C^0-Density of Stable Diffeomorphisms & Flows.

By E.C. Zeeman.

October 1973.

Mathematics Institute
University of Warwick
Coventry.

C^o-DENSITY OF STABLE DIFFEOMORPHISMS & FLOWS

By E.C. Zeeman

1. Introduction.

One of the major objectives of dynamical systems is to find a dense set of "nice" diffeomorphisms in the space $\mathcal{D}$ of all diffeomorphisms of a manifold M. Also to find a dense set in the space $\mathcal{H}$ of all vector fields, that integrate to "nice" flows. Here dense usually means with respect to the C^1-topology, which seems to be the natural topology to put on $\mathcal{D}$ and $\mathcal{H}$. "Nice" means conceptually-graspable, easily-describable, countably-classifiable, etc. Nice used to mean structually stable, until Smale showed the latter were not dense [6] . The search for a nice dense set is still open.

However recently Smale [5] and Shub [3] have proved a lesser, but very striking, result, that if the C^o-topology is used for $\mathcal{D}$ instead of the C^1, then indeed there is a very nice dense set. I would like to suggest calling this nice set <u>Smale diffeomorphisms</u>, because they are only slightly more complicated than Morse-Smale diffeomorphisms. In fact they satisfy Axiom A and the no-cycle condition, and have basic sets that are generalised horseshoes, which, for convenience, we call <u>shoes</u>. Shoes are finite sets or Cantor sets, and we show below that it suffices to use shoes that lie in 2-dimensional submanifolds of the n-manifold M.

The same result holds for flows. It seems that the flow version should be important for applied mathematics, because modelling within tolerance of experimental accuracy is in effect merely C^o-approximation. Therefore any ordinary differential equation can be approximated by a

Smale flow. An interesting property of Smale flows, from the point of view of applied mathematics, is the fact that the only attractors are fixed points and closed orbits. This would seem to justify, in retrospect, the enormous attention paid to oscillators in the literature on differential equations, compared with the scant attention paid to any more complicated kind of attractor.

What then are the disadvantages, if any, of this result ? At first the mixture between C^0 and C^1 seems unaesthetic, but if that is the way the mathematics runs then we must follow. Admittedly it seems a shame to mess up the purity of an Anosov diffeomorphism by approximating it with a Smale diffeomorphism, but then this should be viewed in the same spirit as approximating irrationals by rationals, or Shrödinger operators by operators with discrete spectrum. A more serious critism of the result is that it does not seem to extend easily to parametrised systems. The best kinds of density theorem lead to an open-dense set, which becomes the 0-stratum of a stratification, which in turn leads to a theory of bifurcations and catastrophes. The Smale diffeomorphisms and flows are paradoxically C^1-open and C^0-dense but unfortunately are not open-dense in either topology.

2. Smale diffeomorphisms.

Let M be a compact closed C^∞-manifold. Let $\mathcal{D}$ be the space of C^∞-diffeomorphisms of M, with the C^0-topology. Let $f \in \mathcal{D}$. A basic set S of f is called a <u>shoe</u> if the germ of f at S is diffeomorphic to one of a countable family of germs that we explicitly construct below, in sections 4 – 7. Define f to be a <u>Smale diffeomorphism</u> if:

Axiom 1. The non-wandering set is a finite union of shoes.

Axiom 2. The no-cycle condition. In other words the relation between shoes given by $S_1 < S_2$ if inset (S_1) meets outset (S_2) contains no cycles, and therefore generates a partial ordering.

Remark. We call a Smale diffeomorphism transversal if the insets and outsets of the shoes cut transversally. In fact transversality with Axiom 1 implies Axiom 2 (see [4]). Hyperbolic periodic orbits, including hyperbolic fixed points, are special examples of shoes. If a Smale diffeomorphism is transversal, and if all the shoes are periodic orbits, then it is Morse-Smale. Therefore Smale diffeomorphisms include, and are a strict generalisation of, Morse-Smale diffeomorphisms. Moreover they not only have the added advantage of C^o-density, but also retain the following "nice" properties.

Theorem A. Smale diffeomorphisms are C^o-dense in $\mathcal{D}$.

Theorem B. Transversal Smale diffeomorphisms are C^1-stable.

