filtering with respect to q (the degree of the elements of the right-hand complex). The augmentation of K induces a homomorphism between the two spectral sequences. The homomorphism of the E^0 terms is $D \to P \otimes L$, or can be written

$$\Sigma \mathfrak{F}\tau \otimes \tau \xrightarrow[\epsilon_K]{} \Sigma P \otimes \tau,$$

where the direct sum is taken over all $\tau \in L$. The d^0 differentials of the spectral sequences are d_K, d_P respectively, and operate on the left of the tensor products only. Therefore the homomorphism of the E^1 terms is

$$\Sigma H_*(\mathfrak{F}\tau) \otimes \tau \xrightarrow[\epsilon_K]{} \Sigma Z \otimes \tau.$$

But by the acyclicity of each $\mathfrak{F}\tau$ this is an isomorphism. Therefore ϵ_K induces an isomorphism on all the E^r terms, $1 \leqslant r < \infty$. The spectral sequences converge, because they arise from double complexes having non-zero terms only in the positive quadrant $p, q \geqslant 0$ (therefore the filtration is restricted; see ((8) Theorem 10.3.9)). Therefore ϵ_K induces an isomorphism on the E^∞ terms, and hence an isomorphism of homology.

$$H_*(D) \xrightarrow[\epsilon_K]{\cong} H_*(P \otimes L) = H_*(L).$$

The theorem is proved. We observe in passing that since the E^1 terms are concentrated on the axis $p = 0$, the spectral sequences collapse $E^2 = E^\infty = H_*(D)$.

Corollary 1.1. *If $\mathfrak{F}$ is left acyclic the augmentations induce a homomorphism*

$$\Upsilon = \epsilon_L \epsilon_K^{-1} : H_*(L) \to H_*(K).$$

Corollary 1.2. *If $\mathfrak{F}$ is acyclic Υ is an isomorphism*

$$\Upsilon : H_*(L) \xrightarrow{\cong} H_*(K).$$

Corollary 1.3. *The homology groups of an oriented simplicial complex and its total complex are isomorphic.*

Proof. Use Example 1 and Corollary 1.2.

Corollary 1.4. *The singular and simplicial homology groups of a polyhedron are isomorphic.*

Proof. Use Examples 2 or 4 and Corollary 1.2, and the isomorphism ${}^sH_*(X, \alpha) \xrightarrow{\cong} {}^sH_*(X)$. Similarly we have by Example 3:

Corollary 1.5. *The singular-cubical and simplicial homology groups of a polyhedron are isomorphic.*

Remark. Other methods of proving Corollaries 1.4 and 1.5 are well known, and do in fact give the same isomorphism. By comparison our method enjoys certain advantages. First, it is natural and no choice is involved. Secondly, it is free from orientation troubles. Thirdly, if we generalize the simplicial to Čech theory, the isomorphism is seen to be a special case of the homomorphism Υ from singular to Čech homology (Corollary 2.7). Fourthly, the method generalizes to give an isomorphism between the singular and simplicial spectral sequences of a simplicial map ((**12**) Theorem 7).

COROLLARY 1.6. (Dowker's Theorem (**4**).) *If K, L are the complexes arising from a relation* $\mathbf{R}$, *then* $H_*(K) \cong H_*(L)$.

Proof. Use Example 7 and Corollary 1.2. The advantage of the proof here is that it avoids having to pass to the derived complexes, and admits of easy relativization (see Lemma 3).

4. Equivalences between homology and cohomology theories

Notation for categories

We have already denoted by $\mathfrak{C}$ the category of geometric chain complexes and chain maps. Let $\mathfrak{K}$ be the category of simplicial complexes and simplicial maps. Let $\mathfrak{G}$, $\mathfrak{G}_*$, and $\mathfrak{R}$ denote the categories of (Abelian) groups, graded groups, and associative commutative rings, respectively, together with structure-preserving homomorphisms. If $R \in \mathfrak{R}$ let $\mathfrak{C}_R$, $\mathfrak{R}_R^*$ be, respectively, the categories of cochain complexes of R-modules and cochain maps, graded associative skew-commutative rings over R and homomorphisms.

Let $\mathfrak{A}$ be the category of non-empty topological spaces and continuous maps. Suppose that $f: X \to X'$ is a continuous map, and that α, α' are open coverings of X, X' respectively. We say that a simplicial map $f_\alpha : N(\alpha) \to N(\alpha')$ between the nerves is an *approximation* to f if

$$\sup f_\alpha a \supset f(\sup a)$$

for each vertex $a \in N(\alpha)$; and consequently

$$\sup f_\alpha \sigma \supset f(\sup \sigma)$$

for each cell $\sigma \in N(\alpha)$. We now define a category $\mathfrak{A}_{\text{cov}}$: an object (X, α) of $\mathfrak{A}_{\text{cov}}$ consists of a space X together with an open covering α of X; a map $(f, f_\alpha) : (X, \alpha) \to (X', \alpha')$ of $\mathfrak{A}_{\text{cov}}$ consists of a continuous map $f: X \to X'$ together with an approximation $f_\alpha : N(\alpha) \to N(\alpha')$.

Let $\mathfrak{X}$, $\mathfrak{Y}$ denote arbitrary categories.

Functorial facing relations

Let K, L be covariant functors $\mathfrak{X} \to \mathfrak{C}$ from an arbitrary category $\mathfrak{X}$ to the category $\mathfrak{C}$ of geometric chain complexes. We say $\mathfrak{F}$ is a *functorial* facing relation on $\mathfrak{X}$ between K and L if

(i) for each object X in $\mathfrak{X}$ there is a facing relation $\mathfrak{F}X$ between KX and LX; and

(ii) if $f: X \to X'$ is a map of $\mathfrak{X}$, and $\sigma \otimes \tau \in \mathfrak{F}X$, then $(Kf)\sigma \otimes (Lf)\tau \in \mathfrak{F}X'$.

We say that $\mathfrak{F}$ is left (or right) acyclic if each $\mathfrak{F}X$ is. A functorial facing relation gives rise to a covariant functor D from $\mathfrak{X}$ to the category of double chain complexes and maps. If $H : \mathfrak{C} \to \mathfrak{G}_*$ is the homology functor then we have functors HK, HL, and HD from $\mathfrak{X}$ to $\mathfrak{G}_*$. We apply these ideas to three of the facing relations of § 2.

EXAMPLE 4. The acyclic *total-simplicial* : *singular* facing relation is functorial.

$\mathfrak{F}$ is defined on $\mathfrak{K}$, the category of simplicial complexes and simplicial maps. The functor $K : \mathfrak{K} \to \mathfrak{C}$ assigns to each complex M its total complex $KM = N(M)$. The functor $L : \mathfrak{K} \to \mathfrak{C}$ assigns to each complex M, with star covering α, its α-small singular complex $LM = S(|M|, \alpha)$. The facing relation $\mathfrak{F}M$ between KM and LM is given by

$$\mathfrak{F}M = \{\sigma \otimes \tau;\ |\operatorname{st} \sigma| \supset \operatorname{im} \tau\}.$$

A simplicial map $f : M \to M'$ induces a geometrical chain map $f : KM \to KM'$ between the total complexes. If α' denotes the star covering of M', then f maps an α-small singular simplex of $|M|$ into an α'-small singular simplex of $|M'|$, and so f induces a geometrical chain map $f : LM \to LM'$. (We use the same symbol f to denote the induced chain maps.) If $\sigma \otimes \tau \in \mathfrak{F}M$, then

$$|\operatorname{st} f\sigma| \supset f|\operatorname{st} \sigma| \supset f(\operatorname{im} \tau) = \operatorname{im} f\tau,$$

and so $f\sigma \otimes f\tau \in \mathfrak{F}M'$. Therefore $\mathfrak{F}$ is functorial on $\mathfrak{K}$.

