THE CLASSIFICATION OF ELEMENTARY CATASTROPHES OF CODIMENSION* ≤ 5.

by

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(Notes written and revised by David Trotman)

INTRODUCTION.

These lecture notes are an attempt to give a minimal complete proof of the classification theorem from first principles. All results which are not standard theorems of differential topology are proved. The theorem is stated in Chapter 1 in a form that is useful for applications [12].

The elementary catastrophes are certain singularities of smooth maps $\mathbb{R}^r \to \mathbb{R}^r$. They arise generically from considering the stationary values of r-dimensional families of functions on a manifold, or from considering the fixed points of r-dimensional families of gradient dynamical systems on a manifold. Therefore they are of central importance in the bifurcation theory of ordinary differential equations. In particular the case $r = 4$ is important for applications parametrised by space-time.

The concept of elementary catastrophes, and the recognition of their importance, is due to René Thom [10]. He realized as early as about 1963 that they could be finitely classified for $r \leq 4$, by unfolding certain polynomial germs $(x^3, x^4, x^5, x^6, x^3 \pm xy^2, x^2y+y^4)$. Thom's sources of inspiration were four-fold: firstly Whitney's paper [11] on stable-singularities for $r = 2$, secondly his own work extending these results to $r > 2$, thirdly light caustics, and fourthly biological morphogenesis.

**This paper, giving a complete proof of Thom's classification theorem, seems not to be readily available. In response to many requests from conference participants, Zeeman and his collaborator, David Trotman, agreed to make a revised version of the paper (July, 1975) available for the conference proceedings. I would like to express my appreciation to both Christopher Zeeman and David Trotman.*

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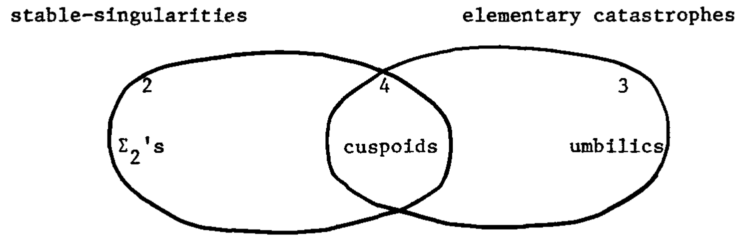
However although Thom had conjectured the classification, it was some years before the conjecture could be proved, because several branches of mathematics had to be developed in order to provide the necessary tools. Indeed the greatest achievement of catastrophe theory to date is to have stimulated these developments in mathematics, notably in the areas of bifurcation, singularities, unfoldings and stratifications. In particular the heart of the proof lies in the concept of unfoldings, which is due to Thom. The key result is that two transversal unfoldings are isomorphic, and for this Thom needed a C^∞ version of the Weierstrass preparation theorem. He persuaded Malgrange [3] to prove this around 1965. Since then several mathematicians, notably Mather, have contributed to giving simpler alternative proofs [4,5,7,8] and the proof we give in Chapter 5 is mainly taken from [1].

The preparation theorem is a way of synthesising the analysis into an algebraic tool; then with this algebraic tool it is possible to construct the geometric diffeomorphism required to prove two unfoldings equivalent. The first person to write down an explicit construction, and therefore a rigorous proof of the classification theorem, was John Mather, in about 1967. The essence of the proof is contained in his published papers [4,5] about more general singularities. However the particular theorem that we need is somewhat buried in these papers, and so in 1967 Mather wrote a delightful unpublished manuscript [6] giving an explicit minimal proof of the classification of the germs of functions that give rise to the elementary catastrophes. The basic idea is to localise functions to germs, and then by determinacy reduce germs to jets, thereby reducing the ∞-dimensional problem in analysis to a finite dimensional problem in algebraic geometry. Regrettably Mather's manuscript was never quite finished, ahtough copies of it have circulated widely. We base Chapters 2, 3, 4, 6 primarily upon his exposition.

Mather's paper is confined to the local problem of classifying germs of functions. To put the theory in a usable form for applications three further steps are necessary. Firstly we need to globalise from germs back to functions again, in order to obtain an open-dense set of functions, that can be used for modeling. For this we need the Thom transversality lemma, and Chapter 8 is based on Levine's exposition [2].

Secondly we have to relate the function germs, as classified by Mather, to the induced elementary catastrophes, which are needed for the applications. For instance the elliptic umbilic starts as an unstable germ $\mathbb{R}^2 \to \mathbb{R}$, which then unfolds to a stable-germ $\mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^3$, or equivalently to a germ $f: \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}$, and eventually induces the elementary catastrophe germ $X_f: \mathbb{R}^3 \to \mathbb{R}^3$. The relation between these is explained in Chapter 7.

Finally in Chapter 9 we verify the stability of the elementary catastrophes, in other words the stability of X_f under perturbations of f. A word of warning here: although the elementary catastrophes are singularities, and are stable, they are different from the classical stable-singularities [1,2,4,5,11]. The unfolded germ is indeed a stable-singularity, but the induced catastrophe germ may not be. The difference can be explained as follows. Let M denote the space of all C^∞ maps $\mathbb{R}^r \to \mathbb{R}^r$, and C the subspace of catastrophe maps. Then $C \neq M$ because not all maps can be induced by a function. Therefore a stable-singularity, such as Σ_2, may appear in M, but not in C, and therefore will not occur as an elementary catastrophe. Conversely an elementary catastrophe, such as an umbilic, may appear in C, and be stable in C, but become unstable if perturbations in M are allowed, and therefore will not occur as a stable-singularity.. For $r = 2$ the two concepts accidentally coincide, because Whitney [11] showed that the only two stable-singularities were the fold and cusp, and these are the two elementary catastrophes. However for $r = 3$ the concepts diverge, and for $r = 4$, for instance, there are 6 stable-singularities and 7 elementary catastrophes, as follows:



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CHAPTER 1. STATING THOM's THEOREM.

Let $f\colon \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$ be a smooth function. Define $M_f \subset \mathbb{R}^{n+r}$ to be given by $(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) = \mathrm{grad}_x f = 0$, where $x_1, \ldots, x_n$ are coordinates for $\mathbb{R}^n$, and $y_1, \ldots, y_r$ are coordinates for $\mathbb{R}^r$. Generically M_f is an r-manifold because it is codimension n, given by n equations. Let $\chi_f\colon M_f^r \to \mathbb{R}^r$ be the map induced by the projection $\mathbb{R}^{n+r} \to \mathbb{R}^r$. We call χ_f the <u>catastrophe map</u> of f. Let $\mathcal{F}$ denote the space of C^∞-functions on $\mathbb{R}^{n+r}$, with the Whitney C^∞-topology. We can now state Thom's theorem.

<u>Theorem</u>. If $r \leq 5$, there is an open dense set $\mathcal{F}_* \subset \mathcal{F}$ which we call <u>generic</u> functions. If f is generic, then

(1) M_f is an r-manifold.

(2) Any singularity of χ_f is equivalent to one of a finite number of types called <u>elementary catastrophes</u>.

(c) χ_f is locally stable at all points of M_f with respect to small perturbations of f.

The number of elementary catastrophes depends only upon r, as follows:

r	1	2	3	4	5	6	7
elem. cats.	1	2	5	7	11	∞	∞

Here equivalence means the following: two maps $\chi\colon M \to N$ and $\chi'\colon M' \to N'$ are equivalent if $\exists$ diffeomorphisms h, k such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\chi"] \arrow[d, "h"'] & N \arrow[d, "k"] \\ M' \arrow[r, "\chi'"'] & N' \end{tikzcd}$$

Now suppose the maps χ, χ' have singularities at x, x' respectively. Then the singularities are equivalent if the above definition holds locally, with $hx = x'$.

Remarks. The reason for keeping $r \leq 5$ is that for $r > 5$ the classification becomes infinite, because there are equivalence classes of singularities depending upon a continuous parameter. One can obtain a finite classification under topological equivalence, but for applications the smooth classification in low dimensions is more important. The theorem remains true when $\mathbb{R}^{n+r}$ is replaced by a bundle over an arbitrary r-manifold, with fibre an arbitrary n-manifold.

The theorem stated above is a classification theorem: we classify the types of singularity that 'most' X_f can have. We find that if X_f has a singularity at $(x,y) \in \mathbb{R}^{n+r} \cap M_f$, and if η is the germ at (x,y) of $f|\mathbb{R}^n x\, y$, then the equivalence class of X_f at (x,y) depends only upon the (right) equivalence class of η (Theorem 7.8). This result is hard and requires an application of the Malgrange Preparation Theorem, itself a consequence of the Division Theorem (Chapter 5), and study of the category of unfoldings of a germ η (Chapter 6).

To use it we have first to classify germs η of C^∞ functions $\mathbb{R}^n,0 \to \mathbb{R},0$. We use two related integer invariants, determinacy and codimension, and the jacobian ideal $\Delta(\eta)$ (the ideal spanned by $\frac{\partial \eta}{\partial x_1}, \ldots, \frac{\partial \eta}{\partial x_n}$ in the local ring E of germs at 0 of C^∞ functions $\mathbb{R}^n \to \mathbb{R}$). The determinacy of a germ η is the least integer k such that if any germ ξ has the same k-jet as η then ξ is right equivalent to η. Theorem 2.9 gives necessary and sufficient conditions for k-determinacy in terms of Δ. Defining the codimension of η as the dimension of m/Δ, where m is the unique maximal ideal of E, we use this theorem to show that $\det \eta - 2 \leq \operatorname{cod} \eta$ in Lemma 3.1. If $r \leq 5$ and $f \in F_*$ then if $\eta = f|\mathbb{R}^n x\, y$, for any $y \in \mathbb{R}^r$, we have $\operatorname{cod} \eta \leq r$. Hence since we can restrict to $\operatorname{cod} \eta \leq 5$ we need only look at 7-determined germs in the vector space J^7 of 7-jets. We must restrict to $r \leq 5$, for if $\operatorname{cod} \eta \geq 7$ there are equivalence classes depending upon a continuous parameter, and the definition of F_* ensures that if $r = 6$ then each of these equivalence classes contains an $f|\mathbb{R}^n x\, y$ for some $y \in \mathbb{R}^r$ and $f \in F_*$.

The 7-jets of codimension ≥ 6 form a closed algebraic variety Σ in J^7, and the partition by codimension of $J^7-\Sigma$ forms a regular stratification (Chapters 3 and 8). We in fact use a condition implied by a-regularity (Definition 8.2). This is necessary to show that F_* is open in F. That it is dense follows from Thom's transversality lemma; and transversality gives that M_f is an r-manifold for $f \in F_*$ (Chapter 8).

The classification of germs of codimension ≤ 5 is completed in Chapter 4 and in Chapter 7 the connection is made with catastrophe germs. Finally in Chapter 9 we show the local stability of X_f.

CHAPTER 2. DETERMINACY.

Definition. Suppose $C^\infty(M,Q)$ is the space of C^∞ maps $M \to Q$, where M and Q are C^∞ manifolds. If $x \in M$ and f and $g \in C^\infty(M,Q)$ let $f \sim g$ if $\exists$ a neighborhood N of x such that $f|N = g|N$. The equivalence class $[f]$ is called a germ, the germ of f at x.

Let E_n be the set of germs at 0 of C^∞ functions $\mathbb{R}^n \to \mathbb{R}$. It is a real vector space of infinite dimension, and a ring with a 1, the 1 being the germ at 0 of the constant function taking the value $1 \in \mathbb{R}$. Addition, multiplication, and scalar multiplication are induced pointwise from the structure in $\mathbb{R}$.

Definition. A local ring is a commutative ring with a 1 with a unique maximal ideal.

We shall show that E_n is a local ring with maximal ideal m_n being the set of germs at 0 of C^∞ functions vanishing at 0 (written as functions $\mathbb{R}^n,0 \to \mathbb{R},0$).

Lemma 2.1. m_n is a maximal ideal of E_n.

Proof. Suppose $\eta \in E_n$ and $\eta \notin m_n$. We claim that the ideal generated by m_n and η, $(m_n,\eta)_{E_n}$, is equal to E_n.

Let the function $e \in \eta$, i.e, η is the germ at 0 of e, and choose a neighborhood U of 0 in $\mathbb{R}^n$ such that $e \neq 0$ on U. Then $1/e$ exists on U. Let ξ be the germ $[1/e]$, then $\xi\eta = [1/e]\cdot[e] = [1/e\cdot e] = [1] = 1$. Also $\xi\eta \in (m_n,\eta)_{E_n}$. Thus $(m_n,\eta)_{E_n} = E_n$.

<u>Lemma 2.2</u>. m_n is the unique maximal ideal of E_n.

<u>Proof</u>. Given $I \subsetneqq E_n$, we claim $I \subset m_n$. If not $\exists \eta \in I - m_n$, and then as in Lemma 2.1 an inverse exists in E_n. $1 = 1/\eta\cdot\eta \in I$, and so $I = E_n$.

Lemma 2.1 and Lemma 2.2 show that E_n is a local ring.

Let G_n be the set of germs at 0 of C^∞ diffeomorphisms $\mathbb{R}^n,0 \to \mathbb{R}^n,0$. G_n is a group with multiplication induced by composition. We shall drop suffices and use E, m and G, when referring to E_n, m_n and G_n rather than E_s when $n \neq s$, etc. Given $\alpha_1, \ldots, \alpha_r \in E$, we let $(\alpha_1,\ldots,\alpha_r)_E$ be the ideal generated by $\{\alpha_i\} = \{\sum_{i=1}^{r} \varepsilon_i\alpha_i : \varepsilon_i \in E\}$, and drop the suffix if there is no risk of confusion. Choose coordinates $x_1, \ldots, x_n$ in $\mathbb{R}^n$ (linear or curvilinear). The symbol 'x_i' will be used ambiguously as:

(i) coordinate of $x = (x_1,\ldots,x_n)$, $x_i \in \mathbb{R}$.

(ii) function $x_i\colon \mathbb{R}^n,0 \to \mathbb{R},0$.

(iii) the germ at 0 of this function in $m \subset E$.

(iv) the k-jet of that germ (see below).

<u>Lemma 2.3</u>. $m = (x_1,\ldots,x_n)_E$

= ideal of E generated by the germs x_i.

<u>Proof</u>. Given $\eta \in m$, represent η by $e\colon \mathbb{R}^n,0 \longrightarrow \mathbb{R},0$. $\forall\, x \in \mathbb{R}^n$,

$$\begin{aligned} e(x) &= \int_0^1 \frac{\partial e}{\partial t}(tx)dt \\ &= \int_0^1 \sum_{i=1}^{n} \frac{\partial e}{\partial x_i}(tx)x_i(x)dt \\ &= \sum_{i=1}^{n} e_i(x)x_i(x). \end{aligned}$$

$e = \sum_{i=1}^{n} e_i x_i$ as functions and so $\eta = \sum_{i=1}^{n} e_i x_i$ as germs. Thus $m \subset (x_1,\ldots,x_n)$. $(x_1,\ldots,x_n) \subset m$ because each $x_i \in m$.

Corollary 2.4. m^k is the ideal generated by all monomials in x_i of degree k.

Corollary 2.5. m^k is a finitely generated $\mathcal{E}$-module.

We let $\mathcal{J}^k$ be the quotient $\mathcal{E}/m^{k+1}$, and let J^k be m/m^{k+1}. j^k denotes the canonical projection $\mathcal{E} \to \mathcal{J}^k$.

Lemma 2.6. $\mathcal{J}^k$ is 1) a local ring with maximal ideal J^k,

2) a finite-dimensional real vector space (generated by monomials in $\{x_i\}$, of degree $\le k$).

Proof. 1) $\mathcal{J}^k$ is a quotient ring of $\mathcal{E}$ and thus is a commutative ring with a 1. There is a 1-1 correspondence between ideals:

$$\begin{array}{ccc} \mathcal{E} & & \mathcal{E}/m^{k+1} = \mathcal{J}^k \\ \cup & & \cup \\ I & \longleftrightarrow & I/m^{k+1} \\ \cup & & \\ m^{k+1} & & \end{array}$$

So $\mathcal{J}^k$ is a local ring.