Theorem C. Smale diffeomorphisms satisfy the Morse-inequalities.

The proof of A is due to Smale [7] and Shub [3] , together with the addition described in Section 8 below; the proof of B is a corollary of Robbin's theorem [1] ; and the proof of C is due to Zeeman [8,9] following Smale [4]. Shub and Williams have also shown that transversal Smale diffeomorphisms are C^o-dense, but have not published the proof yet.

3. Smale flows.

Let $\mathcal{H}$ be the space of C^1-vector fields on M, with the C^o-topology. Let $X \in \mathcal{H}$, and let φ be the resulting flow. A basic set S of φ is called a solenoid of the germ of the phase portrait of φ at S is diffeomorphic to that of the suspension of a shoe. Define X to be a Smale system, and φ to be a Smale flow, if

Axiom 1. The non-wandering set of φ is a finite union of hyperbolic fixed points and solenoids.

Axiom 2. The no-cycle condition.

Then we have as above :

Theorem A'. Smale systems are C^o-dense in $\mathcal{H}$.

Theorem B'. Transversal Smale systems are structurally stable.

Theorem C'. Smale flows satisfy the Morse-inequalities.

The proof of A' is promised in [3] and sketched in Section 9 below. B' is due to Robinson [2] and C' in [8, 9].

4. Shoes.

A shoe will have an ambient dimension n, and an index r, where $o \leq r \leq n$, and we shall call it an n-shoe of index r. Geometrically the index r represents the dimension of the outset. The attracting (r=o) and repelling (r=n) cases are easy to define, and so we do them first before proceeding to the more meaty saddle ($o < r < n$) cases.

Define n-shoe of index 0 = an attracting n-shoe

= the germ in n-dimensions of an attracting hyperbolic periodic orbit.

Define n-shoe of index n = a repelling n-shoe

= the germ in n-dimensions of a repelling hyperbolic periodic orbit.

In particular where the period = 1 we have fixed points. Let α^n, ρ^n denote the germs in n-dimensions of attracting, repelling hyperbolic fixed points.

Now we come to the saddle shoes, which we can reduce to the 2-dimensional case as follows.

Let $n \geq 2$ and $o < r < n$. Define :

$$\text{n-shoe of index r} = (\text{2-shoe of index 1}) \times \rho^{r-1} \times \alpha^{n-r-1},$$

where the right hand side denotes the topological product of the three germs. There remains to classify, by explicit construction, the 2-shoes of index 1, of which the Smale horseshoe is the classic prototype.

5. Preshoes.

Define a preshoe P = (P,V,H,f) where P is a manifold and $f : V \rightarrow H$ a diffeomorphism between submanifolds.

Example 1. The classical horseshoe [5]. Let $P = R^2$, V = square, H = the horseshoe, f = the classical map. (Figure 1).

Figure 1.

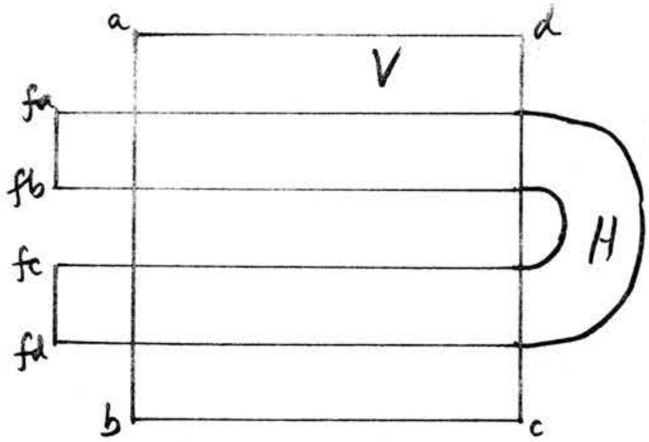
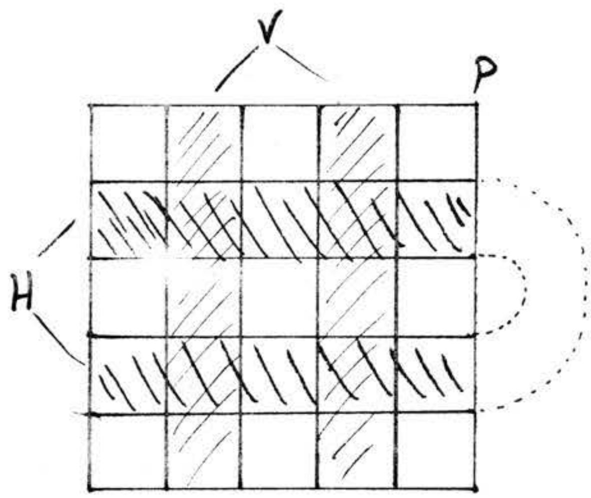


Figure 2.