EXAMPLE 5. The left acyclic *Čech* : *singular* facing relation is functorial.

$\mathfrak{F}$ is defined on $\mathfrak{A}_{\text{cov}}$. The functor $K : \mathfrak{A}_{\text{cov}} \to \mathfrak{C}$ assigns to the object (X, α) the nerve $K(X, \alpha) = N(\alpha)$, and assigns to the map (f, f_α) the approximation f_α between the nerves. The functor $L : \mathfrak{A}_{\text{cov}} \to \mathfrak{C}$ assigns to the object (X, α) the α-small singular complex $L(X, \alpha) = S(X, \alpha)$, and assigns to the map (f, f_α) the geometric chain map between the singular chain complexes induced by f, which we also denote by f. The facing relation $\mathfrak{F}(X, \alpha)$ between $K(X, \alpha)$ and $L(X, \alpha)$ is given by

$$\mathfrak{F}(X, \alpha) = \{\sigma \otimes \tau;\ \sup \sigma \supset \operatorname{im} \tau\}.$$

The facing relation is functorial, because if $\sigma \otimes \tau \in \mathfrak{F}$ then

$$\sup f_\alpha \sigma \supset f(\sup \sigma) \supset f(\operatorname{im} \tau) = \operatorname{im} f\tau,$$

and so $f_\alpha \sigma \otimes f\tau \in \mathfrak{F}(X', \alpha')$.

EXAMPLE 6. The acyclic *Čech* : *Vietoris* facing relation is functorial.

$\mathfrak{F}$ is defined on $\mathfrak{A}_{\text{cov}}$. The functor K assigns to the object (X, α) the nerve $K(X, \alpha) = N(\alpha)$, and assigns to the map (f, f_α) the approximation f_α between the nerves. The functor L assigns to (X, α) the α-small Vietoris complex $L(X,\alpha) = V(X,\alpha)$, and assigns to the map (f,f_α) the geometric chain map between the Vietoris complexes induced by f, which we also denote by f. The facing relation $\mathfrak{F}(X,\alpha)$ between $K(X,\alpha)$ and $L(X,\alpha)$ is given by

$$\mathfrak{F}(X,\alpha) = \{\sigma\otimes\tau;\ \sup\sigma\supset\tau\}.$$

$\mathfrak{F}$ is functorial, because if $\sigma\otimes\tau\in\mathfrak{F}(X,\alpha)$ then $\sup f_\alpha\sigma\supset f(\sup\sigma)\supset f\tau$, and so $f_\alpha\sigma\otimes f\tau\in\mathfrak{F}(X',\alpha')$.

Limiting processes

Having introduced the category $\mathfrak{A}_{\mathrm{cov}}$, we mention here the usual limiting device for getting rid of the coverings. Let $J:\mathfrak{A}_{\mathrm{cov}}\to\mathfrak{Y}$ be a functor into some category $\mathfrak{Y}$. We say that J is *independent of approximation* if any two maps (f,f_α^1) and (f,f_α^2) from (X,α) to (X',α') in $\mathfrak{A}_{\mathrm{cov}}$ with the same underlying continuous map $f:X\to X'$ have the same image under J.

LEMMA 1. *If $\mathfrak{Y}$ is a category in which inverse (direct) limits exist, then a covariant (contravariant) functor $J:\mathfrak{A}_{\mathrm{cov}}\to\mathfrak{Y}$ which is independent of approximation induces a covariant (contravariant) functor* $\lim J:\mathfrak{A}\to\mathfrak{Y}$.

The proof is classical and due to Čech (see ((7) Chapters VIII and IX)).

THEOREM 2. *If $\mathfrak{F}$ is a left acyclic functorial facing relation between the covariant functors $K,L:\mathfrak{X}\to\mathfrak{C}$, then the augmentation of K induces a natural equivalence between the functors $HD\xrightarrow[\epsilon_K]{\cong}HL$ from $\mathfrak{X}$ to $\mathfrak{G}_*$.*

Proof. Let $f:X\to X'$ be a map of $\mathfrak{X}$. The commutative diagrams

$$\begin{array}{ccc} DX & \xrightarrow{\epsilon_K} & LX \\ \downarrow{\scriptstyle Df} & & \downarrow{\scriptstyle Lf} \\ DX' & \xrightarrow{\epsilon_K} & LX' \end{array}$$

yield the commutative diagrams

$$\begin{array}{ccc} HDX & \xrightarrow[\epsilon_K]{\cong} & HLX \\ \downarrow{\scriptstyle HDf} & & \downarrow{\scriptstyle HLf} \\ HDX' & \xrightarrow[\epsilon_K]{\cong} & HLX' \end{array}$$

which comprise the theorem.

COROLLARY 2.1. *If $\mathfrak{F}$ is a left acyclic functorial facing relation between K and L, the augmentations induce a natural transformation between the functors $\Upsilon:HL\to HK$.*

COROLLARY 2.2. *If* $\mathfrak{F}$ *is acyclic then* Υ *is a natural equivalence* $\Upsilon : HL \xrightarrow{\cong} HK$.

Multiplicative structure

If K is a geometric chain complex, a *multiplicative structure* on K is a chain map (not in general geometric) $m : K \to K \otimes K$ carried by the diagonal carrier $\Delta\sigma = \bar{\sigma} \otimes \bar{\sigma}$ $(\sigma \in K)$, and such that $\epsilon_{K \otimes K} m = \epsilon_K$. We say that K is *multiplicative* if it possesses a multiplicative structure. Suppose now that $K : \mathfrak{X} \to \mathfrak{C}$ is a functor from an arbitrary category $\mathfrak{X}$ into the category $\mathfrak{C}$ of geometric chain complexes. We say that K is *functorially multiplicative* if for each object $X \in \mathfrak{X}$ there is a multiplicative structure mX on KX, such that for each map $f : X \to X'$ in $\mathfrak{X}$ the diagram

$$\begin{array}{ccc} KX & \xrightarrow[mX]{} & KX \otimes KX \\ \Big\downarrow{\scriptstyle Kf} & & \Big\downarrow{\scriptstyle Kf \otimes Kf} \\ KX' & \xrightarrow[mX']{} & KX' \otimes KX' \end{array}$$

is commutative.

It is well known (8) that all the examples of geometric chain complexes in § 2 are multiplicative, although not all are functorially multiplicative. For instance Example ii is not functorially multiplicative, because a multiplicative structure on the chain complex of an oriented simplicial complex depends upon some local ordering of the vertices (nor, as we have pointed out, does Example ii give a functor into $\mathfrak{C}$). However, all the other examples are functorially multiplicative.

For instance, consider Example iii: in this case the functor $N : \mathfrak{K} \to \mathfrak{C}$ assigns to each simplicial complex K its total complex $N(K)$, and assigns to each simplicial map the corresponding geometric chain map between the total complexes. The multiplicative structure m on $N(K)$ is defined on the cells of $N(K)$ and extended linearly: if $\sigma = (x_0 x_1 \dots x_p)$ is a cell of $N(K)$, define

$$m\sigma = \sum_{i=0}^{p} (x_0 x_1 \dots x_i) \otimes (x_i x_{i+1} \dots x_p).$$

If $f : K \to K'$ is a simplicial map, and if $fx_i = y_i$, $i = 0, 1, \dots, p$, then the image of σ under the corresponding geometric chain map is $f\sigma = (y_1 y_2 \dots y_p)$. Therefore $mf\sigma = fm\sigma$, since order is preserved by f. Consequently N is functorially multiplicative.