2) $\mathcal{J}^k$ is a quotient vector space of $\mathcal{E}$ and is finite-dimensional. For given $\eta \in \mathcal{E}$, the Taylor expansion at 0 is,

$$\eta = \eta_0 + \eta_1 + \dots + \eta_k + \rho_{k+1},$$

where η_j is a homogeneous polynomial in $\{x_i\}$ of degree j, with coefficients the corresponding partial derivatives at 0, and $\rho_{k+1} \in m^{k+1}$.

Definition. The k-jet of $\eta = j^k\eta = \eta_0 + \dots + \eta_k$

$=$ Taylor series cut off at k.

$\mathcal{J}^k$ and J^k are spaces of k-jets, or jet spaces.

Definition. If $\eta, \xi \in \mathcal{E}$ we say they are right equivalent ($\sim$) if they belong to the same G-orbit. $\eta \sim \xi \Leftrightarrow \exists\gamma \in G$ such that $\eta = \xi\gamma$.

__Definition__. If $\eta, \xi \in E$ we say they are __k-equivalent__ ($\overset{k}{\sim}$) if they have the same k-jet. $\eta \overset{k}{\sim} \xi \Leftrightarrow j^k\eta = j^k\xi$.

__Definition__. $\eta \in E$ is __k-determinate__ if $\forall \xi \in E$, $\eta \overset{k}{\sim} \xi \Rightarrow \eta \sim \xi$. Clearly η k-determinate $\Rightarrow \eta$ i-determinate $\forall i \geq k$. The __determinacy__ of η is the least k such that η is k-determinate. We write __det__ η.

__Lemma 2.7__. If η is k-determinate then

1) $\eta \overset{k}{\sim} \xi \Rightarrow \xi$ k-determinate,

2) $\eta \sim \xi \Rightarrow \xi$ k-determinate.

__Proof__. 1) follows at once from 2), which we shall prove. Assume $\eta \sim \xi$, i.e. $\eta = \xi\gamma_1$, some $\gamma_1 \in G$. Suppose $\xi \overset{k}{\sim} \nu$, i.e. $j^k\xi = j^k\nu$, i.e. $j^k(\eta\gamma_1^{-1}) = j^k\nu$.

Then $j^k\eta = j^k(\eta\gamma_1^{-1}\gamma_1) = j^k(\eta\gamma_1^{-1})\cdot j^k(\gamma_1) = j^k\nu\cdot j^k\gamma_1 = j^k(\nu\gamma_1)$. So $\eta \overset{k}{\sim} \nu\gamma_1$, which $\Rightarrow \eta \sim \nu\gamma_1$, i.e. $\eta = \nu\gamma_1\gamma_2$ some $\gamma_2 \in G$. Then $\xi\gamma_1 = \nu\gamma_1\gamma_2$, and $\xi = \nu\gamma_1\gamma_2\gamma_1^{-1}$, i.e. $\xi \sim \nu$. So 2) is proved.

__Definition__. If $\eta \in E$, choose coordinates $\{x_i\}$ for $\mathbb{R}^n$, and let $\Delta = \Delta(\eta) = (\frac{\partial\eta}{\partial x_1},\ldots,\frac{\partial\eta}{\partial x_n})_E$. Δ is independent of the choice of coordinates. For if $\Delta_x = (\frac{\partial\eta}{\partial x_i})$ and $\Delta_y = (\frac{\partial\eta}{\partial y_j})$, $\frac{\partial\eta}{\partial y_j} = \sum_{i=1}^{n} \frac{\partial\eta}{\partial x_i}\frac{\partial x_i}{\partial y_j} \in \Delta_x$ and so $\Delta_y \subset \Delta_x$. ($\frac{\partial\eta}{\partial x_i} \in \Delta_x$, each i, and $\frac{\partial x_i}{\partial y_j} \in E$, each i, j). Similarly $\Delta_x \subset \Delta_y$, so $\Delta_x = \Delta_y$.

__Lemma 2.8__. If $\eta \in E - m$, and $\eta' = \eta - \eta(0) \in m$, then $\Delta(\eta) = \Delta(\eta')$, and η is k-determinate $\Leftrightarrow \eta'$ is k-determinate.

__Proof__. $\Delta(\eta) = \Delta(\eta')$ is trivial. $\eta \overset{k}{\sim} \xi \Leftrightarrow \begin{cases} \eta' \overset{k}{\sim} \xi', \text{ trivially.} \\ \eta(0) = \xi(0). \end{cases}$

$$\text{Also } \eta = \xi\gamma \Leftrightarrow \begin{cases} \eta' = \xi'\gamma, \ \gamma \in G \\ \eta(0) = \xi(0). \end{cases}$$

$$\text{Thus } \eta \sim \xi \Leftrightarrow \begin{cases} \eta' \sim \xi' \\ \eta(0) = \xi(0). \end{cases}$$

So from now on we shall suppose $\eta \in m$.

Theorem 2.9. If $\eta \in m$ and $\Delta = \Delta(\eta)$, then

$$m^{k+1} \subset m^2\Delta \Rightarrow \eta \text{ is k-determinate } \Rightarrow m^{k+1} \subset m\Delta.$$

Proof. We shall use the following form of Nakayama's Lemma:

Lemma 2.10. If A is a local ring, a its maximal ideal, and M, N are A-modules (contained in some larger A-module) with M finitely generated over A, then $M \subset N + aM \Rightarrow M \subset N$.

Sublemma. $\lambda \in A,\ \lambda \notin a \Rightarrow \lambda^{-1} \in A$.

Proof. λA is an ideal $\not\subset a$. So $\lambda A = A \ni 1$, $\exists\mu$ such that $\lambda\mu = 1$.

Proof of Lemma 2.10. We shall first prove the special case of $N = 0$, i.e., $M \subset aM \Rightarrow M = 0$. Let $v_1, \ldots, v_r$ generate M. $v_i \in aM$ by hypothesis,

$$\text{so } v_i = \sum_{j=1}^{r} \lambda_{ij} v_j \quad (\lambda_{ij} \in a)$$

or $\sum_{j=1}^{r} (\delta_{ij} - \lambda_{ij}) v_j = 0$, i.e. $(I-\Lambda)v = 0$, where Λ is an $(r\times r)$-matrix (λ_{ij}), and $v = \begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix}$. The determinant $|I - \Lambda| = 1 + \lambda$, some $\lambda \in a$. Now $1 + \lambda \notin a$, else $1 \in a$ and $a = A$. So $(1+\lambda)^{-1}$ exists by the sublemma. Then $(I-\Lambda)^{-1}$ exists, giving $v = 0$ and $M = 0$.

To prove the general case consider the quotient by N, $(M+N)/N \subset N/N + (aM+N)/N$. We claim the R.H.S. $= a(M+N)/N$. (*) Then by the special case, $(M+N)/N = 0$, giving $M \subset N$. Q.E.D.

The A-module structure on $(M+N)/N$ is induced by that on $M + N$ by $\lambda(v+N) = \lambda v + N$.

$$\begin{aligned} a(M+N)/N &= \{\lambda(v+N) : \lambda \in a,\ v \in M\} \\ &= \{\lambda v + N : \lambda \in a,\ v \in M\} \\ &= (aM+N)/N, \text{ proving (*)}. \end{aligned}$$

Continuing the proof of Theorem 2.9, we assume $m^{k+1} \subset m^2\Delta$, and must show that $\eta \overset{k}{\sim} \xi \Rightarrow \eta \sim \xi$. The idea of the proof is to change η into ξ continuously with the assumption $\eta \overset{k}{\sim} \xi$. Let Φ denote the germ at $0 \times \mathbb{R}$ of a function $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ given by $\Phi(x,t) = (1-t)\eta(x) + t\xi(x)$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$. Let

$$\phi^t(x) = \Phi(x,t) = \begin{cases} \eta(x) & t = 0 \\ \xi(x) & t = 1. \end{cases}$$

Lemma 1. Fixing t_0, $0 \le t_0 \le 1$, $\exists$ a family $\Gamma^t \in G$ defined for t in a neighborhood of t_0 in $\mathbb{R}$ such that 1) Γ^{t_0} = identity
2) $\phi^t \Gamma^t = \phi^{t_o}$.

Lemma 1 will give $\eta \sim \xi$: Using compactness and connectedness of $[0,1]$, cover by a finite number of neighborhoods as in Lemma 1, then pick $\{t_i\}$ in the overlaps, and construct γ satisfying $\eta = \xi\gamma$ by a finite composition of $\{\Gamma^{t_i}\}$, i.e. $\eta = \phi^0 \sim \ldots \sim \phi^1 = \xi$.

Lemma 2. For $0 \le t_o \le 1$, $\exists$ a germ Γ at (p,t_o) of C^∞ maps $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ satisfying (a) $\Gamma(x,t_o) = x$,

(b) $\Gamma(0,t) = 0$,

(c) $\Phi(\Gamma(x,t),t) = \Phi(x,t_o)$,

for all (x,t) in some neighborhood of $(0,t_o)$.

Lemma 2 will give Lemma 1: Define $\Gamma^t(x) = \Gamma(x,t)$ from a neighborhood of 0 in $\mathbb{R}^n$ to $\mathbb{R}^n$; Γ^t is a germ of C^∞ maps $\mathbb{R}^n,0 \to \mathbb{R}^n,0$ by (b); Γ^{t_o} is the identity by (a). C^∞ diffeomorphisms are open in the space of C^∞ maps $\mathbb{R}^n,0 \to \mathbb{R}^n,0$ (because they correspond to maps with Jacobian of maximal rank, i.e. to the non-vanishing of a certain determinant), and so $\exists$ a neighborhood of t_o such that Γ^t is a germ of diffeomorphisms for t in that neighborhood, i.e. $\Gamma^t \in G$.

Lemma 3. (c) in Lemma 2 is equivalent to,

$$(c') \quad \sum_{i=1}^{n} \frac{\partial\Phi}{\partial x_i}(\Gamma(x,t),t)\,\frac{\partial\Gamma_i}{\partial t}(x,t) + \frac{\partial\Phi}{\partial t}(\Gamma(x,t),t) = 0.$$

(c) $\Rightarrow$ (c'): by differentiation with respect to t.

$$(c') \Rightarrow (c):\quad 0 = \int_{t_o}^{t} (c')dt = \Phi(\Gamma(x,t),t) - \Phi(\Gamma(x,t_o),t_o)$$
$$= \Phi(\Gamma(x,t),t) - \Phi(x,t_o) \text{ by (a) in Lemma 2.}$$

Thus we have (c).

Lemma 4. For $0 \le t_o \le 1$, $\exists$ a germ Ψ at $(0,t_o)$ of a C^∞ map $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$

satisfying (d) $\Psi(0,t) = 0$,

$$\text{(e)}\quad \sum_{i=1}^{n} \frac{\partial\Phi}{\partial x_i}(x,t)\Psi_i(x,t) + \frac{\partial\Phi}{\partial t}(x,t) = 0,$$

for all (x,t) in some neighborhood of $(0,t_o)$.

Lemma 4 $\Rightarrow$ Lemmas 3 and 2: The existence theorem for ordinary differential equations gives a solution $\Gamma(x,t)$ of $\frac{\partial\Gamma}{\partial t} = \Psi(\Gamma,t)$, with initial condition $\Gamma(x,t_o) = x$ (i.e. (a) of Lemma 2). In (e) put $x = \Gamma(x,t)$ to give (c'). (d) $\Rightarrow \Gamma = 0$ is a solution, i.e. $\Gamma(0,t) = 0$ for all t in some neighborhood of t_o, which is (b).

Let A denote the ring of germs at $(0,t_o)$ of C^∞ functions $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$. Projection $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ induces an embedding $E \subset A$ by composition. Let $\Omega = (\frac{\partial\Phi}{\partial x_1},\ldots,\frac{\partial\Phi}{\partial x_n})_A$.

<u>Lemma 5</u>. $m^{k+1} \subset m^2\Delta \Rightarrow m^{k+1} \subset m^2\Omega$.

Lemma 5 $\Rightarrow$ Lemma 4 as follows:

$$\begin{aligned}\frac{\partial\Phi}{\partial t} = \xi - \eta &\in m^{k+1} \qquad (\eta \overset{k}{\sim} \xi)\\ &\subset m^2\Omega.\end{aligned}$$

Thus $\frac{\partial\Phi}{\partial t} = \sum_j \mu_j\omega_j$, $\mu_j \in m^2$, $\omega_j \in \Omega$. (finite sum)

$$= \sum_{ij} \mu_j a_{ij}\frac{\partial\Phi}{\partial x_i}, \text{ where } \omega_j = \sum_i a_{ij}\frac{\partial\Phi}{\partial x_i},\ a_{ij} \in A.$$

$$= -\sum_i \Psi_i \frac{\partial\Phi}{\partial x_i}, \text{ setting } \Psi_i = -\sum_j \mu_j a_{ij} \in A.$$

This gives (e).

Now $\mu_j = \mu_j(x)$ and $a_{ij} = a_{ij}(x,t)$. $\Psi = \{\Psi_i\}$ is a germ at $(0,t_o)$ of a map $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, and $\Psi_i(0,t) = 0$ as each $\mu_j(0) = 0$, so (d) holds for Ψ.

<u>Proof of Lemma 5</u>. (and hence the completion of the proof of a sufficient condition for k-determinacy)

$$\begin{aligned}\frac{\partial\Phi}{\partial x_i} &= \frac{\partial\eta}{\partial x_i} + t\frac{\partial}{\partial x_i}(\xi-\eta)\\ &\in \frac{\partial\eta}{\partial x_i} + Am^k \qquad (t \in A,\ \xi - \eta \in m^{k+1})\end{aligned}$$

i.e. $$\frac{\partial\eta}{\partial x_i} \in \frac{\partial\Phi}{\partial x_i} + Am^k \subset \Omega + Am^k.$$

So $\Delta \subset \Omega + Am^k$.

Denote the maximal ideal of A by a, i.e. those germs vanishing at $(0,t_o)$. Then $m \subset a$. Now

$$
\begin{aligned}
Am^{k+1} &\subset Am^2 \qquad \text{(hypothesis)} \\
&\subset Am^2(\Omega + Am^k) \\
&= m^2\Omega + Am^{k+2} \\
&\subset m^2\Omega + aAm^{k+1}.
\end{aligned}
$$

Now apply Nakayama's Lemma 2.10 for A, a, M, N where $M = Am^{k+1}$ is finitely generated by monomials in $\{x_i\}$ of degree $k+1$ by Corollary 2.4, and $N = m^2\Omega$. This gives $Am^{k+1} \subset m^2\,\Omega$. In particular $m^{k+1} \subset m^2\Omega$, completing Lemma 5.

Now we prove that $m^{k+1} \subset m\Delta$ is a necessary condition of k-determinacy. $\exists$ a natural map $m \xrightarrow{\pi} J^{k+1} \longrightarrow J^k$, $\pi = j^{k+1}/m$.

$$\eta \longmapsto j^{k+1}\eta \mapsto j^k\eta$$

Let $P = \{\xi \in m: \eta \overset{k}{\sim} \xi\}$, and $Q = \{\xi \in m: \eta \sim \xi\} = \text{orbit } \eta G$.

Assuming that η is k-determinate then $P \subset Q$, so that $\pi P \subset \pi Q$. (*)

$P = \eta + m^{k+1}$, so $\pi P = z + m^{k+1}/m^{k+2} = z + \pi m^{k+1}$. (Letting $z = j^{k+1}\eta$). The tangent plane to πP at z, $T_z(\pi P) = \pi m^{k+1}$.

Let G^k denote the k-jets of germs belonging to G; G^k is a finite-dimensional Lie group. Now $j^{k+1}(\eta\gamma) = j^{k+1}(\eta)j^{k+1}(\gamma)$ for $\gamma \in G$, i.e. π is equivariant with respect to G, G^{k+1}. So $\pi Q = \pi(\eta G) = zG^{k+1}$, an orbit under a Lie group, and hence is a manifold. In particular $T_z(\pi Q)$ exists.

<u>Lemma 2.11</u>. $T_z(\pi Q) = \pi(m\Delta)$.

Now (*) gives $T_z(\pi P) \subset T_z(\pi Q)$. Then Lemma 2.11 gives $\pi m^{k+1} \subset \pi(m\Delta)$, i.e. $m^{k+1} \subset m\Delta + m^{k+2}$. Apply Nakayama's Lemma 2.10 with $A = E$, $a = m$, $M = m^{k+1}$, $N = m\Delta$, using Lemmas 2.1, 2.2 and Corollary 2.4, to yield $m^{k+1} \subset m\Delta$.

