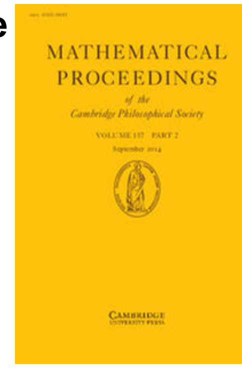

On Contractible Open Manifolds

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By an open manifold we mean a non-compact space, that is triangulable by a countable complex which is a combinatorial manifold without boundary (see next section). The obvious example is Euclidean n-space, which we denote by E^n. We prove:

THEOREM. *If M^n is a contractible open manifold, then $M^n \times E^2$ is piecewise linearly* homeomorphic to E^{n+2}.*

QUESTION. *Can this be improved to $M^n \times E^1 = E^{n+1}$?*

In Theorem 2 of (3), the first author gave an affirmative answer to this question when $n = 3$, provided that each compact subset of M can be embedded in E^3, or provided that the Poincaré Conjecture holds for dimension 3. The above Theorem is an improvement of Theorem 4 of (3).

In higher dimensions, $n \geqslant 5$, if M happens to be the interior of a bounded compact manifold N, then Curtis (1) has noted, using the solution to the higher dimensional Poincaré Conjecture, that

$$N^n \times I^1 = I^{n+1},$$

where I^n is the n-cube. Hence, taking interiors, we find that $M^n \times E^1 = E^{n+1}$. Examples of such manifolds (different from E^n) have been given by Newman (6), Poenaru (7), Mazur (2) and Curtis (1). However this argument is not applicable to a contractible open manifold that is not homeomorphic to the interior of a bounded manifold, such as the 3-manifolds of Whitehead (9) and McMillan (4). In the case $n = 3$, Poenaru (8) has shown that a bounded, compact contractible 3-manifold yields I^5 when multiplied by I^2.

1. *Notation.* By a *combinatorial n-manifold* M^n we mean, as usual, a simplicial complex whose closed vertex stars are combinatorial n-balls. We consider here only manifolds which are either compact or without boundary. We say M is *finite* if the complex is finite. We use the same symbol M to denote the underlying topological manifold, and we denote the interior of M by M° and the boundary by $\dot{M}$.

A subspace X of M is called a combinatorial subspace if it underlies some subcomplex of some subdivision of M. Call $r = \dim M - \dim X$ the *codimension* of X, and write this as a left superscript $X = {}^rX$.

Call X *inessential* in M, written $X \simeq 0$ in M, if the inclusion map $X \to M$ is homotopic to a constant map. Call X *trivial* in M if X is contained in a combinatorial n-ball in M. Clearly if X is trivial in M then it is inessential, but not conversely, even though the codimension of X may be large.

* We are grateful to the Referee for pointing out the application of Newman's Theorem in Lemma 4, which enabled us to improve our result from *homeomorphism* to *piecewise linear homeomorphism*.

For example, think of a simple closed curve (of codimension 2) in a solid torus, that is inessential but links itself around the torus, and is therefore non-trivial. Similarly one can construct an n-sphere S^n (of codimension $n+1$) in $S^1 \times B^{2n}$ that is inessential and non-trivial, by linking two S^n's locally, and then connecting them by a pipe round the S^1. For a more detailed discussion, see Zeeman (12).

In a contractible open manifold, of course, every subspace is inessential. If the manifold is not a Euclidean space then it contains a finite combinatorial subspace of codimension at least one which is inessential but non-trivial (see Lemma 4 and § 3). In most examples there is such a subspace which is geometrically significant. For example, in the interiors of the bounded contractible 4-manifolds of Poenaru and Mazur, one can find such a subspace (of codimension 1) by isotopically shoving the boundary into the interior. In Whitehead's original example (9) of a contractible open 3-manifold, he constructed a simple closed curve (of codimension 2) with the above property. On the other hand we shall show in the Corollary to Lemma 3, that any compact combinatorial subspace of codimension 3 in a contractible open manifold must be trivial.

2. *Collapsing.* Suppose $X \subset Y$ are combinatorial subspaces of M. We say X *expands* to Y, written $X \nearrow Y$, or, equivalently, Y collapses to X, written $Y \searrow X$, if, in some subdivision of M, there are subcomplexes K, L, covering X, Y, such that K expands to L by a finite sequence of elementary expansions in the sense of Whitehead (10).

Lemma 1. *If $X \nearrow Y$ in M°, and if X is trivial in M°, then so is Y.*

Proof. By induction it suffices to examine an elementary expansion. Therefore suppose K, L triangulate X, Y, and that x is a vertex and Q^q a simplex, such that $K \cup xQ = L$, $K \cap xQ = x\dot{Q}$. Let y be the barycentre of Q. Suppose B is a given n-ball, $X \subset B \subset M^\circ$. Let B_1 be a regular neighbourhood of B in M°, also an n-ball by Whitehead ((10), Theorem 23). For some interior point z of the interval xy, we have $xz\dot{Q} \subset B_1$.