A similar argument shows that Examples iv, v, and vi are functorially multiplicative, and an analogous formula for cubes shows the same for Example vii (see ((8) 141, 361, and 367)).

Cohomology ring

Let R be the coefficient ring. The cochain complex of K with coefficients in R is defined in the usual way by† $K \pitchfork R = \sum_p K_p \pitchfork R$, with coboundary $\delta = \partial \pitchfork 1$. A multiplicative structure m on K induces a ring structure on $K \pitchfork R$ as follows: the cup product between a, $b \in K \pitchfork R$ is given by $a \cup b = \mu(a \otimes b) m$, where $\mu : R \otimes R \to R$ is the multiplication of R. The ring structure on $K \pitchfork R$ is associative, and it induces an associative skew-commutative ring structure on the cohomology group $H^*(K; R)$ of $K \pitchfork R$.

If both K and L are multiplicative, there is an induced ring structure on $(K \otimes L) \pitchfork R$. The multiplicative property $\epsilon m = \epsilon$ ensures that the homomorphism $\epsilon^K : L \pitchfork R \to (K \otimes L) \pitchfork R$ induced by the augmentation of K is a ring homomorphism. If $\mathfrak{F}$ is a facing relation between K and L, generating the subcomplex $D \subset K \otimes L$, the facing condition and the carriage of m by Δ together ensure that the annihilator $\mathbf{A}D$ of D is an ideal of $(K \otimes L) \pitchfork R$. Therefore

$$D \pitchfork R \cong \frac{(K \otimes L) \pitchfork R}{\mathbf{A}D}$$

becomes a ring, and the induced homomorphism $\epsilon^K : L \pitchfork R \to D \pitchfork R$ is a ring homomorphism. Applying the universal coefficient theorem and Theorem 1, we have:

LEMMA 2. *If K and L are multiplicative complexes and $\mathfrak{F}$ is a left acyclic facing relation between them, then $\epsilon^K : H^*(L; R) \xrightarrow{\cong} H^*(D; R)$ is a ring isomorphism, and $\Upsilon^* : H^*(K; R) \to H^*(L; R)$ is a ring homomorphism. If the facing relation and the multiplicative structures are functorial on a category $\mathfrak{X}$, there is a natural equivalence and natural transformation, respectively, between the corresponding functors from $\mathfrak{X}$ to $\mathfrak{R}^*_R$.*

Relativization

Let $\mathfrak{C}^+$ denote the category of chain complexes and chain maps. We can embed $\mathfrak{C}$ in $\mathfrak{C}^+$ because a geometric chain complex is in particular a chain complex. If $\mathfrak{X}$ is any other category with subobjects, denote by $\mathfrak{X}^+$ the category of pairs of objects (X, A), where $X \supset A$ in $\mathfrak{X}$, and pairs of maps (f, g) such that f induces g. We can embed $\mathfrak{X}$ in $\mathfrak{X}^+$ by formally adding an empty object $\emptyset$ to $\mathfrak{X}$ (if there is not one already) and identifying $X = (X, \emptyset)$. If $K : \mathfrak{X} \to \mathfrak{C}$ is a functor sending subobjects to subcomplexes, we can *relativize* K; that is to say we can extend K to a functor $\mathfrak{X}^+ \to \mathfrak{C}^+$ in the usual way by defining $K(X, A)$ to be the quotient complex KX/KA, and $K(f, g)$ to be the chain map induced by Kf.

† $A \pitchfork B$ stands for the group of homomorphisms of A into B, usually written Hom (A,B). If $A = \Sigma A_p$ is a graded group, $A \pitchfork B$ stands for the graded group $\Sigma A_p \pitchfork B$.

LEMMA 3. *If the functors K and L on $\mathfrak{X}$ can be relativized, then the functor D arising from a functorial facing relation $\mathfrak{F}$ between K and L can also be relativized. If $\mathfrak{F}$ is left acyclic the natural transformations Υ, Υ^* can be extended to $\mathfrak{X}^+$. If $\mathfrak{F}$ is acyclic then Υ, Υ^* are natural equivalences on $\mathfrak{X}^+$.*

Proof. We relativize the functor D; therefore the sequence

$$0 \to DA \to DX \to D(X, A) \to 0$$

is split exact. The augmentation of K maps this into the split exact sequence

$$0 \to LA \to LX \to L(X, A) \to 0.$$

Passing to homology, the five lemma ensures that when $\mathfrak{F}$ is left acyclic the homomorphism between the resulting exact sequences is an isomorphism. Consequently the natural equivalence $\epsilon_K : HD \xrightarrow{\cong} HL$ extends to $\mathfrak{X}^+$, and we may therefore extend Υ to $\mathfrak{X}^+$. If $\mathfrak{F}$ is acyclic, the extension of $\epsilon_L : HD \xrightarrow{\cong} HK$ shows that the extension of Υ is a natural equivalence.

The dual result holds similarly since the application of the functor $\pitchfork R$ leaves the above sequences split exact.

COROLLARY 2.3. *There are natural equivalences between simplicial and singular-simplicial homology and cohomology theories on the category $\mathfrak{K}^+$ of simplicial pairs and simplicial maps.*

Proof. Use Example 4, Corollary 2.2, and Lemmas 2 and 3. Similarly, by using Example 3 adapted in the same way that Example 4 is an adaptation of Example 2, we have:

COROLLARY 2.4. *There are natural equivalences between simplicial and singular-cubical homology and cohomology theories on $\mathfrak{K}^+$.*

COROLLARY 2.5. *There is a natural equivalence between Čech and Vietoris homology theories on the category $\mathfrak{A}^+$ of pairs of topological spaces and continuous maps.*

Proof. Use Example 6. In the category $\mathfrak{A}_{\text{cov}}$ define $(A, \alpha) \subset (X, \xi)$ if $A \subset X$ and $\alpha = \xi | A$, the covering ξ cut down to A. Therefore $N(\alpha) \subset N(\xi)$ and $V(A, \alpha) \subset V(X, \xi)$, so that we can relativize the equivalence

$$HL \xrightarrow[\Upsilon]{\cong} HK : \mathfrak{A}^+_{\text{cov}} \to \mathfrak{G}_*.$$

As is well known, HK is independent of approximation because any two simplicial approximations are contiguous and induce the same homology homomorphism; while HL is independent of approximation by definition. The result follows by Lemma 1 because the inverse limit of isomorphisms is an isomorphism.

COROLLARY 2.6. *There is a natural equivalence between Čech and Alexander cohomology theories on the category* $\mathfrak{A}^+$.

Proof. Apply Lemma 2 to the preceding case and take direct limits.

Definition. A covering α of a space X is said to be *locally acyclic* if the singular complex $S(U^0 \cap U^1 \cap \ldots \cap U^p)$ of every non-empty intersection of a finite number of sets of α is acyclic. The space X is *locally acyclic* if every covering has a locally acyclic refinement. The pair (X, A) is *locally acyclic* if every covering of X has a refinement α such that both α and $\alpha | A$ are locally acyclic. In particular a polyhedral pair is locally acyclic.