<u>Proof (of 2.11)</u>. Suppose $\gamma \in G$. As $\mathbb{R}^n$ is additive we can write $\gamma = 1 + \delta$, where 1 is the germ of the identity map, and δ is the germ at 0 of a C^∞ map $\mathbb{R}^n,0 \to \mathbb{R}^n,0$. Join 1 to γ by a continuous path of map-germs, $\gamma^t = 1 + t\delta$,

$0 \leq t \leq 1$. When $t = 0$ or 1, γ^t is a diffeomorphism-germ. Diffeomorphisms are open in the space of C maps, and so $\exists\ t_o > 0$ such that $\gamma^t \in G$, $o \leq t \leq t_o$.

Then $\{\gamma^t\}$ is a path in G starting at 1,

$\{\eta\gamma^t\}$ is a path in Q starting at η,

$\{\pi\eta\gamma^t\}$ is a path in πQ starting at z.

The tangent to the path at $t = 0$ is given by

$$\frac{d}{dt}(\pi\eta\gamma^t)_{t=0} = \pi\left[\frac{d}{dt}\underset{\eta\gamma^t}{\eta(1+t\delta)}\Big|_{t=0}\right]$$

Now $\delta = (\delta_1,\ldots,\delta_n)$ where δ_i is a germ of a C^∞ function $\mathbb{R}^n,0 \to \mathbb{R},0$. (Remember m is a ring and a vector space so we can define differentiation).

$$\text{So } \frac{d}{dt}(\pi\eta\gamma^t)_{t=0} = \pi\left[\sum_{i=1}^{n}\frac{\partial\eta}{\partial x_i}(1+t\delta)\cdot\delta_i\Big|_{t=0}\right]$$

$$= \pi\left[\sum_{i=1}^{n}\frac{\partial\eta}{\partial x_i}\cdot\delta_i\right]$$

$$\in \pi(m\Delta). \qquad (\delta_i \in m,\ \frac{\partial\eta}{\partial x_i} \in \Delta).$$

This tangent is in $T_z(\pi Q)$; moreover any tangent in $T_z(\pi Q)$ arises from a path in πQ, so from a path in G^{k+1}, so from a path in G starting at 1. Allowing δ to vary in G gives all such paths. Hence $T_z(\pi Q) \subset \pi(m\Delta)$. Given $\xi \in m\Delta$, we can write $\xi = \sum_{i=1}^{n}\frac{\partial\eta}{\partial x_i}\delta_i$, $\delta_i \in m$. The δ_i assemble into δ determining a path in G.

Hence $\pi(m\Delta) \subset T_z(\pi Q)$, and we have $T_z(\pi Q) = \pi(m\Delta)$.

<u>Corollary 2.12</u>. η is finitely determinate $\Leftrightarrow m^k \subset \Delta$, some k.

<u>Proof</u>. '$\Rightarrow$' follows as η k-determinate $\Rightarrow m^{k+1} \subset m\Delta \subset \Delta$.

'$\Leftarrow$'. $m^k \subset \Delta$, so $m^{k+2} \subset m^2\Delta$, and η is $(k+1)$-determinate.

<u>Corollary 2.13</u>. $\eta \in m - m^2 \Rightarrow \eta$ is 1-determinate.

<u>Proof</u>. $\eta'(0) \neq 0$, i.e. some $\frac{\partial\eta}{\partial x_i} \notin m$, so $\Delta = E$.

$m^2\Delta = m^2$ and then η is 1-determinate by Theorem 2.9.

So we may effectively assume $\eta \in m^2$ from now on.

<u>Definition</u>. With chosen coordinates $\{x_i\}$, the <u>essence</u> of η (with respect to this coordinate system) is the least k for which $j^k\eta$ contains all the x_i. We write <u>ess</u> η.

<u>Corollary 2.14</u>. $\det \eta \geq \text{ess}\ \eta$ (with respect to any coordinate system).

<u>Proof</u>. $k < \text{ess}\ \eta \Rightarrow j^k\eta$ does not contain x_i, some i. Let $\xi = j^k\eta$ as a germ. So $\Delta(\xi) \not\ni$ any power of x_i,

$$\not\supset m^k,\ \forall k.$$

Thus ξ is not finitely determinate (Corollary 2.12). But $\eta \overset{k}{\sim} \xi$, so if η were k-determinate, Lemma 2.7 would give a contradiction, i.e. $k < \det \eta$.

<u>Counterexample 1</u>. Let $\eta = x^{k+1}$, $n = 1$. Then $\Delta = (x^k) = m^k$, and $m\Delta = m^{k+1}$. Det $\eta \geq \text{ess}\ \eta = k + 1$, (Corollary 2.14). η is not k-determinate and so the implication, η k-determinate $\Rightarrow m^{k+1} \subset m\Delta$ in Theorem 2.9 is not reversible.

<u>Counterexample 2</u>. D. Siersma found $\eta = \dfrac{x^3}{3} + xy^3$, $n = 2$. Here $\Delta = (x^2+y^3, xy^2)$.

$$m^2 = (x^2, xy, y^2), \text{ so } m^2\Delta = x^4 + \cancel{x^2y^3},\ x^3y + \cancel{xy^4},\ x^2y^2 + y^5,\ x^3y^2,\ x^2y^3,\ xy^4,$$

$$\ni \cancel{x^3y^2} + xy^5,\ \cancel{x^2y^3} + y^6,\ x^2y^4,\ x^3y^3,\ x^4y^2,\ x^5y,\ x^6,$$

$$\supset m^6 \quad (\ \text{5-determinate})$$

$$\not\ni y^5,\ \not\supset m^5.$$

Det $\eta \geq \text{ess}\ \eta = 4$. By computation it <u>is</u> 4-determinate, and so the implication $m^{k+1} \subset m^2\Delta \Rightarrow \eta$ k-determinate is not reversible.

CHAPTER 3. CODIMENSION.

Remember that we work in m^2 using Corollary 2.13.

<u>Definition</u>. The codimension of $\eta = \dim_{\mathbb{R}} m/\Delta(\eta)$. We write $\text{cod}\ \eta$. The definition makes sense because if $\eta \in m^2$, each $\dfrac{\partial\eta}{\partial x_i} \in m$ and so $\Delta(\eta) \subset m$. If η were in $m - m^2$, $\Delta(\eta) = E$ and by convention $\text{cod}\ \eta = 0$.

<u>Lemma 3.1</u>. Either both $\text{cod}\ \eta$ and $\det \eta$ are infinite, or both are finite and $\det \eta - 2 \leq \text{cod}\ \eta$.

Proof. $\eta \in m^2 \Rightarrow \Delta \subset m$. $\quad (\Delta = \Delta(\eta))$

We have a descending sequence of vector subspaces of m,

$$m = m + \Delta \supset m^2 + \Delta \supset m^3 + \Delta \supset \ldots \supset m^k + \Delta \supset \ldots. \tag{3.2}$$

Either (i) $\exists\ k$ such that $m^{k-1} + \Delta = m^k + \Delta$, and k is the least such, or

(ii) $\nexists$ such a k.

Case (i): $m^{k-1} \subset m^k + \Delta$, and we may apply Nakayama's Lemma 2.10 yielding $m^{k-1} \subset \Delta$, so $m^{k+1} \subset m^2\Delta$. By Theorem 2.9 η is k-determinate, so $\det \eta \leq k$, i.e. $\det \eta$ is finite. Now $\operatorname{cod} \eta = \dim m/\Delta \leq \dim m/m^{k-1}$, and m/m^{k-1} is finitely generated, by monomials in $\{x_i\}$ of degree ≥ 1 and $< k - 1$. So $\operatorname{cod} \eta$ is finite. Now $m^{k-1} + \Delta = \Delta$, and so the above sequence (3.2) descends strictly to the $m^{k-1} + \Delta$ term, and we have,

$$\underbrace{m/\Delta \supsetneq (m^2+\Delta)/\Delta \supsetneq \ldots \supsetneq (m^{k-1}+\Delta)/\Delta = 0}_{\longleftarrow\ k-2 \text{ steps } \longrightarrow}$$

Hence $\operatorname{cod} \eta = \dim m/\Delta \geq k - 2 \geq \det \eta - 2$, as required.

Case (ii): If $\det \eta$ is finite, then $m^k \subset \Delta$ for some k (Corollary 2.12). Then $m^k + \Delta = \Delta = m^{k+1} + \Delta$, and we are in Case (i). So $\det \eta$ is infinite. $m/\Delta \supset (m^2+\Delta)/\Delta \supset \ldots$ is a strictly decreasing sequence and so $\operatorname{cod} \eta$ ($= \dim m/\Delta$) is infinity.

Let $\Gamma_c = \{\eta \in m^2 : \operatorname{cod} \eta = c\}$ (a 'c-stratum' of m^2), and let $\Omega_c = \{\eta \in m^2 : \operatorname{cod} \eta \leq c\}$, and $\Sigma_c = \{\eta \in m^2 : \operatorname{cod} \eta \geq c\}$, so that

$$m^2 = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_c \cup \ldots \cup \Gamma_\infty. \quad \text{(disjoint union)}$$

Let Γ_c^k, Ω_c^k, Σ_c^k be the images of Γ_c, Ω_c, Σ_c under the map $\pi: m^2 \to I^k$ ($\pi = j^k|m^2$), where I^k is defined as m^2/m^{k+1} just as J^k is m/m^{k+1}.

Theorem 3.3. If $0 \leq c \leq k - 2$, then $I^k = \Omega_c^k \cup \Sigma_{c+1}^k$ (disjoint union), and Σ_{c+1}^k is a (closed) real algebraic variety.

Remark. Both statements are false for $c > k - 2$.

Lemma 3.4. $\operatorname{Dim} E/m^{k+1} = \frac{(n+k)!}{n!k!}$, $\forall\ n, k \geq 0$.

Proof. If $n = 0$, $E = \mathbb{R}$, $m = 0$; L.H.S. = 1 = R.H.S. $\forall k$. If $k = 0$, $E/m = \mathbb{R}$; L.H.S. = 1 = R.H.S. $\forall n$. Use induction on $n + k$.

Then E/m^{k+1} = polynomials of degree $\le k$ in $x_1, \dots, x_n$

= (polynomials of degree k in $x_1, \dots, x_{n-1}$)

+ x_n (polynomials of degree $k - 1$ in $x_1, \dots, x_n$)

So $\dim E/m^{k+1} = \dfrac{(n+k-1)!}{(n-k)!k!} + \dfrac{(n+k-1)!}{n!(k-1)!}$ (by induction)

$= \dfrac{(n+k)!}{n!k!}$

Proof of Theorem 3.3. We define an invariant $\tau(z)$ for $z \in I^k = m^2/m^{k+1}$. Choose $\eta \in \pi^{-1}z$. $\eta \in m^2$, so $\Delta(\eta) = \Delta \subset m$. Define $\tau(z) = \dim m/(\Delta+m^k)$. We claim that $\tau(z)$ is independent of the choice of η. Let η' be another choice, $\Delta(\eta') = \Delta'$. Then $\eta - \eta' \in m^{k+1}$, so $\dfrac{\partial\eta}{\partial x_i} - \dfrac{\partial\eta'}{\partial x_i} \in m^k$, and $\dfrac{\partial\eta}{\partial x_i} \in \Delta' + m^k$. Hence $\Delta \subset \Delta' + m^k$ and $\Delta + m^k \subset \Delta' + m^k$.

$\Delta' + m^k \subset \Delta + m^k$ by symmetry.

Hence $\Delta + m^k = \Delta' + m^k$ and $\tau(z)$ is well defined.

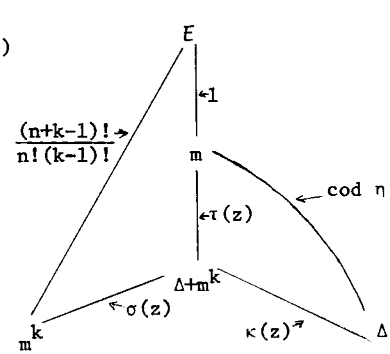
We claim that,

$$(3.5)\quad \begin{cases} \text{(i)} & \tau(z) \le c \Rightarrow \operatorname{cod}\eta = \tau(z), \text{ so } z \in \Omega^k_c. \\ \text{(ii)} & \tau(z) > c \Rightarrow \operatorname{cod}\eta > c, \quad \text{so } z \in \Sigma^k_{c+1}. \quad (\operatorname{cod}\eta \text{ perhaps } \ne \tau(z)) \end{cases}$$

Because (i) and (ii) are disjoint, I^k is the disjoint union of Ω^k_c and Σ^k_{c+1}, once we have shown (i) and (ii) hold.

We have

(Lemma 3.4)



Note that $\tau(z)$ is finite, although $\operatorname{cod}\eta$ may be infinite.

$$\left.\begin{array}{lll}\text{Case (ii):} & \operatorname{cod} \eta \geq \tau(z) & \text{(from the diagram} \\ & > c & \text{(hypothesis of (ii))}\end{array}\right\} \quad \text{Thus (ii) holds.}$$

$$\begin{array}{lll}\text{Case (i):} & k - 2 \geq c & \text{(hypothesos of the theorem)} \\ & \geq \tau(z) & \text{(hypothesis of (i))}\end{array}$$

We have a sequence,

$$0 = m/m = m/\Delta + m \longleftarrow m/\Delta + m^2 \longleftarrow \cdots \longleftarrow m/\Delta + m^k$$
$$\longleftarrow \text{k-1 steps} \longrightarrow$$

$k - 2 \geq \tau(z) = \dim m/\Delta + m^k$, so one step must collapse, i.e. $\Delta + m^{i-1} = \Delta + m^i$, for some $i \leq k$, i.e. $m^{i-1} \subset \Delta + m^i$. Nakayama's Lemma 2.10 $\Rightarrow (m^k \subset) m^{i-1} \subset \Delta$. Therefore $\Delta + m^k = \Delta$, and so $\kappa(z) = 0$ where $\kappa(z) = \dim (\Delta + m^k)/\Delta$ as in the diagram. We observe that $\tau(z) = \operatorname{cod} \eta$, and so (i) holds.

Now $\sigma(z) = \frac{(n+k-1)!}{n!(k-1)!} - 1 - \tau(z)$, from the diagram. If $\tau(z) > c$, then $\sigma(z) < \frac{(n+k-1)!}{n!(k-1)!} - 1 - c = K$, say. $\Sigma^k_{c+1} = \{z \in I^k\colon \operatorname{cod} \eta > c\}$

$$= \{z \in I^k\colon \tau(z) > c\} \qquad (c \leq k - 2)$$

$= \{z \in I^k\colon \sigma(z) < K\}$, which we shall show is an algebraic variety (real).

If $x_1, \ldots, x_n$ are coordinates for $\mathbb{R}^n$, let the monomials of degree $\leq k$ in $\{x_i\}$ be $\{X_j\}$ as below:

$$\begin{array}{ccccccccc} X_1 & X_2 & X_3 & \cdots & X_{n+1} & X_{n+2} & X_{n+3} & \cdots & X_\beta \\ 1 & x_1 & x_2 & \cdots & x_n & x_1^2 & x_1x_2 & \cdots & x_n^k \end{array} \qquad \left(\beta = \frac{(n+k)!}{n!k!}\right)$$

Now J^k is the space of polynomials in $\{x_i\}$ of degree $\leq k$ with coefficients in $\mathbb{R}$ and no constant term. $z \in I^k$ can be written $z = \sum_{j=n+2}^{\beta} a_j X_j$ $(a_j \in \mathbb{R})$. Because $\frac{\partial z}{\partial x_i}$ is a polynomial of degree $k - 1$ with no constant term it belongs to J^{k-1}, so $\frac{\partial z}{\partial x_i} = \sum_{j=2}^{\bar\beta} a_{ij} X_j$, $(\bar\beta = \frac{(n+k-1)!}{n!(k-1)!})$, where each a_{ij} is an integer multiplied by some a_k.