Example 2. An alternative formulation of the horseshoe. Let P = square, V = two vertical rectangles, H = two horizontal rectangles, as shown in Figure 2, and f = the linear part of the classical horseshoe map.

We continue with definitions. Given a preshoe P, define the derived preshoe $P' = (P', V', H', f')$ by

$$P' = V \cap H$$
$$V' = f^{-1}P' \cap P'$$
$$H' = P' \cap fP'$$
$$f' = f|P' .$$

Defined the r^{th} derived preshoe $P^r = (P^{r-1})'$. Let $S = \bigcap_r P^r$. Define the induced shoe to be the set S together with the germ of f at S. Examples 1 and 2 both induce the same shoe, S being the Cantor set of the horseshoe. To construct the general 2-shoe index 1 we shall generalise Example 2. For this we need algebraic data. For simplicity we shall confine ourselves to the orientable case, but the theory extends both to non-orientable manifolds, and to orientation reversing diffeomorphisms of orientable manifolds [9] .

6. Indecomposable matrices.

Let G be a q x q matrix over the non-zero integers. Given a permutation π of (1,...,q), let πG denote the result of applying π to both rows and columns of G. We say G is decomposable if, for some π ,

$$\pi G = \left(\begin{array}{c|c} G_1 & 0 \\ \hline * & G_2 \end{array}\right)$$

where G_1, G_2 are square submatrices, and * indicates possibly non-zero elements. Otherwise call G indecomposable. For example a cyclic matrix is indecomposable, where by a cyclic matrix we mean the matrix associated with a cyclic permutation of (1,...,q).

One can easily show [8] that any matrix G can be decomposed into indecomposable factors

$$G = \begin{pmatrix} G_1 & & & 0 \\ & G_2 & & \\ & & \ddots & \\ * & & & G_r \end{pmatrix} .$$

The factors are unique (up to permuation of rows and columns within each factor). Furthermore the non-zero elements of * induce a partial ordering amongst the factors.

7. Construction of the shoe from algebraic data.

Define the algebraic data for a preshoe to be a pair (A^+, A^-) of $q \times q$ matrices over the non-negative integers, such that

$$G = A^+ + A^-$$

is indecomposable.

We now proceed to construct a preshoe from this data. Denote the sums of rows and columns of G by

$$v_i = \sum_j G_{ij} \quad , \quad h_j = \sum_i G_{ij} \; .$$

Indecomposability implies v_i, $h_i \geqslant 1$, for all i. Let $P_1, P_2, \ldots, P_q$ be a set of q disjoint rectangles. It is convenient to think of them sitting on $\mathbb{R}^2$, so that they have parallel axes. For each i, in P_i let V_i be the union of v_i disjoint vertical subrectangles, where vertical means they stretch from top to bottom and do not meet the sides of P_i (as in Figure 2), and let H_i be the union of h_i disjoint horizontal subrectangles, where horizontal means they stretch from side to side and do not meet the top and bottom of P_i. For instance Example 2 above would arise from the data $q = 1$, $A^+ = A^- = (1)$, $G = (2)$. Let $P = \bigcup P_i$, $V = \bigcup V_i$, $H = \bigcup H_i$. There remains to construct the diffeomorphism $f : V \rightarrow H$. This is going to map each vertical rectangle onto some horizontal rectangle by expanding horizontally and contracting vertically. There are exactly two possible ways to map each rectangle, either preserve the orientation of both axes, or reverse the orientation of both axes (since we have agreed to preserve the 2-dimensional orientation). Call these two ways positive and negative.

To define f, choose G_{ij} of the V_i-rectanlges and map them onto H_j-rectangles, and since $G_{ij} = A^+_{ij} + A^-_{ij}$ we can map A^+_{ij} positively and A^-_{ij} negatively.