COROLLARY 2.7. *There are natural transformations* $\Upsilon : {}^sH_* \to \check{H}_*$ *from singular homology to Čech homology, and* $\Upsilon^* : \check{H}^* \to {}^sH^*$ *from Čech cohomology to singular cohomology on the category* $\mathfrak{A}^+$. *On the subcategory of locally acyclic pairs and continuous maps the transformations become equivalences.*

Proof. Use Example 5, Corollary 2.1, and Lemmas 1, 2, and 3. The taking of limits offers no difficulties because the singular and Čech functors are independent of approximation. The local acyclicity of a pair (X, A) provides a cofinal subset of coverings upon which the facing relation is acyclic, and to which we may apply Corollary 2.2.

Remark 1. The homomorphism Υ^* (and similarly Υ) is in reality part of a more detailed structure linking Čech and singular cohomology, namely the spectral ring of Theorem 4 in § 6. The geometrical interpretation of the E_2 term will provide a more delicate test for Υ^* to be an isomorphism than local acyclicity.

Remark 2. The second way in which we shall generalize Υ^* (and similarly Υ) is to establish (in (**12**), Theorem 5) a similar transformation $\Upsilon^* : \check{E}^*(f) \to {}^sE^*(f)$ between the Čech and the singular spectral rings of a continuous map.

5. The spectral sequence of a facing relation

We use the term *stack* to denote a local coefficient system for a geometric chain complex, in which each simplex has its own personal coefficient group. The notion crops up in sheaf theory: it generalizes Steenrod's local coefficients, and is due independently to Wylie. Recall that we may regard a geometric chain complex K (or, more precisely, its partially ordered set of cells) as a category. Recall that $\mathfrak{G}$ denotes the category of Abelian groups and homomorphisms.

Definition. A *covariant stack* on a geometric chain complex K is a covariant functor $\mathfrak{H} : K \to \mathfrak{G}$.

We can then form the *homology group* $H_*(K;\mathfrak{H})$ *of* K *with coefficients in* $\mathfrak{H}$ as follows. The stack $\mathfrak{H}$ assigns to each $\sigma\in K$ a group $\mathfrak{H}\sigma$, and to each pair $\sigma^i \succ \sigma^j$ a homomorphism $\mathfrak{H}^{ji}:\mathfrak{H}\sigma^i\to\mathfrak{H}\sigma^j$ satisfying the associative law $\mathfrak{H}^{kj}\mathfrak{H}^{ji}=\mathfrak{H}^{ki}$. We define the chain complex to be the direct sum $C(K;\mathfrak{H})=\sum_{\sigma\in K}\mathfrak{H}\sigma$. A p-chain may be written as a finite sum $\Sigma h^i\sigma_p^i$, where $h^i\in\mathfrak{H}\sigma_p^i$. The boundary is given by extending linearly the formula

$$\partial(h^i\sigma_p^i)=\sum_j \eta^{ij}\mathfrak{H}^{ji}(h^i)\,\sigma_{p-1}^j,$$

where η^{ij} is the incidence number between σ_p^i and σ_{p-1}^j in K. We verify that $\partial^2=0$ and define the homology group in the usual way.

Dually we can form the *cohomology ring* $H^*(K;\mathfrak{H})$ *with coefficients in the contravariant stack* $\mathfrak{H}:K\to\mathfrak{R}$ *of rings on* K, provided K is multiplicative. A cochain may be thought of as a function on the cells of K, the image of σ lying in $\mathfrak{H}\sigma$. The multiplication between cochains is given by the same formula $a\cup b=\mu(a\otimes b)m$ as in § 4, provided we interpret μ correctly, as follows. If $\sigma^i\in K$, then $m\sigma^i$ is linearly dependent on cells of type $\sigma^j\otimes\sigma^k$, where σ^j,σ^k are faces of σ^i. Therefore $(a\otimes b)m\sigma^i$ is linearly dependent on terms of type $h^j\otimes h^k$, where $h^j\in\mathfrak{H}\sigma^j$, $h^k\in\mathfrak{H}\sigma^k$. Define $\mu(h^j\otimes h^k)$ by the product $\mathfrak{H}^{ij}(h^j).\mathfrak{H}^{ik}(h^k)$ in the ring $\mathfrak{H}\sigma^i$, and extend μ linearly. Therefore $(a\cup b)\sigma^i\in\mathfrak{H}\sigma^i$, and $a\cup b$ is a cochain, which is what we wanted to prove.

Facet homology

Suppose $\mathfrak{F}$ is a facing relation between K and L. The right facets form a covariant stack $\mathfrak{F}.:K\to\mathfrak{C}$ on K. The homomorphisms of the stack are given by inclusions between facets. If we compose this functor with the homology functor

$$K\xrightarrow{\mathfrak{F}.}\mathfrak{C}\xrightarrow{H}\mathfrak{G}_*$$

we obtain the graded covariant stack on K of *right facet homology*. We denote this stack by $H_*(\mathfrak{F})=\sum_q H_q(\mathfrak{F})$. Dually the graded contravariant stack $H^*(\mathfrak{F};R)$ on K of *right facet cohomology* is given by the composition of functors

$$K\xrightarrow{\mathfrak{F}.}\mathfrak{C}\xrightarrow{\#R}\mathfrak{C}_R\xrightarrow{H}\mathfrak{R}_R^*.$$

Remark. Similarly there are left facet stacks on L. We have not bothered to include in the notation the fact that $H_*(\mathfrak{F})$ is right rather than left facet homology, partly because its use as a stack on K is sufficient implication, and partly because in this paper all the facing relations are left acyclic. Therefore the left facet homology is none other than ordinary integer coefficients for L (which observation is essentially the proof of Theorem 1).

Notation for spectral sequences

We now define the category $\mathfrak{E}$ of *canonical homology spectral sequences*, or, more briefly, *spectral sequences*, and structure-preserving homomorphisms. An *object* of $\mathfrak{E}$ is a spectral sequence E, which consists of a sequence $\{E^r, d^r\}$, $2 \leqslant r \leqslant \infty$, of bigraded differential groups, together with a filtered graded group H, with the following properties:

(i) For each r, $2 < r \leqslant \infty$, $E^r = \sum_{p,q} E^r_{p,q}$, and $E^r_{p,q} = 0$ unless $p, q \geqslant 0$. We call p the filtering degree, q the complementary degree, and $n = p+q$ the total degree.

(ii) For each r, $2 \leqslant r < \infty$, d^r is a differential on E^r of degree $(-r, r-1)$ in p, q, and $E^{r+1} = H(E^r, d^r)$. Formally $d^\infty = 0$.

(iii) The sequence converges, in the sense that $E^r_{p,q} = E^\infty_{p,q}$ for $r > \max(p, q+1)$.

(iv) H is filtered by p, and graded by n, and the associated bigraded group $\mathrm{Gr}(H) = E^\infty$.

As notation, we say that the sequence runs $E^2_{p,q} \underset{p}{\Longrightarrow} H_n$.

A *map* of $\mathfrak{E}$ from E to E' is a collection of structure-preserving homomorphisms $f^r: E^r \to E'^r$ and $f: H \to H'$, that commute with the differentials d^r, and are compatible with the isomorphisms $E^{r+1} = H(E^r, d^r)$ and $\mathrm{Gr}(H) = E^\infty$.