Just as Δ is the ideal of E generated by $\{\frac{\partial \eta}{\partial x_i}\}$, so $(\Delta + m^k)/m^k$ is the ideal of J^{k-1} generated by $\{\frac{\partial z}{\partial x_i}\}$. Now J^k as a vector space has a basis $X_2, \ldots, X_\beta$. $(\Delta + m^k)/m^k$ is now the vector subspace of J^{k-1} spanned by

$\{\frac{\partial z}{\partial x_i} X_j\}$. Let each $\frac{\partial z}{\partial x_i} X_j = \sum_{k=2}^{\bar{\beta}} a_{ij,k} X_k$, where each $a_{ij,k}$ is some $a_{\ell m}$.

We put M = the matrix $(a_{ij,k})$

= the coordinates of vectors spanning $(\Delta + m^k)/m^k$.

Now $\sigma(z) < K \Leftrightarrow \dim (\Delta+m^k)/m^k < K$

$\Leftrightarrow$ rank of $M < K$

$\Leftrightarrow$ all K-minors of M vanish.

And so Σ^k_{c+1} is given by polynomials in the $\{a_{ij,k}\}$, k.e. by polynomials in the $\{a_i\}$, each $a_i \in \mathbb{R}$. Hence Σ^k_{c+1} is a real algebraic variety in the real vector space I^k of dimension $\frac{(n+k)!}{n!k!} - n - 1$, itself a subspace of J^k which is $(\frac{(n+k)!}{n!k!} - 1)$-dimensional.

Corollary. I^k is the disjoint union $\Gamma^k_0 \cup \Gamma^k_1 \cup \ldots \cup \Gamma^k_{k-2} \cup \Sigma^k_{k-1}$, and each Γ^k_c is the difference $\Sigma^k_c - \Sigma^k_{c+1}$ between 2 algebraic varieties.

Recall that the map $\pi\colon m^2 \to I^k$ is equivariant with respect to G, G^k;

$$\eta \longmapsto z$$

also the image of the orbit ηG is zG^k, a submanifold of I^k, as in the proof of Theorem 2.9.

Theorem 3.7. Let $\eta \in m^2$ and cod $\eta = c$ where $0 \le c \le k - 2$. Then zG^k is a submanifold of I^k of codimension c.

Proof. By Lemma 2.11, $T_z(zG^k) = \pi(m\Delta)$. $(\Delta = \Delta(\eta))$

By Lemma 3.1, $\det \eta - 2 \le \operatorname{cod} \eta = c \le k - 2$, by the hypotheses. So $\det \eta \le k$, i. e. η is k-determinate. By Theorem 2.9, $m^{k+1} \subset m\Delta$.

The codimension of zG^k in $I^k = \dim I^k - \dim \pi(m\Delta)$

$= \dim m^2/m^{k+1} - \dim m\Delta/m^{k+1}$

$= \dim m^2/m\Delta$.

Now $m/m\Delta = m/m^2 + m^2/m\Delta$, so $\dim m^2/m\Delta = \dim m/m\Delta - \dim m/m^2$. So the codimension of zG^k in $I^k = \dim m/m\Delta - \dim m/m^2$

$= \dim m/\Delta + \dim \Delta/m\Delta - \dim m/m^2$

$= \quad c \quad + \quad n \quad - \quad n \quad ,$

using the following lemma.

Lemma 3.8. If $\eta \in m^2$ and cod $\eta < \infty$, then $\dim \Delta/m\Delta = n$.

This completes the proof of the theorem.

Proof of Lemma 3.8. Since Δ is the ideal of E generated by $\{\frac{\partial\eta}{\partial x_i}\}$, every $\xi \in \Delta$ can be written as $\xi = \sum_{i=1}^{n} \alpha_i \frac{\partial\eta}{\partial x_i}$ where $\alpha_i \in E$, $\alpha_i = a_i + \mu_i$, $\mu_i \in m$, $a_i \in \mathbb{R}$. Then $\xi = \sum_{i=1}^{n} a_i \frac{\partial\eta}{\partial x_i}$ mod $m\Delta$. So $\{\frac{\partial\eta}{\partial x_i}\}$ span Δ over $\mathbb{R}$, mod $m\Delta$, and $\dim \Delta/m\Delta \le n$. It remains to prove $\dim \Delta/m\Delta \ge n$.

Suppose not, i.e. that $\dim \Delta/m\Delta < n$. Then $\{\frac{\partial\eta}{\partial x_i}\}$ are linearly dependent mod $m\Delta$. $\exists\ a_1, \ldots, a_n \in \mathbb{R}$, not all zero, such that

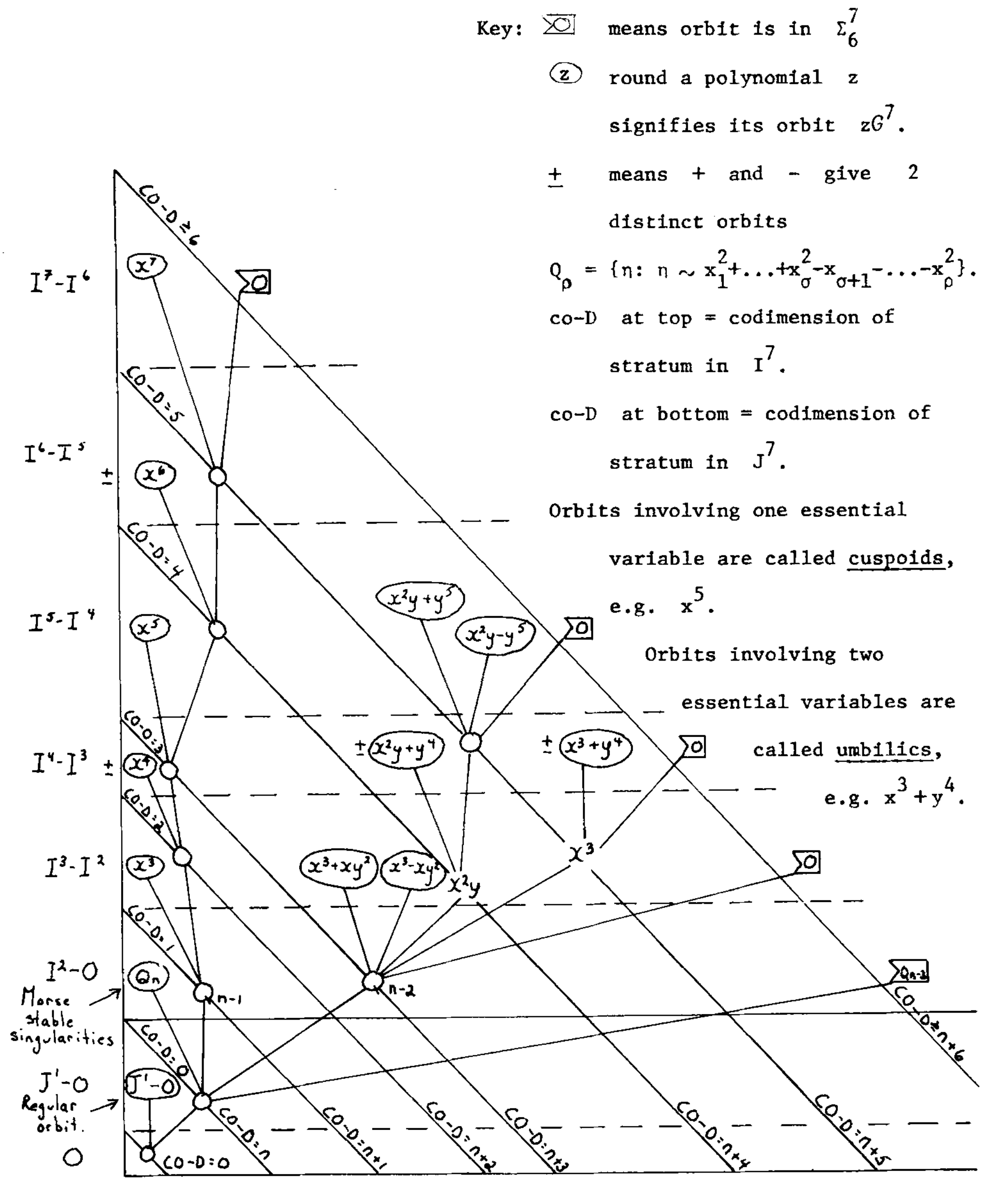
$$\sum_{i=1}^{n} a_i \frac{\partial\eta}{\partial x_i} = \sum_{i=1}^{n} \mu_i \frac{\partial\eta}{\partial x_i} \in m\Delta, \text{ some } \{\mu_i\} \in m.$$

Then $X\eta = \sum_{i=1}^{n} (a_i - \mu_i) \frac{\partial\eta}{\partial x_i} = 0$ where $X = \sum_{i=1}^{n} (a_i - \mu_i) \frac{\partial}{\partial x_i}$ is a vector field on a neighborhood of 0 in $\mathbb{R}^n$. X is nonzero at 0 because $\{\mu_i\} \in m$ and so vanish at 0 and $\{a_i\}$ are not all zero.

Change local coordinates so that $X = \frac{\partial}{\partial y_1}$ where $\{y_i\}$ are the new coordinates. Then $\frac{\partial\eta}{\partial y_1} = 0$. So $\eta = \eta(y_2, \ldots, y_n)$. Ess $\eta = \infty$ with respect to $\{y_i\}$. But det $\eta \ge$ ess η, by Corollary 2.14. By Lemma 3.1., cod $\eta = \infty$, . We have shown that $\dim \Delta/m\Delta = n$.

Theorem 3.7 justifies the notation cod η, as an abbreviation for codimension.

CHAPTER 4. CLASSIFICATION



<u>**Diagram 4.1.**</u> Classification of I^7 (and J^7).

This chapter will complete the above classification of I^7 as in Diagram 4.1. Supposing we already have our classification it follows that:

Theorem 4.2. In $I^7 - \Sigma_6^7$ there are exactly $16n - 7$ orbits under G^7.

Proof. We merely add the orbits in Diagram 4.1.

Stable singularities	$n + 1$
Cuspoids	$7n$
Umbilics	$8(n-1)$
	$16n - 7$

Corollary 4.3. If $0 \le c \le 5$, Γ_c^7 is a submanifold of I^7 of codimension c.

Proof. Γ_c^7 is the union of a finite number of orbits by Theorem 4.2. By Theorem 3.7, each of these is a submanifold of codimension c.

We note that our classification also gives that Σ_6^7 is the union of a finite number of parts each of codimension ≥ 6 in I^7. (See Diagram 4.1)

Theorem 4.4. $I^7 = \Gamma_0^7 \cup \Gamma_1^7 \cup \Gamma_2^7 \cup \Gamma_3^7 \cup \Gamma_4^7 \cup \Gamma_5^7 \cup \Gamma_6^7$ (disjoint) and each Γ_c^7 is of codimension c in I^7 and Σ_6^7 is of codimension 6 in I^7.

We shall now proceed with the classification.

Lemma 4.5. Let $\pi: m \to J^1 = m/m^2 \cong \mathbb{R}^n$, where $\pi = j^1|m$. Then $\pi^{-1}(J^1-0)$ is the orbit of regular germs.

Proof. Given $\eta \in m$, if $j^1\eta \neq 0$, then $\eta = \eta_1 +$ higher terms, where η_1 is a nonzero linear term. Then $\Delta = E$, as in Lemma 2.2. $m^2\Delta = m^2$ and by Theorem 2.9, η is 1-determinate. So $\eta \sim \eta_1 = \sum_{i=1}^{n} a_i x_i = y_j$ for some linear change of coordinates. Thus $\eta \sim x_1$ by the linear map sending y_j to x_1, and $\eta \in$ orbit of x_1 which as a function is regular.

These regular germs are precisely those with no singularity, or rather are not singularities. We observe that $J^7 = J^1 \times J^7/J^1$. Lemma 4.5 tells us that $(J^1-0) \times J^7/J^1$ is the regular orbit. The remainder, $0 \times J^7/J^1 = m^2/m^8 = I^7$, are the irregular orbits which we must classify.

$\eta \in m^2 \Rightarrow \eta = q +$ higher terms, where q is a quadratic form in

$\{x_i\}$, say

$$q(x_1,\dots,x_n) = \sum_{ij} a_{ij}x_ix_j \qquad (a_{ij} = a_{ji})$$

Write A for the matrix (a_{ij}), which is symmetric, and define rank η, the <u>Hermitian rank</u> of η or of $j^k\eta$ $(k \geq 2)$ to be rank A. Then $0 \leq \text{rank}\,\eta \leq n$.

<u>Lemma 4.6</u>. Let rank $\eta = \rho$. By an elementary theorem of linear algebra there is a linear change of coordinates such that $q = y_1^2 + y_2^2 + \dots + y_\sigma^2 - y_{\sigma+1}^2 - \dots - y_\rho^2$.

<u>Corollary 4.7</u>. $\eta \sim (x_1^2 + \dots - x_\rho^2)$ + higher terms, if rank $\eta = \rho$.

Let $Q_\rho = \{q\colon \text{Hermitian rank of } q = \rho\}$, in $I^2 = m^2/m^3$ which is diffeomorphic to $R^{\frac{1}{2}n(n+1)}$ because it is the linear space of all quadratic forms with coordinates $\{a_{ij}\}$, $i \leq j$. Then $I^2 = Q_n \cup Q_{n-1} \cup \dots \cup Q_0$.

<u>Lemma 4.8</u>. $Q_{n-\lambda}$ is a submanifold of I^2 of codimension $\frac{1}{2}\lambda(\lambda+1)$.

<u>Proof</u>. Each Q_ρ is a submanifold because each component is an orbit under the action of the general linear group.

Choose $q \in Q_\rho$. By Lemma 4.6. we may assume that $q = x_1^2 + \dots - x_\rho^2$. Then the associated matrix is $\begin{pmatrix} E & 0 \\ \underbrace{0}_{\rho} & \underbrace{0}_{\lambda} \end{pmatrix}$ $(\rho = n-\lambda)$, where $E = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{pmatrix}$. Suppose q' has matrix $\begin{pmatrix} A & B \\ \underbrace{B'}_{\rho} & \underbrace{C}_{\lambda} \end{pmatrix}$. $\exists$ a neighborhood N of q in I^2 such that if $q' \in N$, then $|A| \neq 0$. There rank q' = rank $\begin{pmatrix} A & B \\ B' & C \end{pmatrix}$

$$= \text{rank} \begin{pmatrix} A^{-1} & 0 \\ -B'A^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ B' & C \end{pmatrix}$$

$$= \text{rank} \begin{pmatrix} I & A^{-1}B \\ 0 & C - B'A^{-1}B \end{pmatrix}$$

Thus rank $q' = \rho \Leftrightarrow C = B'A^{-1}B$, i.e. the entries of C are determined by the entries of A and B. Then $Q\rho \cap N$ only has the freedom of the entries of A and B.

So the codimension of $Q_\rho = Q_{n-\lambda}$ is $\frac{1}{2}\lambda(\lambda+1)$, which is the number of free entries in symmetric C.

Now $I^7 = I^2 \times m^3/m^8 = (Q_n \times m^3/m^8) \cup \{(Q_\rho \times m^3/m^8)\}_{\rho<n}$. $Q_n \times m^3/m^8$ is the union of orbits of stable singularities (studied in Morse theory) and by Lemma 4.8. is an open set in I^7. It is in fact Γ_0^7 (clear).

Suppose now that rank $\eta = \rho$ and that as in Lemma 4.6. we have chosen coordinates $x_1, \ldots, x_n$ so that $\eta = x_1^2 + \ldots - x_\rho^2$ + higher terms. We call $x_1, \ldots, x_\rho$ the <u>dummy</u> variables and $x_{\rho+1}, \ldots, x_n$ the <u>essential</u> variables. The following lemma justifies these terms.

<u>Lemma 4.9</u>. (Reduction Lemma) Let $\eta \in m^2$ and $j^2\eta = q = x_1^2 + \ldots - x_\rho^2$. Then $\forall k, \exists \eta' \in m^2$ such that $\eta \sim \eta'$, and $j^k\eta' = q + p(x_{\rho+1},\ldots,x_n)$ where p is a polynomial in only the essential variables with $3 \le$ degree of monomials of $p \le k$.