Remark 1. There is a finite element of choice here of choosing which rectangles to map to which. Therefore each algebraic data (A^+, A^-) gives rise to a finite number of shoes. If however $2 \leq r \leq n - 2$, the resulting shoes are diffeomorphic.

Let S be the induced shoe arising from the preshoe of this construction. One can prove the following lemmas [9] .

Lemma 1. If G is a cyclic matrix then S is a saddle hyperbolic periodic orbit of period q.

Lemma 2. If G is a non-cyclic indecomposable matrix, then S is a saddle hyperbolic Cantor set, and $f \mid S$ is a subshift of finite type. Therefore $f \mid S$ is topologically transitive and the periodic orbits are dense in S. Therefore S is a basic set, provided that the germ of f at S is suitably embedded into a global diffeomorphism of a manifold. In other words the general shoe is very like the classical horseshoe.

Remark 2. We have completed the definition of saddle shoes. We have classified shoes in the sense of presenting a countable list of the possible algebraic data, namely the set of pairs of matrices (A^+, A^-)

such that G is indecomposable. However this list contains much redundancy, and we have not classified shoes in the sense of stating criteria for when two data give rise to diffeomorphic shoes. This is an open question.

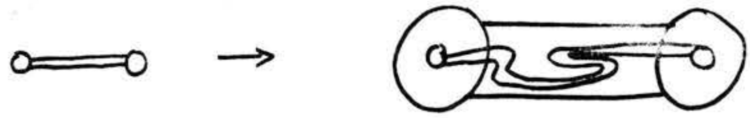
Remark 3. If G is decomposable, we call the resulting germ S a composite shoe. Then the unique decomposition of G into indecomposable factors decomposes S into a finite union of shoes, which are partially ordered by the partial ordering of factors.

Remark 4. The matrix $G = A^+ + A^-$ is the geometric sum, and is related to the geometry, as we have described above. Meanwhile the algebraic sum $A = A^+ - A^-$ is significantly related to the algebraic structure of the manifold. In fact A has interesting invariants, called signatures, which are related to the homology of the inset and outset of the shoe, and which give rise to the Morse-inequalities of Theorems C and C' (see [9]).

8. C^0-density of diffeomorphisms.

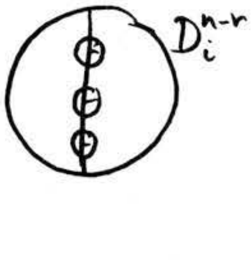
The proof of Theorem A in section 2 is too long to give here, and we confine ourselves to sketching the main steps. The proof is to be found in Smale [7] , Shub [3] , and [9] . In addition to the Smale-Shub procedure, it is necessary to show that our classified list of 2-shoes, and hence the resulting Smale diffeomorphisms, are sufficient to approximate any given diffeomorphism of an n-manifold. There are 3 main steps to the construction.

Step (1). Let $f : M \to M$ be the given diffeomorphism. Choose a smooth triangulation K of M, with mesh as fine as the desired approximation - all diffeotopies will be this fine. Choose a smooth tubulation of K, that is a tubular neighbourhood T^r of the r-skeleton of K, for each r, giving a handle decomposition of M. Step (1) of Shub [3] is to diffeotop f onto a new diffeomorphism, which, for convenience, we shall still call f, with the property $fT^r \subset \text{int}(T^r)$, for each r.



Step (2) of Smale [7] is to further diffeotop f so that the image of each r-handle, where it crosses r-handles, crosses them linearly, expanding r-dimensions, and contracting (n-r)-dimensions. In particular r = 0 gives attracting, and r = n repelling, periodic orbits.

Step (3) If $o < r < n$, each r-handle can be written as a product of disks, $h_i^r = D_i^r \times D_i^{n-r}$. At the centre of D_i^{n-r} choose an oriented frame, and let d_i^{n-r} denote the diameter along the first axis. Inside D_i^{n-r} there are a finite number of subdisks, which are sections of images of r-handles. Each subdisk inherits by f a diameter and frame. We show in [9], that f can be followed by a diffeotopy that strings the subdisks like beads along the diameter, each with its little diameter in line, and with the remainder of its axes parallel (the first axis along the diamater may have its orientation reversed, due to an orientation obstruction). Similarly for the disks D_i^r.