Dually we define the category $\mathfrak{E}^*$ of *canonical cohomology spectral sequences of rings*, or, more briefly, *spectral rings*, and structure-preserving homomorphisms. An *object* of $\mathfrak{E}^*$ is a spectral ring, E^*, which consists of a sequence $\{E_r, d_r\}$, $2 \leqslant r \leqslant \infty$, of bigraded associative skew-commutative rings with derivation, together with a filtered graded associative skew-commutative ring H^*, with the following properties:

(i) For each r, $2 \leqslant r \leqslant \infty$, $E_r = \sum_{p,q} E_r^{p,q}$, and $E_r^{p,q} = 0$ unless $p, q \geqslant 0$. Again p is the filtering, q the complementary, and $n = p+q$ the total degree.

(ii) For each r, $2 \leqslant r < \infty$, d_r is a derivation on E_r of degree $(r, -r+1)$ in p, q, and $E_{r+1} = H(E_r, d_r)$. Formally $d_\infty = 0$.

(iii) The sequence converges for the same values as before; namely $E_r^{p,q} = E_\infty^{p,q}$ for $r > \max(p, q+1)$.

(iv) H^* is filtered by p, and graded by n, and the associated bigraded ring $\mathrm{Gr}(H^*) = E_\infty$. We say that the spectral ring runs

$$E_2^{p,q} \overset{p}{\Longrightarrow} H^n.$$

As in the dual case, a *map* of $\mathfrak{E}^*$ from E^* to E'^* is a collection of structure-preserving homomorphisms, commuting with the differentials, and compatible with the isomorphisms.

THEOREM 3. *A left acyclic facing relation* $\mathfrak{F}$ *between* K *and* L *gives rise to a spectral sequence* E *running* $H_p(K; H_q(\mathfrak{F})) \underset{p}{\Longrightarrow} H_n(L)$. *The augmentation of* L *induces a homomorphism* $E^2_{p,0} \to H_p(K)$, *which is an isomorphism if the right facets are connected, and which when composed with the edge homomorphism* $H_p(L) \to E^2_{p,0}$ *gives* Υ. *If* $\mathfrak{F}$ *is acyclic the spectral sequence collapses to the isomorphism* $\Upsilon : H_*(L) \cong E^\infty = E^2 \cong H_*(K)$. *If* $\mathfrak{F}$ *is functorial on* $\mathfrak{X}$ *then* $E : \mathfrak{X} \to \mathfrak{E}$ *is a covariant functor.*

Proof. Filter with respect to p the double complex D generated by $\mathfrak{F}$, and form the spectral sequence E running $H_p(H_q(D)) \underset{p}{\Longrightarrow} H_n(D)$. The term $E^0 = D = C(K; \mathfrak{F})$. The differential d^0 operates only on the second factor, so that $E^1 \cong C(K; H_*(\mathfrak{F}))$. The differential d^1 may be identified with the boundary ∂ of this chain complex, because d^1 is induced by $\partial_K \otimes 1$, and in the formula for ∂ the incidence numbers arise from ∂_K, while the stack homomorphisms $H_*(\mathfrak{F})^{ji}$ are induced by the inclusion $\mathfrak{F}\sigma^i \subset \mathfrak{F}\sigma^j$, or in other words by the identity $1 : L \to L$. Therefore $E^2_{p,q} \cong H_p(K; H_q(\mathfrak{F}))$ as desired. The isomorphism $H_n(D) \xrightarrow{\cong} H_n(L)$ is given by Theorem 1.

The augmentation of L induces the augmentation $\mathfrak{F}\sigma \to P$ of right facets, and hence $H_0(\mathfrak{F}\sigma) \to Z$; therefore it induces a stack homomorphism $H_0(\mathfrak{F}) \to Z$, onto the simple stack of integers, and a homology homomorphism

$$E^2_{p,q} = H_p(K; H_q(\mathfrak{F})) \to H_n(K).$$

If the right facets are connected the required isomorphism may be traced back to the isomorphism $H_0(\mathfrak{F}\sigma) \xrightarrow{\cong} Z$.

To identify the homomorphism $\Upsilon = \epsilon_L \epsilon_K^{-1}$ with the spectral sequence structure, we observe that in the diagram

$$H_p(L) \underset{\epsilon_K}{\xleftarrow{\cong}} H_p(D) \underset{e}{\longrightarrow} E^2_{p,0} \underset{\epsilon_L}{\longrightarrow} H_p(K)$$

the edge homomorphism e is induced by inclusion, so that the composition $\epsilon_L e = \epsilon_L$ in the sense that it is also a homomorphism induced by the augmentation of L.

If $\mathfrak{F}$ is acyclic the spectral sequence collapses as in Theorem 1. If $\mathfrak{F}$ is functorial, then the functorial quality of E follows from that of D and Theorem 2. The proof of Theorem 3 is complete.

DUAL THEOREM 3. *R is the coefficient ring. A left acyclic facing relation* $\mathfrak{F}$ *between multiplicative complexes* K *and* L *gives rise to a spectral ring* E^* *running* $H^p(K; H^q(\mathfrak{F}; R)) \xrightarrow{p} H^n(L; R)$. *The augmentation of* L *induces a homomorphism* $H^p(K; R) \to E_2^{p,0}$, *which is an isomorphism if the right facets are connected, and which when composed with the edge homomorphism* $E_2^{p,0} \to H^p(L; R)$ *gives* Υ^*. *If* $\mathfrak{F}$ *is acyclic the spectral ring collapses to the isomorphism* $\Upsilon^* : H^*(K; R) \cong E_2 = E_\infty \cong H^*(L; R)$. *If* $\mathfrak{F}$ *is functorial on* $\mathfrak{X}$ *then* $E^* : \mathfrak{X} \times \mathfrak{R} \to \mathfrak{E}^*$ *is a functor contravariant on* $\mathfrak{X}$ *and covariant on* $\mathfrak{R}$.

Proof. Let E^* be the spectral ring arising from the double cochain complex $D \pitchfork R$, filtering with respect to p. From the isomorphism $(K \otimes L) \pitchfork R \cong K \pitchfork (L \pitchfork R)$ we can identify $E_0 = D \pitchfork R = \prod_{\sigma \in K} \mathfrak{F}\sigma \pitchfork R$; this is the ring of cochains on K with coefficients in the stack $\{\mathfrak{F}\sigma \pitchfork R\}$. Since Π is an exact functor, $E_1 = \prod_{\sigma \in K} H^*(\mathfrak{F}\sigma; R)$, and so the E_2 term is the cohomology ring of K with coefficients in the right facet cohomology stack. The rest of the proof echoes that of Theorem 3.

6. The Čech singular spectral ring of a space

In this section we apply the analysis of the preceding section to the Čech singular facing relation of Example 5. Since limits are involved we concentrate on cohomology, the direct limit of a system of spectral rings being a spectral ring because the direct limit functor is exact. Therefore we obtain for an arbitrary pair of spaces a spectral ring. The E_2 term can be identified with the cohomology of the pair with coefficients in a presheaf. The presheaf measures the local singularities of the space by means of local singular cohomology, and reduces to the simple sheaf of rings R if the pair is locally acyclic. In Lemma 6 we improve upon the last remark and give a more delicate condition than local acyclicity which is necessary and sufficient for the sheaf to be simple. When the sheaf is simple the spectral ring collapses to an isomorphism between Čech and singular cohomology; in particular Cartan showed in (**1**), Exposé 20, that this occurred when the space is HLC.