<u>Proof</u>. Use induction on k. The lemma is true for $k = 2$. Suppose it is true for $k - 1$. In I^k, $j^k\eta' = q + p(x_{+1},\ldots,x_n) + \eta_k(x_1,\ldots,x_n)$, where $3 \le$degree of monomials of $p \le k - 1$, and η_k is homogeneous of degree k. Write $\eta_k = 2x_1P_1$ (all terms containing x_1; P_1 a homogeneous polynomial in $x_1, \ldots, x_n$ of degree $k - 1$)

$+ 2x_2P_2$ (all terms containing x_2, not x_1)

$+ 2x_3P_3$ (all terms containing x_3, not x_1, x_2)

$+ \ldots - 2x_{\sigma+1}P_{\sigma+1} - \ldots - 2x_\rho P_\rho$

$+ p_1(x_{\rho+1},\ldots,x_n)$ (all terms not containing dummy variables).

First incorporate the 2's and -'s into the $\{P_i\}$. Then let $y_i = \begin{cases} x_i + P_i & i \le \rho \\ x_i & i > \rho \end{cases}$

If $i \le \rho$, $y_i^2 = (x_i + P_i)^2 = x_i^2 + 2x_iP_i$ because monomials of degree $> k$ vanish in I^k. So $j^k\eta' = y_1^2 + \ldots - y_\rho^2 + p(y_{\rho+1}, \ldots, y_n) + p_1(y_{\rho+1}, \ldots, y_n)$, completing the lemma.

<u>Addendum 4.10</u>. The function $\eta \longmapsto p$ is well-defined because the construction is explicit.

Lemma 4.11. If rank $\eta \geq n-3$, then cod $\eta \geq 6$.

Proof. Either η is not finitely determinate, in which case cod $\eta = \infty$, (Lemma 3.1), or η is k-determinate, some k, i.e. $\eta \sim j^k\eta$, and $j^k\eta \sim q +$ p (essentials), by Lemma 4.9. Then cod η = cod (q+p). $\Delta(q+p) = (2x_1,\ldots,-2x_\rho,\frac{\partial p}{\partial x_{\rho+1}},\ldots,\frac{\partial p}{\partial x_n})$.

So cod $\eta = \dim m/\Delta(q+p)$

$\geq \dim m/(\Delta(q+p)+m^3)$

= number of the missing linear and quadratic terms in the essentials.

If n-rank $\eta = \lambda$, all λ linear terms are missing, as too are at least all but λ of the $\frac{1}{2}\lambda(\lambda+1)$ quadratic terms. So cod $\eta \geq \lambda + \frac{1}{2}\lambda(\lambda+1) - \lambda = \frac{1}{2}\lambda(\lambda+1)$. If rank $\eta \leq n-3$, then $\lambda \geq 3$ and cod $\eta \geq 6$.

We have that $\bigcup_{\rho\leq n-3} \{Q_\rho \times m^3/m^8\}$ consists of η with cod $\eta \geq 6$. By Lemma 4.8., this subspace has codimension 6 in I^7. It remains to investigate $Q_{n-1} \times m^3/m^8$ and $Q_{n-2} \times m^3/m^8$.

Lemma 4.12. (Classifying cuspoids) If rank $\eta = n-1$, then $\eta \sim q + x_n^k$, $3 \leq k \leq 7$, or cod $\eta \geq 6$.

Proof. By the reduction lemma 4.9., $\eta \sim \eta'$ where $j^7\eta' = q$ and p is a polynomial $p(x_n)$ with $3 \leq$ degree of monomials of $p \leq 7$. Let k be the least degree appearing, so that $p = a_k x_n^k + \ldots$. Then $j^k\eta'$ is k-determinate, because $\Delta(j^k\eta') = (x_1,\ldots,x_{n-1},x_n^{k-1})$ and so $m^2\Delta \supset m^{k+1}$, and we can use Theorem 2.9. Thus $\eta' \sim j^k\eta' = q + a_k x_n^k$

$= q + y_n^k$, changing coordinates so that $|a_k|^{1/k}x_n = y_n$.

If k is odd changing coordinates $y_n \to -y_n$ makes $\eta' = q - y_n^k \sim q + y_n^k$.

Classify as q + p where	p =	x_n^3	$\pm x_n^4$	x_n^5	$\pm x_n^6$	x_n^7	0
and cod(q+p)	=	1	2	3	4	5	.

Lemma 4.13. The cuspoids η with cod $\eta \geq 6$ form a submanifold of I^7 of codimension 6.

Proof. If η is a cuspoid, $j^2\eta = q = x_1^2 + \ldots - x_{n-1}^2$.

Write $m^3/m^8 = R \times S$, where R is the set of polynomials involving one of $x_1, \ldots, x_{n-1}$ and such that $3 \le$ degree of monomials in $r \in R \le 7$, and S is the set of polynomials in x_n only, so that $S \cong \mathbb{R}^5$. Then $j^7\eta = q + r + s$, $r \in R$, $s \in S$. The reduction lemma 4.9. gave a (unique algebraic) map $\theta: R \to S$ such that $\eta \sim \eta'$, and $j^7\eta' = q + 0 + (\theta r + s)$.

$$\begin{aligned} \operatorname{cod} \eta \ge 6 &\Leftrightarrow \operatorname{cod} \eta' \ge 6 \\ &\Leftrightarrow \theta r + s = 0 \\ &\Leftrightarrow s = -\theta r \\ &\Leftrightarrow (r,s) \in M_\theta, \end{aligned}$$

where M_θ is the graph of $-\theta$, and is a submanifold of $R \times S$ of codimension 5. (θ is algebraic and so graph $\theta \cong$ source of θ.) As q varies through Q_{n-1} we find that the required set of cuspoids η with $\operatorname{cod} \eta \ge 6$ form a bundle over Q_{n-1} (of codimension 1 in m^2/m^3 by Lemma 4.8) with fibre M_θ which has codimension 5 in m^3/m^8. Thus the bundle has codimension 6 in $m^2/m^8 = I^7$.

Now we classify the umbilics, $Q_{n-2} \times m^3/m^8$. Let $\eta \in m^2$ be such that $j^2\eta = q$, and $q = x_1^2 + \ldots - x_{n-2}^2$. By the reduction lemma 4.9., $\eta \sim \eta'$ where $j^3\eta' = q + p$ and p is a homogeneous cubic in x_{n-1}, x_n.

In place of x_{n-1}, x_n we shall use x, y respectively, for clarity. Note that Lemma 4.12., which classifies cuspoids, has been interpreted in this way in Diagram 4.1 with x replacing x_n.

Let $(x,y) \in \mathbb{R}^2$. The space of cubic forms in x, y is, $\{(a_1x^3+a_2x^2y+a_3xy^2+a_4y^3): a_1,a_2,a_3,a_4 \in \mathbb{R}\} = \mathbb{R}^4$. The action of $GL(2,\mathbb{R})$ on $\mathbb{R}^2$ induces an action on $\mathbb{R}^4$.

<u>Lemma 4.14</u>. There are 5 $GL(2,\mathbb{R})$-orbits in $\mathbb{R}^4$, and so each $p \in \mathbb{R}^4$ is equivalent to one of 5 forms:

	dimension	codimension
(1) $x^3 + y^3$ hyperbolic umbilic	4	0
(2) $x^3 - xy^2$ elliptic umbilic	4	0
(3) x^2y parabolic umbilic	3	1
(4) x^3 symbolic umbilic	2	2
(5) 0	0	4

Proof. Consider the roots x, y of $p(x,y) = 0$, $p \in \mathbb{R}^4$.

There are 5 cases (1) 2 complex, 1 real

(2) 3 real distinct

(3) 3 real, 2 same

(4) 3 real equal

(5) 3 equal to zero

Case (4): $p = (a_1x+a_2y)^3 = u^3$ by changing coordinates, $\begin{cases} u = a_1x + a_2y \\ v = \text{independent.} \end{cases}$

$\sim x^3$

Case (3): $p = u^2v$ where u, v are independent linear forms in x, y.

$\sim x^2y$

Case (2): $p = d_1d_2d_3$, product of 3 linear forms, $d_i = a_ix + b_iy$. We have $k_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \neq 0$ because the root of $d_2 \neq$ the root of d_3. Let

$$\left.\begin{aligned} u + v &= k_1d_1 = u' \\ u - v &= k_2d_2 = v' \end{aligned}\right\} \quad (*)$$

We claim this is a nonsingular coordinate change.

$u,v \longmapsto u', v'$ has a change of basis matrix with determinant $= \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$.

$x,y \longmapsto u', v'$ has a change of basis matrix with determinant $= k_1k_2 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

$= k_1k_2k_3 \neq 0$

Adding (*),
$$\begin{aligned} 2u &= k_1d_1 + k_2d_2 \\ &= (a_2b_3-a_3b_2)(a_1x+b_1y) + (a_3b_1-a_1b_3)(a_2x+b_2y) \\ &= x(a_1a_2b_3-a_1a_3b_2+a_2a_3b_1-a_1a_2b_3) + y(\ldots) \\ &= a_3x(a_2b_1-a_1b_2) + b_3y(a_2b_1-a_1b_2) \\ &= -k_3(a_3x+b_3y) \\ &= -k_3d_3. \end{aligned}$$

So $u^3 - uv^2 \sim 2u(u^2-v^2) = -k_1k_2k_3d_1d_2d_3 \sim p$. Thus $p \sim x^3 - xy^2$.

Case (1): This is the same as Case (2) except that $a_2 = \bar{a}_1$, $b_2 = \bar{b}_1$ and a_3, b_3 are real. $d_2 = \bar{d}_1$, $k_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = \begin{vmatrix} \bar{a}_1 & \bar{b}_1 \\ a_3 & b_3 \end{vmatrix} = -\bar{k}_2$,

$k_3 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1\bar{b}_1 - \bar{a}_1 b_1 = it,\ t \in \mathbb{R}$. Change coordinates, $\left.\begin{array}{l} iu + v = k_1 d_1 \\ iu - v = k_2 d_2 \end{array}\right\} (\ast)$.

We claim this is a real change. Adding, $2iu = k_3 d_3 = itd_3$ and td_3 is real. Subtracting, $2v = k_1 d_1 - k_2 d_2 = k_1 d_1 + \bar{k}_1 \bar{d}_1$, which is real. So both u and v are real. It is a non-singular change because $\begin{vmatrix} i & 1 \\ i & -1 \end{vmatrix} = -2i \neq 0$. The

product of $(\ast)$ is $2u(-u^2-v^2) = k_1 k_2 tp \sim p$. So $p \sim 2(u^3+uv^2)$, absorbing $-$ into the u-coordinate. $2(u^3+uv^2) \sim 2(u^3+3uv^2)$ absorbing $3^{\frac{1}{2}}$ into v.

$$\begin{aligned} &= u'^3 + v'^3 \quad \text{with} \quad \left.\begin{array}{l} u' = u + v \\ v' = u - v \end{array}\right\} \\ &\sim x^3 + y^3. \end{aligned}$$

By calculation $x^3 + y^3$ and $x^3 - xy^2$ are both 3-determinate and both $\operatorname{cod}(x^3+y^3)$ and $\operatorname{cod}(x^3-xy^2)$ equal 3. Thus the orbits corresponding to these are of codimension 3 in I^7 by Theorem 3.7.

<u>Lemma 4.15</u>. If $\eta = q + p$, $q \in Q_{n-2}$, $p = x^2y$ + higher terms, then either

(1) $\eta \sim q + (x^2y \pm y^4)$ and $\operatorname{cod} \eta = 4$ (the parabolic umbilic)

or (2) $\eta \sim q + (x^2y \pm y^5)$ and $\operatorname{cod} \eta = 5$.

or (3) η belongs to Σ_6^7.

<u>Proof</u>. If $k \geq 4$, then if $p = x^2y \pm y^k$, $\operatorname{cod} p = k = \det p$.

<u>Lemma 4.16</u>. If $k \geq 4$ and $j^{k-1}p = x^2y$ then $p \sim x^2y \pm y^k$, or $p \sim p'$ and $j^k p' = x^2y$.

Lemma 4.16. clearly gives Lemma 4.15.

<u>Proof of Lemma 4.16</u>. $j^k p = x^2y$ + a polynomial of degree k

$$= x^2y + ax^k + 2xyP + by^k,$$

where P is a homogeneous polynomial of degree $k - 2 \geq 2$.

$(x+P)^2(y+ax^{k-2}) = (x^2+2xP)(y+ax^{k-2}) = x^2y + 2xyP + ax^k$ in I^k. Put $u = (x+P)$ and $v = y + ax^{k-2}$; $v^k = y^k$ in I^k. So $j^k p = u^2v + bv^k$. There are two cases. $b \neq 0$: $j^k p \sim u^2v \pm v^k$ absorbing $|b|^{1/k}$ into v, and absorbing $1/|b|^{1/2k}$ into u. $b = 0$: $j^k p = u^2v \sim x^2y$.

Lemma 4.17. If $\eta = q + p$, $p \in Q_{n-2}$ and $p = x^3$ + higher terms in x, y, then either (1) $\eta \sim q + x^3 \pm y^4$ and $\operatorname{cod} \eta = 5$

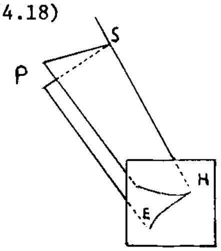
or (2) $\eta \in \Sigma_6^7$.

Proof. Calculation shows that $x^3 \pm y^4 = p'$ is 4-determinate and $\operatorname{cod} p' = 5$. $j^4p = x^3 + a_0x^4 + a_1x^3y + a_2x^2y^2 + a_3xy^3 + a_4y^4$. $a_4 \neq 0$: Put $v = y + \frac{a_3x}{4a_4}$. Then $j^4p = x^3 + 3x^2P + a_4v^4$, where P is a homogeneous polynomial of degree 2 in x, v. In I^4 $j^4p = (x+P)^3 + a_4v^4$

$\sim u^3 \pm v^4$, putting $u = x + P$ and absorbing $|a_4|^{1/4}$ into v.

$a_4 = 0$: As above we find that $j^4p \sim x^3 + xy^3$, which is 4-determinate as stated in Chapter 2. (This is Siersma's germ) In any case a short calculation gives $\operatorname{cod} \eta = \operatorname{cod}(x^3+xy^3) = 6$, so $\eta \in \Sigma_6^7$.

Lemma 4.14 and a straightforward calculation produce the following facts. The symbolic umbilic (S) is a twisted cubic curve of dimension 1 in R^3. The parabolic umbilic (P) is a quartic surface with a cusp edge along S. The elliptic umbilic (E) is inside the cusp. The hyperbolic umbilic (H) is outside the cusp. (4.18)



CHAPTER 5. THE PREPARATION THEOREM.

This chapter is self-contained and is devoted to proving a major result, the Preparation Theorem, which we need for Chapter 6.

The words "near 0" will always be understood to mean "in some neighborhood of 0."

Theorem 5.1. (Division Theorem) Let D be a C^∞ function defined near 0, from $\mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}$, such that $D(t,0) = d(t)t^k$ where $d(0) \neq 0$ and d is

C^∞ near 0 in $\mathbb{R}$. Then given any $C^\infty E: \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ defined near 0, $\exists$ C^∞ functions q and r such that: (1) $E = qD + r$ near 0 in $\mathbb{R}\times\mathbb{R}^n$, where (2) $r(t,x) = \sum_{i=0}^{k-1} r_i(x)t^i$ for $(t,x) \in \mathbb{R}\times\mathbb{R}^n$ near 0.

Notation. Let $P_k: \mathbb{R}\times\mathbb{R}^k \to \mathbb{R}$ be the polynomial $P_k(t,\lambda) = t^k + \sum_{i=0}^{k-1} \lambda_i t^i$.

Theorem 5.2. (Polynomial Division Theorem) Let $E(t,x)$ be a $\mathbb{C}$-valued C^∞ function defined near 0 in $\mathbb{R}\times\mathbb{R}^n$. Then $\exists$ $\mathbb{C}$-valued C^∞ functions $q(t,x,\lambda)$ and $r(t,x,\lambda)$ defined near 0 in $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^k$ satisfying:

(1) $E(t,x) = q(t,x,\lambda)P_k(t,\lambda) + r(t,x,\lambda)$, and

(2) $r(t,x,\lambda) = \sum_{i=0}^{k-1} r_i(x,\lambda)t^i$,

where each r_i is a C^∞ function defined near 0 in $\mathbb{R}^n\times\mathbb{R}^k$. Moreover if E is $\mathbb{R}$-valued, then q and r may be chosen $\mathbb{R}$-valued.