There remains to verify that after the three steps the diffeomorphism is Smale. Let $h^r = \bigcup_i h_i^r$, and let h^r also denote the n-dimensional preshoe

$$h^r = (h^r,\ f^{-1}h^r \cap h^r,\ h^r \cap fh^r,\ f|(f^{-1}h^r \cap h^r)).$$

Let H^r denote the induced composite n-shoe of index r. For each i, let k_i^2 denote the product of the two diameters of D_i^r, D_i^{n-r}, and let k^2 denote the corresponding 2-preshoe, and K^2 the induced composite 2-shoe of index 1. Then by Step (3), $H^r = K^2 \times \rho^{r-1} \times \alpha^{n-r-1}$.

Let G_{ij} be the number of components of $fh_i^r \cap h_j^r$. Of these let A_{ij}^+ be the number of those with diameters oriented coherently, and A_{ij}^- the number of those with diameters reversed. The (A^+, A^-) is the data for the preshoe K^2. In general G is not indecomposable; however the decomposition of G into indecomposable factors gives the shoes in H^r, and hence, for all r, the basic sets of f.

Finally the ordering of shoes, first by dimension, and then by the partial ordering of factors of G, for each r, ensures the no-cycle condition.

9. C^0-density of flows.

The analogous theorem for flows, Theorem A' of section 3, was announced by Shub and Hirsch in [3] and further details are also given in [9]. Given a flow, the basic idea is to choose a finite set of (n-1)-dimensional disks transverse to the flow so that each orbit is cut by at least one disk. The flow itself gives a map f between disks. Approximate f as above. There are three main technical difficulties in this procedure, which are awkward but soluble, as follows.

(1) f is not continuous; a discontinuity occurs whereever an orbit flows from an interior point of one disk to a boundary point of another disk. However all such discontinuity points can be made wandering, and so do not interfere with the non-wandering set.

(2) It may not be possible to have $fT^r \subset \text{int } T^r$, because the image of an r-handle may cross the boundary of a disk. However this difficulty can be tackled by making the boundaries in general position relative to one another, with respect to the flow, and triangulating them first. By careful construction the diffeotopy of the flow can be arranged to ensure

$$fT^r \subset \text{int}(T^r) \cup (\partial T^r \cap \partial T^{r-1}),$$

which is sufficient to permit Step (2) above.

(3) The fixed points require special treatment. By an initial approximation we may assume there are only a finite number of fixed points, and that these are hyperbolic. Round each sink and source choose a small (n-1)-sphere, and add these to the family of disks above. For each saddle choose two small (n-1)-dimensional cylinders, one transversal to the outset, and the other to the inset. The map f is defined on all the disks, spheres and cylinders except on the spheres surrounding sinks, and the interiors of the cylinders tranversal to the insets.

The non-wandering set of the approximated flow consists of the fixed points and the orbits through the shoes of f. But the orbits through a shoe are diffeomorphic to the solenoid suspending that shoe, and so the approximated flow can be shown to be Smale.

References

1. J. Robbin, A structural stability theorem, Annals of Math. 94 (1971) 447 - 493.
2. C. Robinson, (to appear).
3. M. Shub, Structurally stable diffeomorphisms are dense, (Santa Cruz preprint 1972).
4. S. Smale, Morse inequalities for a dynamical system, Bull. A.M.S. 66 (1960), 43 - 49.
5. S. Smale, A structurally stable differentiable homeomorphism with an infinite number of periodic points, Izdat. Akad. Nauk. Ukrain. SSR, Kiev (1963), 365 - 366.
6. S. Smale, Structurally stable systems are not dense, Am. J. Math. 88, (1966) 491 - 496.
7. S. Smale, Stability and isotopy in discrete dynamical systems, Proc. Int. Symp. Dyn. Systems, (1971) Salvador, Brazil (ed. M.M. Peixoto).
8. E.C. Zeeman, Smale diffeomorphisms and flows, I.H.E.S. Conference in Catastrophes, September 1972.
9. E.C. Zeeman, Morse inequalities for Smale diffeomorphisms and flows, (to appear).