There is a dual theory, which we do not give, of a semi-spectral sequence relating Čech and singular homology. To define a semi-spectral sequence, replace the isomorphisms $H(E^r, d^r) \cong E^{r+1}$ and $\mathrm{Gr}\, H \cong E^\infty$ in the definition of a spectral sequence by homomorphisms $H(E^r, d^r) \to E^{r+1}$ and a monomorphism $\mathrm{Gr}\, H \to E^\infty$. The inverse limit of a system of spectral sequences is semi-spectral, and not in general spectral, because the inverse limit functor is only left exact. If we use a field of coefficients, then the limit does retain its spectrality. The E^2 term is less familiar than its analogue above, since a dual theory of homology sheaves is not well developed.

Sheaves

Suppose $(X, A) \in \mathfrak{A}^+$; that is to say X is a topological space and A a subspace. We shall consider sheaves on X of R-modules and graded R-rings. The *constant presheaf* $R(X, A)$ is given by assigning

(i) to each open set U, $\subset X$, either 0 or R according to whether U meets A or not; and

(ii) if $U \subset V$ either the identity homomorphism on R if U does not meet A, or the zero homomorphism otherwise.

We use the same symbol $R(X, A)$ to denote the resulting *constant sheaf*, and we shall abbreviate both to R when there is no fear of confusion. The stalks of $R(X, A)$ above points of the closure $\bar{A}$ of A are zero, and above other points are isomorphic to R.

The *local singular cohomology presheaf* $\mathfrak{S}_R(X, A) = \mathfrak{S}_R = \sum_p \mathfrak{S}_R^p$ is given by assigning

(i) to each open set $U, \subset X$, the singular cohomology ring

$$^sH^*(U, U \cap A; R);$$

and

(ii) if $U \subset V$, the restriction homomorphism

$$^sH^*(U, U \cap A; R) \to {}^sH^*(V, V \cap A; R).$$

We denote the resulting graded sheaf by $\bar{\mathfrak{S}}_R = \sum_p \bar{\mathfrak{S}}_R^p$. The stalks above interior points of A are zero, and the stalks elsewhere depend upon the local structure of X, and the way in which A is embedded in X; for example think of the curve $\sin 1/x$, $x > 0$, in the Euclidean plane (we discuss this example in a Remark after Lemma 5). The augmentation of the singular complex $S(U, U \cap A)$ induces a presheaf monomorphism $R \to \mathfrak{S}_R$, and a sheaf monomorphism $R \to \bar{\mathfrak{S}}_R$. We say that $\bar{\mathfrak{S}}_R$ is *constant* if $R \xrightarrow{\cong} \bar{\mathfrak{S}}_R$ is an isomorphism. Similarly we may confine our attention to dimension zero, and say that $\bar{\mathfrak{S}}_R^0$ is constant if $R \xrightarrow{\cong} \bar{\mathfrak{S}}_R^0$. Certainly if (X, A) is a polyhedral pair the local acyclicity ensures that $\bar{\mathfrak{S}}_R$ is constant.

Local connectedness

A space X is *locally path-connected* (LPC) if to each $x \in X$ and neighbourhood U of x, there exists a smaller neighbourhood $V = V(x, U)$, $x \in V \subset U$, such that any two points in V may be connected by a path in U. The pair (X, A) is defined to be LPC if V may be chosen such that any two points in $V \cap A$ may also be connected in $U \cap A$. This condition implies not only that X and A are LPC but that the embedding $A \subset X$ is not too bad.

A space X is *homologically locally connected* (HLC) if it is LPC, and to each $x \in X$ and neighbourhood U of x, there exists a smaller neighbourhood $V = V(x, U)$ such that any singular p-cycle in V, $p > 0$, bounds in U. The pair (X, A) is defined to be HLC if it is LPC, and if V may be chosen such that any p-cycle in $V \cap A$, $p > 0$, also bounds in $U \cap A$.

A space X is *cohomologically locally connected* (CLC) *over* R if it is LPC, and to each $x \in X$, neighbourhood U of x, and singular cocycle $f \in S(U) \pitchfork R$ of positive dimension, there exists a smaller neighbourhood $V = V(x, U, f)$ such that $f | V$ cobounds in V, where $f | V$ denotes the restriction of f to a cocycle in $S(V) \pitchfork R$. The pair (X, A) is defined to be CLC over R if it is LPC, and X is CLC over R, and if to each cocycle $g \in S(U \cap A) \pitchfork R$ of positive dimension, there exists a smaller neighbourhood $V = V(x, V, g)$ such that $g \mid V \cap A$ cobounds in $V \cap A$.

The last definition is a new one adapted to our particular needs. The next lemma relates it to the concept of HLC. However, it is not known whether CLC over Z (the integers) implies, or is a weaker condition than, HLC. Local acyclicity, on the other hand, is a stronger condition and clearly implies HLC. Local acyclicity plays much the same role as HLC together with paracompactness, and is therefore useful for spaces which are not too bad locally and yet do not happen to be paracompact.

The space X is *paracompact* if every covering has a locally finite refinement. The pair (X, A) is paracompact if, further, to every covering α of A there is a locally finite covering ξ of X such that $\xi \mid A$ refines α.

The pair (X, A) is *normal* if X is normal, and if every pair of subsets of A closed in A can be enclosed in disjoint open sets of X.

LEMMA 4. *A space X is* HLC *if and only if it is* CLC *over R for all R.*

Proof. Suppose X is HLC. Given x, U, f let $V = V(x, U)$ be constructed by the HLC property. The inclusion $j : V \to U$ induces

$$j_* = 0 : {}^sH_p(V) \to {}^sH_p(U), \quad p > 0,$$

between the singular homology groups, and so also

$$j^* = 0 : {}^sH^p(U; R) \to {}^sH^p(V; R), \quad p > 0.$$

Therefore X is CLC over R for all R.

Conversely, choose R sufficiently large so that for each x, U, p $(p > 0)$ there exists a (group) homomorphism $f = f(x, U, p) : S_p(U) \to R$ with kernel the boundaries of $S_p(U)$. In particular we are given that X is CLC over this R, and, f being a singular cocycle, we may choose $V = V(x, U, f)$. Therefore $f \mid V = \delta g = g\partial$ for some cochain $g : S_{p-1}(V) \to R$. If c is a singular p-cycle in V, then $fc = (f \mid V)c = g\partial c = 0$, so that, by our construction of f, c bounds in U. The lemma is proved.

We now relate these concepts to the local singular cohomology sheaf $\bar{\mathfrak{S}}_R$.

LEMMA 5. (i) $\bar{\mathfrak{S}}^0_R$ *is constant on X if and only if X is* LPC.
(ii) $\bar{\mathfrak{S}}^0_R$ *is constant on X, A, and (X, A) provided (X, A) is* LPC.

Proof. (i) Suppose X is LPC, and let f be a 0-cocycle on a neighbourhood U of x. Let a, b be two points in $V = V(x, U)$; since they are connected by a path in U, and since f is a cocycle, $fa = fb$. Hence $f \mid V$ is a constant, so that f gives rise to an element in the stalk above x which is in the image of the monomorphism $R \to \bar{\mathfrak{S}}^0_R$. Hence $\bar{\mathfrak{S}}^0_R$ is constant on X.

Conversely suppose X is not LPC at x: there exists a neighbourhood U of x such that x is not interior to the path component W of x in U. Choose

$r \neq 0$ in R, and let f be the 0-cocycle on U such that $fW = 0, f(U-W) = r$. The restriction of f to any smaller neighbourhood V does not vanish or cobound. Consequently the cohomology class of f gives rise to a non-zero element in the stalk above x which is not in the image of $R \to \bar{\mathfrak{S}}^0_R$. Therefore $\bar{\mathfrak{S}}^0_R$ is not constant on X.