Note that if E is R-valued we merely equate real parts of (1) in Theorem 5.2 to give the last part.

Proof of Theorem 5.1 using Theorem 5.2. Given D, E we can apply Theorem 5.2 to find q_D, r_D, q_E, r_E such that $D = q_D P_k + r_D$ and $E = q_E P_k + r_E$; let now $r_D(t,x,\lambda) = \sum_{i=0}^{k-1} r_i^D(x,\lambda)t^i$ (*).

Now
$$t^k d(t) = D(t,0) = q_D(t,0)P_k(t,0) + r_D(t,0) \qquad (\lambda = 0)$$
$$= q_D(t,0)t^k + \sum_{i=0}^{k-1} r_i^D(0)t^i.$$

Comparing coefficients of powers of t, $r_i^D(0) = 0$ and $q_D(0) \neq 0$ $(d(0) \neq 0)$. Write $s_i(\lambda) = r_i^D(0,\lambda)$. We claim that $\left|\frac{\partial s_i(0)}{\partial\lambda_j}\right| \neq 0$.

$$t^k d(t) = D(t,0) = q_D(t,0,\lambda)(t^k + \sum_{i=0}^{k-1}\lambda_i t^i) + \sum_{i=0}^{k-1} s_i(\lambda)t^i.$$ Differentiating with respect to λ_j and setting $\lambda = 0$, $0 = \frac{\partial q_D}{\partial\lambda_j}(t,0)t^k + q_D(t,0)t^j + \sum_{i=0}^{k-1}\frac{\partial s_i}{\partial\lambda_j}(0)t^i$. Thus $\frac{\partial s_i}{\partial\lambda_j}(0) = 0$ if $i < j$ and $\frac{\partial s_i}{\partial\lambda_i}(0) = -q_D(0)$. So $(\frac{\partial s_i}{\partial\lambda_j}(0))$ is a lower triangular matrix, and as $q_D(0) \neq 0$, $\left|\frac{\partial s_i}{\partial\lambda_j}\right|(0) \neq 0$.

By the implicit function theorem, $\exists$ C^∞ functions $\theta_i(x)$ $(0 \le i \le k-1)$ such that (a) $r_j^D(x,\theta) \equiv 0$, and (b) $\theta(0) = 0$ (recall $r_j^D(0) = 0$). Let $\bar{q}(t,x) = q_D(t,x,\theta)$ and $P(t,x) = P_k(t,\theta)$. Then $D(t,x) = \bar{q}(t,x)P(t,x)$ (as $r_D(t,x,\theta) \equiv 0$ by (a).) As $\bar{q}(0) = q_D(0) \neq 0$, $P(t,x) = \dfrac{D(t,x)}{\bar{q}(t,x)}$ near 0 in $\mathbb{R} \times \mathbb{R}^n$.

By (*), $E(t,x) = q_E(t,x,\theta)P_k(t,\theta) + r_E(t,x,\theta) = q(t,x)D(t,x) + r(t,x)$, where $q(t,x) = \dfrac{q_E(t,x,\theta)}{\bar{q}(t,x)}$ and $r(t,x) = r_E(t,x,\theta) = \sum\limits_{i=0}^{k-1} r_i^E(x,\theta)t^i$. Finally let $r_i(x) = r_i^E(x,\theta)$.

Suppose $f\colon \mathbb{C} \to \mathbb{C}$, $f = u + iv$ and $u, v\colon \mathbb{C} \to \mathbb{R}$. If $z = x + iy$, then $\dfrac{\partial u}{\partial \bar{z}} = \dfrac{\partial u}{\partial x}\cdot\dfrac{\partial x}{\partial \bar{z}} + \dfrac{\partial u}{\partial y}\cdot\dfrac{\partial y}{\partial \bar{z}} = \dfrac{1}{2}\left[\dfrac{\partial u}{\partial x} + \dfrac{i\partial u}{\partial y}\right]$. A similar result for v gives us that

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \tag{5.3}$$

<u>Lemma 5.4</u>. Let $f\colon \mathbb{C} \to \mathbb{C}$ be C as a function $\mathbb{R}^2 \to \mathbb{R}^2$. Let γ be a simple closed curve in C whose interior is U. Then for $w \in U$,

$$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\,dz + \frac{1}{2\pi i}\iint_U \frac{\partial f}{\partial \bar{z}}(z)\,\frac{dz\wedge d\bar{z}}{z-w}\,.$$

(If f is holomorphic this reduces to the Cauchy Integral Formula since f is holomorphic $\Leftrightarrow \dfrac{\partial f}{\partial \bar{z}} \equiv 0$.)

<u>Proof</u>. Let $w \in U$ and choose $\varepsilon < \min\{|w-z| : z\in\gamma\}$. Let $U_\varepsilon = U -$ (disc radius ε about w), and $\gamma_\varepsilon = \partial U_\varepsilon$.

Recall Green's Theorem for $\mathbb{R}^2$. If $M, N\colon U_\varepsilon \to \mathbb{R}$ are C^∞ on γ_ε, then

$$\int_{\gamma_\varepsilon}(M\,dx + N\,dy) = \iint_{U_\varepsilon}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\wedge dy.$$

Green's Theorem and (5.3) for $f = u + iv$ give

$$\int_{\gamma\varepsilon} f\,dz = \int_{\gamma\varepsilon}(u+iv)(dx+idy) = 2i\iint_{U_\varepsilon}\frac{\partial f}{\partial \bar{z}}\,dx\wedge dy.$$

$2i\,dx\wedge dy = -dz\wedge d\bar{z}$, so $\displaystyle\int_{\gamma_\varepsilon} f\,dz = -\iint_{U_\varepsilon}\frac{\partial f}{\partial \bar{z}}\,dz\wedge d\bar{z}$ (*)

Apply (*) to $\frac{f(z)}{z-w}$, noting that $\frac{1}{z-w}$ is holomorphic on U.

$$-\iint_{U_\varepsilon} \frac{\partial f(z)}{\partial \bar{z}} \frac{dz\wedge d\bar{z}}{z-w} = \int_{\gamma_\varepsilon} \frac{f(z)}{z-w}\, dz = \int_{\gamma} \frac{f(z)}{z-w}\, dz - \int_{C_\varepsilon} \frac{f(z)}{z-w} dz, \qquad (\substack{*\\*})$$

where C_ε is the circle, radius ε, centre w.

With polar coordinates at w, $\int_{C_\varepsilon} \frac{f(z)}{z-w}\, dz = \int_0^{2\pi} f(w+\varepsilon e^{i\theta}) i d\theta$. As $\varepsilon \to 0$, R.H.S. of $(\substack{*\\*}) \to \int_\gamma \frac{f(z)}{z-w}\, dz - 2\pi i f(w)$, and L.H.S. of $(\substack{*\\*}) \to -\iint_U \frac{\partial f(z)}{\partial \bar{z}} \frac{dz\wedge d\bar{z}}{z-w}$. (The limit exists because $\frac{\partial}{\partial \bar{z}}$ is bounded on U, and $\frac{1}{z-w}$ is integrable over U.)

<u>Proof of Theorem 5.2</u>. Let $\tilde{E}(z,x,\lambda)$ be a C^∞ function defined near 0 in $\mathbb{C} \times \mathbb{R}^n \times \mathbb{C}^k$ such that $\tilde{E}(t,x,\lambda) = E(t,x)\ \forall\ t\ \mathbb{R}$, i.e. $\tilde{E}$ is an <u>extension</u> of E. Then $\tilde{E}(w,x,\lambda) = \frac{1}{2\pi i} \int_\gamma \frac{\tilde{E}(z)}{z-w}\, dz + \frac{1}{2\pi i} \iint_U \frac{\partial \tilde{E}(z)}{\partial z} \frac{dz\wedge d\bar{z}}{z-w}$, by Lemma 5.4. Let $P_k(z,\lambda) - P_k(w,\lambda) = (z-w)\sum_{i=0}^{k-1} p_i(z,\lambda) w^i$, i.e. $\frac{P_k(z,\lambda)}{z-w} = \frac{P_k(w,\lambda)}{z-w} + \sum_{i=0}^{k-1} p_i(z,\lambda) w^i$.

In the expression for $\tilde{E}(w,x,\lambda)$ multiply top and bottom inside the integrals by $P_k(z,\lambda)$ and expand $\frac{P_k(z,\lambda)}{z-w}$ giving $\tilde{E} = qP_k + r$ on $\mathbb{C} \times \mathbb{R}^n \times \mathbb{C}^k$ where

$$q(w,x,\lambda) = \frac{1}{2\pi i} \int_\gamma \frac{\tilde{E}(z,x,\lambda)}{P_k(z,\lambda)} \cdot \frac{dz}{(z-w)} + \frac{1}{2\pi i} \iint_U \frac{\partial \tilde{E}(z,x,\lambda).1.dz\wedge d\bar{z}}{\partial \bar{z}\ \ P_k(z,\lambda)(z-w)}$$

and $$r_i(x,\lambda) = \frac{1}{2\pi i} \int_\gamma \frac{\tilde{E}(z,x,\lambda)}{P_k(z,\lambda)} \cdot p_i(z,\lambda) \cdot dz + \frac{1}{2\pi i} \iint_U \frac{\partial \tilde{E}}{\partial \bar{z}}(z,x,\lambda) \cdot \frac{p_i(z,\lambda)}{P_k(z,\lambda)} \cdot dz \wedge d\bar{z},$$

so long as these integrals are well defined and yield C^∞ functions.

The first integral in the definition of both q and r is well-defined and C as long as the zeros of $P_k(z,\lambda)$ do not occur on the curve γ for λ near 0 in $\mathbb{C}^k$. Such a γ is easily chosen.

But U may contain zeros of P_k. So we need $\tilde{E}$ such that $\frac{\partial \tilde{E}}{\partial \bar{z}}$ vanishes on zeros of P_k and for real z to ensure q, r well-defined. As the integrands are bounded we need C^∞ $\tilde{E}$ such that $\frac{\partial \tilde{E}}{\partial \bar{z}}$ vanishes to infinite order on zeros of P_k and for real z to ensure q and r C^∞.

<u>Lemma 1</u>. (Nirenberg Extension Lemma) Let $E(t,x)$ be a C^∞ $\mathbb{C}$-valued function defined near 0 in $\mathbb{R} \times \mathbb{R}^n$. Then $\exists$ a C^∞ $\mathbb{C}$-valued function $\tilde{E}(z,x,\lambda)$ defined near 0 in $\mathbb{C} \times \mathbb{R}^n \times \mathbb{C}^k$ such that,

(1) $\tilde{E}(t,x,\lambda) = E(t,x)\ \forall\ t \in \mathbb{R}$.

(2) $\dfrac{\partial\tilde{E}}{\partial\bar{z}}$ vanishes to infinite order on $\{\operatorname{Im} z = 0\}$.

(3) $\dfrac{\partial\tilde{E}}{\partial\bar{z}}$ vanishes to infinite order on $\{P_k(z,\lambda) = 0\}$.

Lemma 2. (E. Borel's Theorem) Let $f_0, f_1, \ldots$ be a sequence of C^∞ functions on a given neighborhood N of 0 in $\mathbb{R}^n$. Then $\exists$ a C^∞ function $F(t,x)$ on a neighborhood of 0 in $\mathbb{R} \times \mathbb{R}^n$ such that $\dfrac{\partial^i F}{\partial t^i}(0,x) = f_i(x)\ \forall\ i$.

Proof. Let $\rho: \mathbb{R} \xrightarrow{C^\infty} \mathbb{R}$ be such that $\rho(t) = \begin{cases} 1 & |t| \le \frac{1}{2} \\ 0 & |t| \ge 1 \end{cases}$

Let $F(t,x) = \sum\limits_{i=0}^{\infty} \dfrac{t^i}{i!}\rho(\mu_i t) f_i(x)$, where $\{\mu_i\}$ is a rapidly increasing sequence of real numbers tending to ∞, so that F is C^∞ near 0.

(Lemma 2 may be used to show that for any power series about 0 in $\mathbb{R}^n$ $\exists$ a C^∞ real-valued function with its Taylor series at 0 the given power series.)

Lemma 3. Let V, W be complementary subspaces or $\mathbb{R}^n$ $(= V+W)$. Let g, h be C^∞ functions near 0 in $\mathbb{R}^n$, such that for all multi-indices α, $\dfrac{\partial^{|\alpha|}g(x)}{\partial x^\alpha} = \dfrac{\partial^{|\alpha|}h(x)}{\partial x^\alpha}\ \forall\ x \in V \cap W$. Then $\exists$ C^∞ F near 0 in $\mathbb{R}^n$, such that

$$\forall\alpha,\quad \frac{\partial^{|\alpha|}F(x)}{\partial x^\alpha} = \begin{cases} \dfrac{\partial^{|\alpha|}g(x)}{\partial x^\alpha} & x \in V\,. \\[2ex] \dfrac{\partial^{|\alpha|}h(x)}{\partial x^\alpha} & x \in W \end{cases}$$

(A multi-index $\alpha = (a_1,\ldots,a_n)$ and $|\alpha| = a_1 + \ldots + a_n$ so that

$$\frac{\partial^{|\alpha|}g(x)}{\partial x^\alpha} = \frac{\partial^{a_1+\ldots+a_n}g(x)}{\partial x_1^{a_1}\cdots\partial x_n^{a_n}}.)$$

Proof. Without loss of generality $h \equiv 0$, for if F_1 is the required extension for $(g-h)$ and 0, then $F = F_1 + h$ is the required extension for g and h.

Choose coordinates $y_1, \ldots, y_n$ so that $V \equiv y_1 = \ldots = y_j = 0$

and $W \equiv y_{j+1} = \dots = y_k = 0$. Let

$$F(y) = \sum_{|\alpha|=0}^{\infty} \frac{y^\alpha}{\alpha!} \frac{\partial^{|\alpha|} g}{\partial y^\alpha}(0,\dots,0,y_{j+1},\dots,y_n)\rho(\mu_{|\alpha|}\sum_{i=1}^{j} y_i^2), \text{ where } \rho \text{ is as in}$$

$\alpha = (a_1,\dots,a_j,0,\dots,0)$

Lemma 2 and $\{\mu_i\}$ increases to ∞ rapidly enough so that F is C^∞ near 0, If $y \in W$, each term of $\frac{\partial^{|\beta|}F(y)}{\partial y^\beta}$ contains a factor $\frac{\partial^{|\gamma|}g}{\partial y^\gamma}(0,\dots,0,y_{k+1},\dots,y_n)$. Since $(0,\dots,0,y_{k+1},\dots,y_n) \in V \cap W$, this factor $= 0$ $(h \equiv 0)$. So $\frac{\partial^{|\beta|}F(y)}{\partial y^\beta} = 0$.

If $y \in V$, note that

$$\frac{\partial^{|\gamma|}}{\partial y^\gamma}\rho(\mu_{|\alpha|}\sum_{i=1}^{j} y_i^2)\Big|_{y_1=\dots=y_j=0} = \begin{cases} 1 & \gamma = 0, \\ 0 & \gamma \neq 0 \end{cases}$$

and then

$$\frac{\partial^{|\beta|}F(y)}{\partial y^\beta} = \sum_{|\alpha|=0}^{\infty} \frac{\partial^{|\beta|}}{\partial y^\beta}\left[\frac{y^\alpha}{\alpha!}\frac{\partial^{|\alpha|}g(y)}{\partial y^\alpha}\right]\Bigg|_{y_1=\dots=y_j=0}.$$

If $b_i \neq a_i$ some $i \leq j$, then this term is 0. In fact the only nonzero term is $\frac{\partial^{|\beta|}g(y)}{\partial y^\beta}$.

<u>Lemma 4</u>. Let f be a C^∞ $\mathbb{C}$-valued function near 0 in $\mathbb{R}^n$ and let X be a vector field on $\mathbb{R}^n$ with $\mathbb{C}$ coefficients. Then $\exists$ C^∞ $\mathbb{C}$-valued F near 0 in $\mathbb{R} \times \mathbb{R}^n$ so that

(a) $F(0,x) = f(x) \quad \forall\, x \in R^n$.