Choose two simplexes U^q, V^{n-q} in E^n meeting at their barycentres only, at u, say. Since $Q \in M^\circ$, its link is a combinatorial $(n-q-1)$-sphere, and so we can choose a piecewise linear homeomorphism

$$h\colon \operatorname{lk}(Q, M) \to \dot{V}$$

which throws x to a vertex $v \in V$. Define $h\colon Q \to U$ to be an isomorphism, so that $hy = u$. The join gives a piecewise linear homeomorphism

$$h\colon \operatorname{st}(Q, M) \to U\dot{V}.$$

Let W be the face of V opposite v, and w its barycentre. Then $u = hy$, $u' = hz$ are two points on the interval vw. Let f map vu', $u'w$ linearly onto vu, uw, respectively, so that f is a piecewise linear homeomorphism of the interval vw onto itself. Define $f \mid \dot{U}\dot{W} = 1$, and extend f linearly to the join $U\dot{V} = vw\dot{U}\dot{W}$. Define $g\colon M \to M$ to be the piecewise linear homeomorphism given by

$$g \mid \operatorname{st}(Q, M) = h^{-1}fh,$$
$$g \mid M - \operatorname{st}(Q, M) = 1.$$

Then $g \mid X = 1$ and gB_1 is a ball containing Y, and so Y is trivial.

LEMMA 2. *Let M be a finite combinatorial manifold. Given a combinatorial subspace ${}^{r+1}X \simeq 0$ in M°, then there exist combinatorial subspaces ${}^{r}Y$, ${}^{2r}Z$ in M° such that $X \subset Y \searrow Z$.*

The proof is given in Zeeman (11). The argument is long, but the geometrical idea is simple. Y is a cone on X mapped into general position, with singularities of codimension $2r$. We can collapse Y onto the $(2r-1)$-codimensional subcone that contains these singularities. The last step down to codimension $2r$ is achieved by piping the middles of the singularities over the edge of the cone.

LEMMA 3. *Suppose $\{M_i\}$, $i = 1, 2, \ldots$, is a sequence of finite combinatorial n-manifolds, such that each M_i is a combinatorial subspace of M°_{i+1}, and $M_i \simeq 0$ in M°_{i+1}. If ${}^{r+1}X \subset M_i$, $r \geqslant 2$, then X is trivial† in M_{i+n-r}.*

Proof. By induction downwards on r: If $r = n$, then X is empty, and so the lemma is trivially true for all i. Assume the lemma to be true for all X of codimension $> r+1$, $r \geqslant 2$, and for all i. Given ${}^{r+1}X \subset M_i$, then $X \simeq 0$ in M°_{i+1}. Therefore by Lemma 2, $X \subset Y \searrow {}^{2r}Z$ in M°_{i+1}. But $2r > r+1$. Therefore by induction Z is trivial in

$$M^\circ_{(i+1)+n-(2r-1)} \subset M^\circ_{i+n-r}.$$

Since $Z \nearrow Y$, by Lemma 1 Y is trivial in M°_{i+n-r}, and hence also X.

COROLLARY. *If M is a contractible open manifold, then any compact combinatorial subspace of codimension $\geqslant 3$ is trivial.*

Proof. Choose a sequence of finite combinatorial submanifolds M_i to satisfy the hypothesis of Lemma 3, and such that $M = \bigcup M_i$. (For choose a sequence of finite complexes K_i to exhaust M, $M = \bigcup K_i$, and then choose M_i inductively to be a regular neighbourhood of the union of K_i and a piecewise linear image of the cone on M_{i-1}.) If ${}^{r+1}X$ is a compact combinatorial subspace, $r \geqslant 2$, then $X \subset$ some M_i; therefore by the Lemma X is trivial in M°_{i+n-r}, and hence in M.

LEMMA 4. *If $M = \bigcup B_i$, the union of a sequence of combinatorial n-balls, such that each $B_i \subset B^\circ_{i+1}$, then M is piecewise linearly homeomorphic to E^n.*

Proof. We may write $E^n = \bigcup I^n_i$, the union of a sequence of n-cubes, such that each $I^n_i \subset (I^n_{i+1})^\circ$. Let $f_1: B_1 \to I_1$ be a piecewise linear homeomorphism. Suppose, inductively, that we have defined piecewise linear homeomorphisms $f_i: B_i \to I_i$ for each i, $i \leqslant j$, such that $f_i = f_j \mid B_i$ $(i \leqslant j)$. Then we may extend f_j to $f_{j+1}: B_{j+1} \to I_{j+1}$, a piecewise linear homeomorphism by Newman ((5), Theorem 3). The family $\{f_i\}$ defines a piecewise linear homeomorphism $M \to E^n$.

3. *Proof of the theorem.* We are given a contractible open n-manifold M^n. We have to show that $M^n \times E^2 = E^{n+2}$.

Choose the M_i as in the Corollary above. Let D_i be the disk in E^2 of radius i. Then $M_i \times D_i$ is another sequence of manifolds satisfying the hypothesis of Lemma 3. Since M_i is a bounded finite manifold, we can choose an $(n-1)$-dimensional spine T_i such that $M_i \searrow T_i$.

† In fact X is trivial in M_{i+j}, where j is the least integer such that $2^j(r-1) \geqslant n-2$.

Now T_i is of codimension 3 in $M_i \times D_i$, which is of dimension $n+2$. Therefore by Lemma 3, T_i is trivial in $(M_{i+n} \times D_{i+n})^\circ$. But $T_i \nearrow M_i \nearrow M_i \times D_i$. Therefore by Lemma 1, there exists an $(n+2)$-ball B_i, such that

$$M_i \times D_i \subset B_i \subset (M_{i+n} \times D_{i+n})^\circ.$$

Taking the union over all i,

$$M \times E^2 \subset \bigcup B_i \subset M \times E^2.$$

Hence $M \times E^2 = \bigcup_{i=1}^{\infty} B_i = \bigcup_{j=1}^{\infty} B_{jn}$, which is the union of a sequence of balls each in the interior of its successor, and which therefore $= E^{n+2}$ by Lemma 4.

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