(ii) In the relative case suppose (X, A) is LPC. Then $\bar{\mathfrak{S}}^0_R$ is constant on X and A since each is LPC. Let U be a neighbourhood of $x \in X$, and consider the exact sequence

$$0 \to {}^sH^0(U, U \cap A\,;\, R) \to {}^sH^0(U\,;\, R) \to {}^sH^0(U \cap A\,;\, R).$$

By the same argument as in the absolute case the sequence reduces in the limit to

$$0 \to 0 \to R \xrightarrow{\cong} R \quad \text{if} \quad x \in \bar{A},$$

and

$$0 \to R \xrightarrow{\cong} R \to 0 \quad \text{if} \quad x \notin \bar{A}.$$

The second term shows that $\bar{\mathfrak{S}}^0_R$ is also constant on (X, A). Lemma 5 is proved.

Remark. We cannot prove the converse to Lemma 5(ii) because it can happen that $\bar{\mathfrak{S}}^0_R$ is constant on X, A, and (X, A), but (X, A) is not LPC. For example consider the $\sin 1/x$ curve embedded in Euclidean 3-space. In this case, however, the singularities are caught by $\bar{\mathfrak{S}}^1_R(X, A) \neq 0$, as is confirmed by the next lemma.

LEMMA 6. (i) *$\mathfrak{S}_R$ is constant on X if and only if X is* CLC *over R.*

(ii) *$\bar{\mathfrak{S}}_R$ is constant on X, A, and (X, A) if and only if (X, A) is* CLC *over R.*

Proof. (i) The proof is similar to that of Lemma 5. Suppose X is CLC over R. This condition is designed to ensure that the stalks $\bar{\mathfrak{S}}^p_R(X)$ vanish for $p > 0$. Lemma 4 then shows that $\bar{\mathfrak{S}}_R$ is constant on X.

Conversely suppose X is not CLC over R. Then either X is not LPC, whence $\bar{\mathfrak{S}}^0_R$ is not constant, or else there exist an x, U, and p-cocycle f, $p > 0$, such that $f \mid V$ never cobounds for any smaller neighbourhood V. Thus f gives rise to a non-zero element of $\bar{\mathfrak{S}}^p_R$ in the stalk above x implying $\bar{\mathfrak{S}}_R$ not constant.

(ii) In the relative case, suppose (X, A) is CLC over R. Then $\bar{\mathfrak{S}}_R$ is constant on X and A. Let U be a neighbourhood of $x \in X$, and consider the exact triple

$$\begin{array}{ccc} {}^sH^*(U\,;\, R) & \longrightarrow & {}^sH^*(U \cap A\,;\, R) \\ & \nwarrow \qquad \swarrow & \\ & {}^sH^*(U, U \cap A\,;\, R) & \end{array}$$

In the limit this reduces to

$$\begin{array}{ccc} R & \xrightarrow{\cong} & R \\ \nwarrow & & \swarrow \\ & 0 & \end{array} \qquad \text{or} \qquad \begin{array}{ccc} R & \longrightarrow & 0 \\ \cong\nwarrow & & \swarrow \\ & R & \end{array},$$

according to whether x lies in $\bar{A}$ or not. The bottom term shows that $\mathfrak{S}_R$ is constant on (X, A).

Conversely suppose that $\mathfrak{S}_R$ is constant on X and (X, A). Then X is CLC over R by the absolute case. We may reverse the argument about the last three diagrams and deduce from the right-hand term that for $x \in \bar{A}$ the stalk $R \cong \lim_{U} {}^sH^*(U \cap A; R)$. This is an expression of the second condition for the pair (X, A) to be both LPC and CLC over R. The proof of Lemma 6 is complete.

We are now in a position to state and prove the theorem about the Čech-singular spectral ring, which is the purpose of this section.

THEOREM 4. *There is a functor* $E^* : \mathfrak{A}^+ \times \mathfrak{R} \to \mathfrak{E}^*$, *contravariant on the category* $\mathfrak{A}^+$ *of pairs of topological spaces and continuous maps, covariant on the category* $\mathfrak{R}$ *of rings, with values in the category* $\mathfrak{E}^*$ *of spectral rings, and with the following properties:*

If $(X, A) \in \mathfrak{A}^+$ *and* $R \in \mathfrak{R}$ *then the spectral ring* $E^*(X, A; R)$ *runs*

$$H^p(X, A; \mathfrak{S}_R^q) \overset{p}{\Longrightarrow} {}^sH^n(X, A; R),$$

where

(1) *The* E_2 *term is the cohomology ring of the pair* (X, A) *with coefficients in* $\mathfrak{S}_R$, *the local singular cohomology presheaf over* R. *If* (X, A) *is normal and paracompact the sheaf* $\mathfrak{S}_R$ *can be used instead.*

(2) *The* E_∞ *term is related to the singular cohomology ring of* (X, A) *over* R, *suitably filtered.*

(3) *There is a functorial homomorphism* $\check{H}^p(X, A; R) \to E_2^{p,0}$ *from the Čech cohomology ring of* (X, A) *over* R, *which when composed with the edge homomorphism* $E_2^{p,0} \to {}^sH^p(X, A; R)$ *gives the natural transformation* Υ^* *from Čech to singular cohomology.*

(4) *If* (X, A) *is locally acyclic, or normal paracompact* CLC *over* R, *in particular if* (X, A) *is a polyhedral pair, then the spectral ring collapses to the isomorphism* $\Upsilon^* : \check{H}^*(X, A; R) \cong E_2 = E_\infty \cong {}^sH^*(X, A; R)$.

Proof. Relativize by Lemma 3 the functorial left acyclic Čech singular facing relation of Example 5, and apply the dual Theorem 3. We have a functor $E^* : \mathfrak{A}^+_{\text{cov}} \times \mathfrak{R} \to \mathfrak{E}^*$, but before we can take limits over α we must verify:

LEMMA 7. *E^* is independent of approximation.*

Proof. An object (X,A,α) of $\mathfrak{A}^+_{\text{cov}}$ is comprised of a pair (X,A) and a covering α of X. A map $(f,f_\alpha):(X,A,\alpha)\to(X',A',\alpha')$ of $\mathfrak{A}^+_{\text{cov}}$ consists of a continuous map $f:X\to X'$ mapping A to A' and a simplicial approximation $f_\alpha: N(\alpha)\to N(a')$. If (f,f^1_α) and (f,f^2_α) are two such with the same underlying continuous map f, then f^1_α, f^2_α are both carried by the acyclic Čech carrier $\Phi: N(\alpha)\to N(\alpha')$ given by $\Phi\sigma=\{\sigma';\ \sup\sigma'\subset f(\sup\sigma)\}$. Therefore f^1_α, f^2_α differ by a chain homotopy h also carried by Φ. Meanwhile the maps between the singular complexes are the same $f: S(X,\alpha)\to S(X',\alpha')$. If $\sigma\otimes\tau\in\mathfrak{F}(X,\alpha)$ and $\sigma'\in\Phi\sigma$, then

$$\sup\sigma'\supset f(\sup\sigma)\supset f(\operatorname{im}\tau)=\operatorname{im} f\tau,$$

and so $\sigma'\otimes f\tau\in\mathfrak{F}(X',\alpha')$. Therefore $h\otimes f$ carries $D(X,\alpha)$ into $D(X',\alpha')$. Moreover it is easy to verify (and it will be shown in (**12**), Lemma 3) that $h\otimes f$ is a chain homotopy between the chain maps