(b) $\frac{\partial F}{\partial t}$ agrees to infinite order with XF at all $(0,x) \in \mathbb{R} \times \mathbb{R}^n$.

<u>Proof</u>. Try $\overline{F}(t,x) = e^{tX}f = \sum_{k=0}^{\infty} \frac{t^k}{k!} X^k f$. Differentiating termwise at $t = 0$ gives (b). Clearly (a) holds. To ensure that $\overline{F}$ is C^∞ use Lemma 2 to choose C^∞ F such that $F = \sum_{k=0}^{\infty} \frac{t^k}{k!} X^k f\rho(\mu_k t)$.

<u>Proof of Lemma 1</u>. We use induction on k. If $k = 0$, $P_k(z,\lambda) \equiv 1$, so we need C^∞ $\tilde{E}(z,x)$ such that $\tilde{E}(t,x) = E(t,x)$ $\forall\, t \in R$ and $\frac{\partial \tilde{E}}{\partial \bar{z}}(t,x)$ vanishes to infinite order $\forall\, t \in R$. Let $z = s + it$, $2\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial s} + i\frac{\partial}{\partial t}$. (Compare 5.3) Then Lemma 4 with $X = -i\frac{\partial}{\partial s}$ gives such an $\tilde{E}$.

Suppose Lemma 1 is proved for $k - 1$. We show $\exists$ $C^\infty F(z,x,\lambda)$ and $G(z,x,\lambda)$ such that

(1)' F and G agree to infinite order on $\{P_k(z,\lambda) = 0\}$

(2)' F is an extension of E.

(3)' $\frac{\partial F}{\partial \bar{z}}$ vanishes to infinite order on $\{\operatorname{Im} z = 0\}$.

(4)' Let $M = F|\{P_k(z,\lambda) = 0\}\cdot \frac{\partial M}{\partial \bar{z}}$ vanishes to infinite order on $\{\frac{\partial P_k}{\partial z}(z,\lambda) = 0\}$.

(5)' $\frac{\partial G}{\partial \bar{z}}$ vanishes to infinite order on $\{P_k(z,\lambda) = 0\}$.

<u>Existence of F and G proves Lemma 1.</u> Let $u = P(z,\lambda) \equiv P_k(z,\lambda)$ and $\lambda' = (\lambda_1,\ldots,\lambda_{k-1})$. Consider $(z,\lambda_0,\lambda') \to (z,u,\lambda')$ on $\mathbb{C}\times\mathbb{C}\times\mathbb{C}^{k-1}$. This is a valid coordinate change because $\frac{\partial u}{\partial \lambda_0} \equiv 1$. In the new coordinates, $\{P_k(z,\lambda) = 0\}$ is given by $u = 0$. By Lemma 3 $\exists\ \tilde{E}$ agreeing to infinite order with G on $u = 0$ and to infinite order with F on $\operatorname{Im} z = 0$. ($u = 0$ and $\operatorname{Im} z = 0$ intersect transversally in $\mathbb{R}^{2k+2}$.) (2)', (3)' and (5)' now imply $\tilde{E}$ is the desired extension of E.

<u>Existence of F and G.</u> Suppose we have that F exists. In (z,u,λ')-coordinates, $\frac{\partial}{\partial z}$ becomes $\frac{\partial}{\partial z} + \frac{\partial P}{\partial z}\frac{\partial}{\partial u}$, and $\frac{\partial}{\partial \bar{z}}$ becomes $\frac{\partial}{\partial \bar{z}} + \overline{\frac{\partial P}{\partial z}}\frac{\partial}{\partial \bar{u}}$. So in these coordinates we need $G(z,x,u,\lambda')$ such that

(a) $F = G$ to infinite order on $\{u = 0\}$, and

(b) $(\frac{\partial}{\partial \bar{z}} + \overline{\frac{\partial P}{\partial z}}\frac{\partial}{\partial \bar{u}})G = 0$ to infinite order on $\{u = 0\}$.

Let $X = -(\overline{\frac{\partial P}{\partial z}})^{-1}\frac{\partial}{\partial \bar{z}}$. As in Lemma 4 we must find C^∞ G satisfying (a) and

(b') $\frac{\partial G}{\partial \bar{z}} = XG$ to infinite order on $\{u = 0\}$. The formal solution is,

$$G = \sum_{i=0}^{\infty} \frac{(\bar{u})^i X^i M(z,x,\lambda')\rho(\mu_i|\bar{u}|^2)}{i!} \tag{*}$$

As $\frac{\partial M}{\partial \bar{z}} = 0$ to infinite order on $\{\frac{\partial P}{\partial \bar{z}}(z,\lambda') = 0\}$ by (4)', $X^i M$ is C^∞ in $(z,x,\lambda')\ \forall\ i$, so we can choose $\{\mu_i\}$ to increase quickly enough to make G C^∞.

We need only a C^∞ F so that in (z,x,u,λ')-coordinates,

(2)' $F(t,x,u,\lambda') = E(t,x)\ \forall\ t \in R$

(3)' $\frac{\partial F}{\partial \bar{z}} = XF$ to infinite order on $\{\operatorname{Im} z = 0\}$

(4)' If $M = F|\{u=0\}$, $\frac{\partial M}{\partial \bar{z}} = 0$ to infinite order on $\{\frac{\partial P_k}{\partial z} = 0\}$.

Consider $u = 0$ and the coordinate change $\lambda' = (\lambda_1,\ldots,\lambda_{k-1}) \longmapsto (\frac{\lambda_1}{1},\ldots,\frac{\lambda_{k-1}}{k-1}) = \lambda''$. The conditions are now that we find C^∞ $M(z,x,\lambda'')$ such that,

(I) $M(t,x,\lambda'') = E(t,x)$ $\forall$ $t \in R$

(II) $\frac{\partial M}{\partial \bar{z}}$ vanishes to infinite order on $\{\operatorname{Im} z = 0\}$, and

(III) " " " " " " $\{P_{k-1}(z,\lambda'') = 0\}$.

The induction hypothesis gives such a C^∞ $M(z,x,\lambda'')$, and we can view M as a C^∞ function of (z,x,λ').

Let $F(z,x,u,\lambda') = \sum_{i=0}^{\infty} \frac{(\bar{u})^i}{i!} X^i M(z,x,\lambda')\rho(\mu_i|\bar{u}|^2)$. Compare (*). By (III), $X^i M$ is C^∞ in z, x, λ', and so the $\{\mu_i\}$ may be chosen so that F is a C^∞ function satisfying (2)', (3)'. Also, on $u = 0$, $F = M$ and (III) gives (4)'.

The completes the proof of Lemma 1.

The remarks before Lemma 1 state that this suffices to prove the (Polynomial Division Theorem)Theorem 5.2.

Let π be projection $\mathbb{R}^{n+s} \to \mathbb{R}^s$. π induces $\pi^*\colon \mathcal{E}_s \to \mathcal{E}_{n+s}$, where $\mathcal{E}_s$ is the set of germs at 0 of C^∞ functions $\mathbb{R}^s \to \mathbb{R}$, as usual. Let M be an $\mathcal{E}_{n+s}$-module, and let $\underline{M}$ denote the same set regarded as an $\mathcal{E}_s$-module with structure induced by π^*.

<u>Theorem 5.5.</u> (Preparation Theorem) Suppose that

(1) M is a finitely generated $\mathcal{E}_{n+s}$-module,

(2) $M/(\pi^* m_s)M$ is a finite-dimensional real vector space.

Then $\underline{M}$ is finitely generated as an $\mathcal{E}_s$-module.

<u>Proof.</u> There are 2 steps.

Step 1. Let $\pi_1\colon \mathbb{R}^s \times \mathbb{R} \to \mathbb{R}^s$ and $t\colon \mathbb{R}^s \times \mathbb{R} \to \mathbb{R}$ denote the projections. We prove the theorem for $n = 1$, $\pi = \pi_1$. Let $v_1, \ldots, v_p$ be elements of M generating M as an $\mathcal{E}_{s+1}$-module, whose images in $M/(\pi^* m_s)M$ span this vector space. Then any $v \in M$ can be written $v = \sum_{i=1}^{p} a_i v_i + \sum_{i=1}^{p} \alpha_i v_i$ where $a_i \in \mathbb{R}$,

and $\alpha_i \in (\pi^*m_s)E_{s+1}$. In particular $\exists\ a_{ij} \in R$, $\alpha_{ij} \in (\pi^*m_s)E_{s+1}$ $(1 \le i,\ j \le p)$, such that $tv_i = \sum_{j=1}^{p}(a_{ij}+\alpha_{ij})v_j$. Let D be the determinant $|t\delta_{ij} - a_{ij} - \alpha_{ij}|$; by Cramer's rule $Dv_i = 0$, $i = 1, \dots, p$. Expanding the determinant we see that D is regular of order k, some $k \le p$, since $D|(0\times\mathbb{R},\theta)$ is a monic polynomial in t of order p ($\alpha_{ij}=0$ on $0\times\mathbb{R}$). Since $D.M = 0$, M is an $(E_{s+1}/D.E_{s+1})$-module.

Now D is regular of order k (i.e. $D(t,0) = d(t)t^k$, where $d(0) \neq 0$ and d is C^∞ near 0, and D is C^∞ defined near 0 in $\mathbb{R}^s \times \mathbb{R}$) and so using the Division Theorem 5.1., $E_{s+1}/D.\,E_{s+1}$ is finitely generated as an E_s-module.

Since M is finitely generated as an $(E_{s+1}/D.E_{s+1})$-module, it follows that $\underline{M}$ is finitely generated as an E_s-module.

Step 2. We complete the proof of the theorem. Factor π as follows:

$$\mathbb{R}^s \times \mathbb{R}^n \xrightarrow{\pi_n} \dots \xrightarrow{\pi_2} \mathbb{R}^s \quad \mathbb{R} \xrightarrow{\pi_1} \mathbb{R}^s,$$

where $\pi_i : \mathbb{R}^s \times \mathbb{R}^i \to \mathbb{R}^s \times \mathbb{R}^{i-1}$ is the germ of the projection,

$$(y,a_1,\dots,a_i) \longmapsto (y,a_1,\dots,a_{i-1}).$$

For each i, $0 \le i \le n + s$, we give M the E_{s+i}-module structure induced by $(\pi_{i+1}\circ\dots\circ\pi_n)^*$. If $i = 1$ this is the E_s-module structure of $\underline{M}$ since $\pi = \pi_1\circ\dots\circ\pi_n$.

Now we prove by decreasing induction on i that M is finitely generated as an E_{s+i}-module $\forall\ i$, $0 \le i \le n$. By hypothesis, it is true for $i = n$, so it suffices to carry out the inductive step. Assume M is finitely generated as an E_{s+i+1}-module.

$(\pi^*m_s)M = (\pi_1\circ\dots\circ\pi_{i+1})^*(m_s)M$. (On the L.H.S. M is regarded as an E_{n+s}-module, and on the R.H.S. as an E_{s+i+1}-module.) So $(\pi^*m_s)M \subset (\pi^*_{i+1}m_{s+i})M$. In particular $M/(\pi^*_{i+1}m_{s+i})M$ is finitely generated as a real vector space. In particular the hypotheses of the theorem are satisfied for π_{i+1} in place of π. Thus we may apply Step 1 to see that M is finitely generated as an E_{s+i}-module.

This completes the inductive step and also the proof as $i = 0$ is the statement of the theorem.

<u>Definition</u>. Let π be projection $\mathbb{R}^{n+s} \to \mathbb{R}^s$. A <u>mixed homomorphism over</u> π^* <u>of finite type</u> (a <u>mixture</u>) is a diagram:

$$\begin{tikzcd} & B \arrow[d, "\beta"] \\ A \arrow[r, "\alpha"] \arrow[d, dash] & C \arrow[d, dash] \\ E_s \arrow[r, "\pi^*"] & E_{n+s} \end{tikzcd}$$

where A is a finitely generated E_s-module,

B is an E_{n+s}-module,

C is a finitely generated E_{n+s}-module;

α is a module homomorphism over π^*, i.e. $\alpha(\eta a) = (\pi^*\eta)(\alpha a)$, $\eta \in E_s$ and $a \in A$; β is an E_{n+s}-module homomorphism.

<u>Corollary 5.6</u>. $C = \alpha A + \beta B + (\pi^* m_s)C \Rightarrow C = \alpha A + \beta B$.

<u>Proof</u>. Let $C' = C/\beta B$ and $\rho: C \to C'$ be the projection. As C is a finitely generated E_{n+s}-module so is C'. (1)

$(\pi^* m_s)C' = m_s\underline{C}'$, so $C'/(\pi^* m_s)C' = \underline{C}'/m_s\underline{C}'$. (2)

Our hypothesis $\Rightarrow C' = \rho\alpha A + (\pi^* m_s)C' \Rightarrow \underline{C}' = \underline{\rho\alpha A} + m_s\underline{C}'$ (3)

and this $\Rightarrow \underline{C}'/m_s\underline{C}'$ is a finitely generated E_s-module. Choose now a finite base $\{c_i\}$ for $\underline{C'}$ mod $m_s\underline{C}'$ as an E_s-module. Any $c \in \underline{C}'$ can be written,

$$c = \sum_i \eta_i c_i \bmod m_s\underline{C}' \qquad \text{(finite sum)} \qquad \eta_i \in E_s.$$

Now $\eta_i = \eta_i(0) + \eta_i'$, $\eta_i(0) \in \mathbb{R}$, $\eta_i' \in m_s$ in the notation of Lemma 2.8. So $c = \sum_i \eta_i(0)c_i \bmod m_s\underline{C}'$. Because c was arbitrary we have shown that $\underline{C}'/m_s\underline{C}'$ is a finite-dimensional vector space over $\mathbb{R}$, and hence by (2) so is $C'/(\pi^* m_s)C'$. (4)

(1) and (4) for C' are the two hypotheses of the Preparation Theorem 5.5, and so $\underline{C}'$ is a finitely generated E_s-module. We can now apply Nakayama's Lemma 2.10 with $A = E_s$, $\mathfrak{a} = m_s$, $M = \underline{C}'$ and $N = \underline{\rho\alpha A}$ to (3). Therefore $\underline{C}' = \underline{\rho\alpha A}$.

And so $C' = \rho\alpha A$, i.e. $C = \alpha A + \beta B$.

CHAPTER 6. UNFOLDINGS

We defint the category of unfoldings of η, for fixed $\eta \in m^2$. An <u>object</u> (r,f) is a germ $f\colon \mathbb{R}^n \times \mathbb{R}^r, 0 \to \mathbb{R}, 0$ (shorthand for "is a germ f of a C^∞ function $\mathbb{R}^n \times \mathbb{R}^r,0 \to \mathbb{R},0$"), such that $f|\mathbb{R}^n \times 0 = \eta$, i.e.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\eta"] \arrow[d, "1\times 0"'] & \mathbb{R} \arrow[d, "1"] \\ \mathbb{R}^{n+r} \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$ commutes.

A <u>morphism</u> $(\phi,\bar\phi,\varepsilon)\colon (s,g) \to (r,f)$ is a germ $\phi\colon \mathbb{R}^{n+s}, 0 \to \mathbb{R}^{n+r},0$,

$$\begin{tikzcd} \mathbb{R} \arrow[d, "1"'] & \mathbb{R}^n \arrow[l] \arrow[r, "1"] \arrow[d, "1\times 0"] & \mathbb{R}^n \arrow[r] \arrow[d, "1\times 0"'] & \mathbb{R} \arrow[d, "1"] \\ \mathbb{R} & \mathbb{R}^{n+s} \arrow[l] \arrow[r, "\phi"] \arrow[d, "\pi_s"] & \mathbb{R}^{n+r} \arrow[r] \arrow[d, "\pi_r"] & \mathbb{R} \\ & \mathbb{R}^s \arrow[r, "\bar\phi"] & \mathbb{R}^r & \end{tikzcd}$$

a germ $\bar\phi\colon \mathbb{R}^s,0 \to \mathbb{R}^r,0$,

a sheer germ $\varepsilon\colon \mathbb{R}^s,0 \to \mathbb{R},0$, (6.1)

such that $\phi|\mathbb{R}^n \times 0 = 1$, and

if $\pi_r\colon \mathbb{R}^{n+r} \to \mathbb{R}^r$ is projection,

$\pi_r\phi = \bar\phi\pi_s$ and $g = f\phi + \varepsilon\pi_s$.