$$f^1_\alpha\otimes f\simeq f^2_\alpha\otimes f: D(X,\alpha)\to D(X',\alpha').$$

If $\sigma\in N(\alpha|A)$ and $\sigma'\in\Phi\sigma$, then

$$A'\cap\sup\sigma'\supset fA\cap f(\sup\sigma)\supset f(A\cap\sup\sigma)\neq\emptyset,$$

and so $\sigma'\in N(\sigma'|A')$. Therefore h carries $N(\alpha|A)$ into $N(\alpha'|A')$. Consequently $h\otimes f$ carries $D(A,\alpha|A)$ into $D(A',\alpha'|A')$ and induces a homomorphism, h say, of the quotients $D(X,A,\alpha)\to D(X',A',\alpha')$. Also induced upon the quotients is the formula expressing the fact that h is a homotopy operator between the induced chain maps

$$D(f,f^1_\alpha)\simeq D(f,f^2_\alpha): D(X,A,\alpha)\to D(X',A',\alpha').$$

Therefore by ((**2**) 321), the induced homomorphisms between the spectral rings are equal. Lemma 7 is proved.

Returning to the proof of Theorem 4, we can now take direct limits by Lemmas 1 and 7, and obtain the required functor $E^*:\mathfrak{A}^+\times\mathfrak{R}\to\mathfrak{E}^*$.

(1) To identify the E_2 term of the spectral ring $E^*(X,A,\alpha;\ R)$ we first observe that it is the cohomology ring of the relative nerve with coefficients in the right facet cohomology stack, and secondly notice that this is none other than the cohomology of the pair (X,A) with respect to α, with coefficients in the local singular cohomology presheaf $\mathfrak{S}_R$. If β refines α the homomorphism $E^*_2(X,A,\alpha;\ R)\to E^*_2(X,A,\beta;\ R)$ is the unique homomorphism induced by restriction, so that taking direct limits merely gives us the cohomology ring of (X,A) with coefficients in the presheaf $\mathfrak{S}_R$. If, further, (X,A) is normal paracompact then by ((**5**) 68) the cohomology in any presheaf of the zero sheaf is zero, so, by a standard argument, the

homomorphism $\mathfrak{S}_R \to \bar{\mathfrak{S}}_R$ from presheaf to sheaf induces an isomorphism of cohomology.

(2) By the dual Theorem 3, the E_∞ term of $E^*(X, A, \alpha;\, R)$ is related to ${}^sH^*(X, A, \alpha;\, R)$, the α-small relative singular cohomology ring of (X, A). If β refines α, restriction induces an isomorphism

$$ {}^sH^*(X, A, \alpha;\, R) \xrightarrow{\cong} {}^sH^*(X, A, \beta;\, R), $$

and so we can identify the direct limit with ${}^sH^*(X, A;\, R)$.

(3) The presheaf monomorphism $R \to \mathfrak{S}^0_R$ induces a homomorphism $\check{H}(X, A, \alpha;\, R) \to E_2^{p,0}(X, A, \alpha;\, R)$, which in the limit gives the required homomorphism from Čech cohomology. It is functorial because $R \to \mathfrak{S}^0_R$ is induced by augmentation. The identification of Υ^* comes from Corollary 2.7 and the dual Theorem 3.

(4) If (X, A) is locally acyclic, there is a cofinal set of coverings on which $\mathfrak{F}$ is acyclic and the spectral ring collapses. The limit therefore also collapses to the isomorphism Υ^*. If (X, A) is normal paracompact we can calculate E_2 by using coefficients in the sheaf $\bar{\mathfrak{S}}_R$, and if (X, A) is also CLC over R the sheaf isomorphism of Lemma 6 causes an isomorphism $H^*(X, A;\, R) \xrightarrow{\cong} E_2$. The E_2 term is therefore concentrated on the axis $q = 0$, and so the spectral ring collapses. Theorem 4 is proved.

Remark 1. D. B. A. Epstein has shown that the functor E^* satisfies the homotopy axiom, and therefore the spectral ring is an invariant of homotopy type.

Remark 2. The functor E^* could just as well have been defined using singular cubes instead of singular simplexes. A chain equivalence $Q^N(X) \to S(X)$ can be used to define a natural equivalence between the two resulting functors E^*.

Remark 3. If when applying $\pitchfork R$ to D we insert a condition about compact supports, we obtain a functor E^* relating Čech and singular cohomology with compact supports.

EXAMPLE. *The Hawaiian earring.*

Let us examine an example for which the spectral ring E^* is non-trivial. We want a space whose Čech and singular cohomology groups differ. One of the simplest examples is the Hawaiian earring, which is the familiar countable bunch of circles in the Euclidean plane, defined by $X = \bigcup_{n=1}^{\infty} \Gamma_n$, where Γ_n is the circle through the origin with centre $\left(\frac{1}{n}, 0\right)$. X is not a polyhedron because it has a singularity at the origin, where it is not HLC. Since X is 1-dimensional, in all dimensions greater than 1 the Čech and singular cohomology groups vanish (see (**3**)), but in dimension 1 they are non-zero and different, due to the singularity. As we shall see, both groups are embedded in the spectral ring.

Since X is not HLC, we can choose, by Lemma 4, a ring R over which X is not CLC (for instance choose R to contain a subgroup isomorphic to ${}^sH_1(X)$; or perhaps the integers might suffice). Let X_n denote the space X with the circle Γ_n filled in. Since X_n is a polyhedron, $\check{H}^1(X_n; R) = {}^sH^1(X_n; R) =$ a free R-module on $n-1$ generators. Taking limits $X = \cap X_n$, we see that the Čech group $\check{H}^1 = \check{H}^1(X; R)$ is a free R-module with a countable number of generators. Meanwhile the inclusion $X \subset X_n$ embeds ${}^sH^1(X_n; R)$ into the singular group ${}^sH^1 = {}^sH^1(X; R)$. Taking limits, we have that $\Upsilon^* : \check{H}^1 \to {}^sH^1$ is a monomorphism. Complete the exact sequence

$$0 \to \check{H}^1 \xrightarrow[\Upsilon^*]{} {}^sH^1 \to Q \to 0.$$

By our choice of R the group Q does not vanish (for instance if $j : {}^sH_1(X) \subset R$ then $j \in {}^sH_1(X) \pitchfork R = {}^sH^1$ gives rise to a non-zero element of Q).

Since X is normal paracompact, we may use the sheaf $\mathfrak{S}_R$ to calculate the E_2 term of the spectral ring E^*. $\mathfrak{S}_R^0 = R$ because X is LPC. $\mathfrak{S}_R^q = 0$, $q \geqslant 2$, because X is 1-dimensional (see (**3**)). $\mathfrak{S}_R^1$ vanishes everywhere except at the singularity, where the stalk is isomorphic to Q. Hence the only non-zero terms of E_2 are:

$$\left.\begin{aligned} E_2^{0,0} &= R \\ E_2^{1,0} &= \check{H}^1 \end{aligned}\right\} \text{ the Čech cohomology groups of } X,$$
$$E_2^{0,1} = Q.$$

Due to the low dimensionality of X, the spectral ring converges $E_2 = E_\infty$ (but it does not collapse onto the axis $q = 0$). The exact sequence

$$0 \to E_\infty^{1,0} \to {}^sH^1 \to E_\infty^{0,1} \to 0,$$

relating E_∞ to the singular cohomology group of X, is identical with the exact sequence above.

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