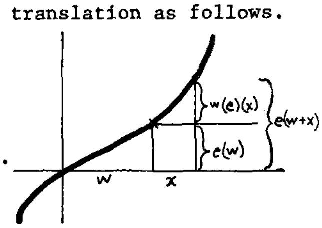
<u>Definition</u>. (r,f) is said to be <u>universal</u> if, $\forall$ (s,g) $\exists$ a morphism, $(s,g) \to (r,f)$.

<u>Definition</u>. $(\phi,\bar\phi,\varepsilon)$ is an <u>isomorphism</u> if it has an inverse. Note that this requires $r = s$, and ϕ and $\bar\phi$ are diffeomorphism-germs, so $(\phi^{-1},\bar\phi^{-1},-\varepsilon\bar\phi^{-1})$ will do.

<u>Prolongation of a germ</u>. Given $\eta \in m^2$, let $z = j^k\eta$. Choose a representative function of η, $e\colon \mathbb{R}^n,0 \to \mathbb{R},0$. $\mathbb{R}^n$ operates on e by translation as follows.

Given $w \in \mathbb{R}^n$, define $w(e)\colon \mathbb{R}^n,0 \to \mathbb{R},0$

$$x \mapsto e(w+x) - e(w).$$

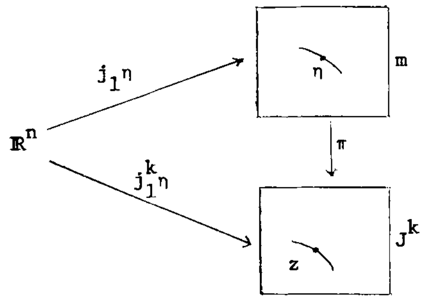


Graph $w(e)$ = graph e with origin moved to $(w,e(w))$.

Denote by j_1e the map obtained: $\mathbb{R}^n,0 \to m,\eta$

$$w \mapsto \text{germ at } 0 \text{ of } w(e).$$

Let $j_1\eta$ denote the germ at 0 of j_1e (we shall show this is unambiguous). $j_1\eta$ is called the <u>natural germ prolongation</u> of η. $j_1^k\eta = \pi o j_1\eta$ is called the <u>natural k-jet prolongation</u> of η, where π is the usual projection $m \to J^k$.



Lemma 6.2. (1) $j_1\eta$ and $j_1^k\eta$ are uniquely determined by η (not by z, necessarily), i.e. they are independent of the choice of e.

(2) If η is (k+1)-determinate, $j_1^k\eta$ is the germ of an embedding $\mathbb{R}^n,0 \to J^k,z$.

(3) The tangent plane $T_z(\text{im } j_1^k\eta)$ lies in $\pi\Delta(\Delta=\Delta(\eta))$ transverse to $\pi(m\Delta)$, and is spanned by $\{j^k \frac{\partial\eta}{\partial x_i}\}$.

Proof. If e and e' are 2 representatives of η, then $e = e'$ on N, some neighborhood of 0 in $\mathbb{R}^n$. $w(e) = w(e')$ if $w + x$, $w \in N$. So $j_1\eta$ is well defined (and clearly $j_1^k\eta$ is too). (1) is proved. (2) follows using (3) and the definition of determinacy. (3). Clearly $T_z(\text{im } j_1^k\eta)$ is spanned by $j^k\{\frac{\partial\eta}{\partial x_i}\}$ which are in $\pi\Delta$. By the definition of Δ (the ideal generated by $\{\frac{\partial\eta}{\partial x_i}\}$), $m\Delta$ and the space spanned by $\{\frac{\partial\eta}{\partial x_i}\}$ are transversal in Δ (use Lemma 3.8). Quotient out by m^{k+1}. Hence $T_z(\text{im } j_1^k\eta)$ is transverse to $\pi(m\Delta)$ in $\pi\Delta$.

We define the k-jet prolongation of an unfolding (r,f) of a germ $\eta \in m^2$ in a similar way. Represent f by a function $\tilde{f}\colon \mathbb{R}^{n+r},0 \to \mathbb{R},0$. Let F be the germ at 0 of the map $\mathbb{R}^{n+r},0 \to J^k,z$

$$(x',y') \mapsto \text{k-jet at } 0 \text{ of the function}$$
$$\mathbb{R}^n,0 \to \mathbb{R},0$$
$$x \mapsto \tilde{f}(x'+x,y') - \tilde{f}(x',y')$$

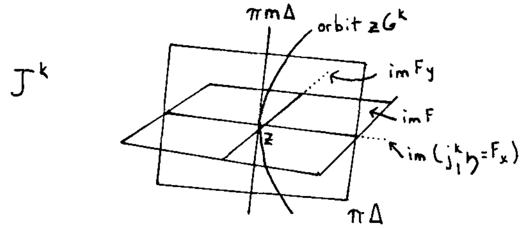
F is the <u>k-jet prolongation of the unfolding</u> (r,f).

<u>Definition</u>. We say the unfolding (r,f) is <u>k-transversal</u> if the germ F is transversal to the orbit zG^k in J^k.

Let $x_1, \ldots, x_n$ be coordinates for $\mathbb{R}^n$ and $y_1, \ldots, y_r$ be coordinates for $\mathbb{R}^r$. Choose $\tilde{f} \in f$, and then for each $j = 1, \ldots, r$ we have a function $\frac{\partial \tilde{f}}{\partial y_j}$ from $\mathbb{R}^{n+r}$, 0 to $\mathbb{R}$, $\frac{\partial \tilde{f}}{\partial y_j}(0,0)$. Let $\partial_j f$ be the germ at p of $\frac{\partial \tilde{f}}{\partial y_j}|\mathbb{R}^n \times 0 - \frac{\partial \tilde{f}}{\partial y_j}(0,0)$. $\partial_j f$ is in m. V_f will denote the vector subspace of m spanned by $\partial_1 f, \ldots, \partial_r f$. (6.3)

<u>Lemma 6.4</u>. An unfolding (r,f) of a germ η is k-transversal $\Leftrightarrow$ $m = \Delta + V_f + m^{k+1}$.

<u>Proof</u>. In J^k, i.e. mod m^{k+1}, the tangent to the orbit zG^k is $m\Delta$ (Lemma 2.11), the tangent to the x-direction of $F(=j_1^k\eta)$ is $T_z(\text{im } j_1^k\eta)$, and these two are transverse in Δ by Lemma 6.2 (3). The tangent to the y-direction of F is V_f (6.3). So F is transversal to $zG^k \Leftrightarrow \Delta + V_f$ span m mod m^{k+1}.



<u>Corollary 6.5</u>. Let η have finite determinacy and $c = \text{cod } \eta$, then $\exists$ an unfolding (c,f), which is k-transversal $\forall\, k > 0$.

<u>Proof</u>. Because $\det \eta$ is finite, so is $\text{cod } \eta = c$ finite by Lemma 3.1; by definition $\text{cod } \eta = \dim m/\Delta$ $(\Delta=\Delta(\eta))$. Choose $u_1, \ldots, u_c \in m$ such that their images in m/Δ form a basis for m/Δ. Define an unfolding (c,f) by,

$$f: \mathbb{R}^n \times \mathbb{R}^c \to \mathbb{R} \qquad (x,y) \to \eta(x) + \sum_{j=1}^{c} y_j u_j(x).$$

Then $\frac{\partial f}{\partial y_j} = u_j(x)$, so $\partial_j f = \frac{\partial f}{\partial y_j}|\mathbb{R}^n\times 0 - \frac{\partial f}{\partial y_j}(0,0) = u_j(x)$. $(u_j \in m)$

By the choise of $\{u_j\}$, $\{\partial_j f\}$ span m/Δ. By 6.3 $m = \Delta + V_f = \Delta + V_f + m^{k+1}$ $\forall\, k > 0$. Now apply Lemma 6.4.

Lemma 6.6. Let η have finite determinacy, with a universal unfolding (r,f). Then (r,f) is k-transversal $\forall\ k > 0$ and $r \geq \operatorname{cod} \eta$.

Proof. Let $c = \operatorname{cod} \eta$ and (c,g) be the unfolding of Corollary 6.5, which is k-transversal $\forall\ k > 0$. By the definition of universality $\exists$ a morphism $(\phi,\bar{\phi},\varepsilon): (c,g) \to (r,f)$. So $g(x,y) = f(\phi(x,y)) + \varepsilon(y)$ where $(x,y) \in \mathbb{R}^n \times \mathbb{R}^c$, by (6.1. $= f(\phi^y x,\bar{\phi}y) + \varepsilon(y)$ with $\phi^y x = \pi_{x'}(\phi(x,y))$, choosing $x_1', \ldots, x_n'$ and $y_1', \ldots, y_r'$ as coordinates for $\mathbb{R}^{n+r}$. Now we have

$$\frac{\partial g}{\partial y_j}(x,0) = \sum_i \frac{\partial f}{\partial x_1'}(\phi^0 x,\bar{\phi}0)\frac{\partial \phi_i^y}{\partial y_j}(x) + \sum_h \frac{\partial f}{\partial y_h'}(\phi^0 x,\bar{\phi}0)\frac{\partial \bar{\phi}_h}{\partial y_j}(0) + \frac{\partial \varepsilon}{\partial y_j}(0).$$

$\phi^0 = \phi|\mathbb{R}^n \times 0 = 1$ and $\bar{\phi}0 = 0$ by 6.1. Also $\dfrac{\partial \phi_i^y}{\partial y_j} \in E$ and $\dfrac{\partial \bar{\phi}_h}{\partial y_j}(0) \in \mathbb{R}$. So the first sum is in Δ, as $\dfrac{\partial f}{\partial x_1'}(x,0) = \dfrac{\partial \eta}{\partial x_i}(x)$, and the h^{th} term in the second sum is $\dfrac{\partial f}{\partial y_h'}(x,0) \times$ constant. Remember $\partial_h f = \dfrac{\partial f}{\partial y_h'}(x,0) - \dfrac{\partial f}{\partial y_h'}(0,0) \in V_f$. So $V_g \subset \Delta + V_f$.

Now $m = \Delta + V_g\ \forall\ k > 0$ by Lemma 6.4.

So $m \subset \Delta + V_f\ \forall\ k > 0$, i.e. (r,f) is k-transversal $\forall\ k > 0$ by Lemma 6.4, $(\Delta, V_f \subset m)$. Also $r \geq \dim V_f \geq \dim m/\Delta = c$, follows at once.

Lemma 6.7. If η is k-determinate and if (r,f) and (r,g) are k-transversal unfoldings of η, then they are isomorphic.

Proof. (r,f) is k-transversal $\Rightarrow m = \Delta + V_f + m^{k+1}$ (Lemma 6.4)
η is k-determinate $\Rightarrow m^{k+1} \subset m\Delta \subset \Delta$ (Theorem 2.9) $\Big\}$ So $m = \Delta + V_f$. (6.8)

Let $\overline{\partial_j f}$ denote the image of $\partial_j f$ in m/Δ. Then (r,f) k-transversal means $\overline{\partial_j f}$ spans m/Δ. (r,f) and (r,g) are isomorphic if $\exists$ a morphism $(\phi,\bar{\phi},\varepsilon): (r,f) \to (r,g)$ where ϕ, $\bar{\phi}$ are diffeomorphisms. We write $f \cong g$.

Lemma 1. It suffices to prove Lemma 6.7 in the special case $\overline{\partial_j f} = \overline{\partial_j g}\ \forall j$.

Proof. We introduce a standard unfolding (r,h) and show that $\exists\ h' \cong h$ such that $\overline{\partial_j h'} = \overline{\partial_j f}$, $j = 1, \ldots, r$. By symmetry $\exists$ also $h'' \cong h$ such that $\overline{\partial_j h''} = \overline{\partial_j g}$, $1 \leq j \leq r$. Assuming the special case of Lemma 6.7, $f \cong h' \cong h \cong h'' \cong g$.

Choose $u_1, \ldots, u_c \in m$ such that $\bar{u}_1, \ldots, \bar{u}_c$ form a base for m/Δ, where $c = \operatorname{cod} \eta$, finite since $\det \eta$ is finite. Define $h: \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^{r-c} \times \mathbb{R}$

$$(x,v,w) \to \eta(x) + \sum_{j=1}^{c} v_j u_j(x) = \eta + vu,$$

where $v = (v_1 \ldots v_c)$, $u = \begin{pmatrix} u_1 \\ \vdots \\ u_c \end{pmatrix}$. $(w_1, \ldots, w_{r-c}$ are disconnected control coordinates, see below.)

Now $\overline{\partial_j f} = \sum_{h=1}^{r} a_{jh} \bar{u}_h$, $a_{jh} \in \mathbb{R}$. Denote the matrix (a_{jh}) by A. A has rank c since $\overline{\partial_j f}$ span m/Δ. Choose a matrix B such that AB is nonsingular, where AB is,

$$\begin{array}{c|c|c|} & c & r-c \\ \hline r & A & B \\ \hline \end{array}$$

(not the matrix product)

Define $\bar{\phi}: \mathbb{R}^r \to \mathbb{R}^c \times \mathbb{R}^{r-c}$, a linear isomorphism

$$y \mapsto (yA, yB).$$

This induces $h': \mathbb{R}^{n+r} \xrightarrow{\phi = 1\times\bar{\phi}} \mathbb{R}^{n+r} \xrightarrow{h} \mathbb{R}$. $(1\times\bar{\phi}, \bar{\phi}, 0): (r,h') \to (r,h)$ is

$$(x,y) \to (x, yA, yB) \mapsto \eta(x) + yAu$$

clearly an isomorphism, $\partial_j h = \begin{cases} u_j(x) & j \le c, \\ 0 & j > c. \end{cases}$ $\partial_j h' = \sum_{h=1}^{r} a_{jh} u_h(x)$. So $\overline{\partial_j h'} = \sum_{h=1}^{r} a_{jh} \bar{u}_h(x) = \overline{\partial_j f}$.

<u>Lemma 2</u>. $m_s \mathcal{E}_{n+s}$ = those germs in $\mathcal{E}_{n+s}$ vanishing on the $\mathbb{R}^n$-axis.

<u>Proof</u>. $\subseteq$: m_s is generated by $\{y_j\}$ which vanish on the $\mathbb{R}^n$-axis, where $x_1, \ldots, x_n$ are coordinates for $\mathbb{R}^n$ and $y_1, \ldots, y_s$ are coordinates for $\mathbb{R}^s$.

$\supseteq$: Suppose the function $\theta(x,y)$ vanishes on the $\mathbb{R}^n$-axis.

$$\theta(x,y) = [\theta(x,ty)]_0^1 = \int_0^1 \frac{\partial\theta}{\partial t}(x,ty)dt = \int_0^1 \sum_j \frac{\partial\theta}{\partial y_j}(x,ty) y_j dt$$

$$= \sum_j y_j \psi_j(x,y), \quad \psi_j \in \mathcal{E}_{n+s}.$$

The continuing proof of Lemma 6.7 now mimics the first half of Theorem 2.9. Let $E^t = (1-t)f + tg$. Then assuming $\overline{\partial_j f} = \overline{\partial_j g}$,

$\overline{\partial_j E^t} = (1-t)\,\overline{\partial_j f} + \overline{t\partial_j g} = \overline{\partial_j f}$. So E^t is k-transversal. For $0 \le t \le 1$ we have a 1-parameter family of k-transversal unfoldings connecting f and g. Fix t_o, $0 \le t_o \le 1$.

Lemma 3. $\exists$ an isomorphism $(\phi^t, \bar{\phi}^t, \varepsilon^t): (r, E^{t_o}) \to (r, E^t)$, $\forall\, t$ in some neighborhood of t_o.

This implies Lemma 6.7 by the compactness and connectedness of $[0,1]$ (Cf. 2.9).

Lemma 4. $\exists$ a germ ϕ at $(0,t_o)$ of a map $\mathbb{R}^{n+r} \times \mathbb{R},\ 0 \times \mathbb{R} \to \mathbb{R}^{n+r},\ 0$.
" " " $\bar{\phi}$ " " " " " $\mathbb{R}^r \times \mathbb{R},\ 0 \times \mathbb{R} \to \mathbb{R}^r,\ 0$.
" " " ε " " " " function $\mathbb{R}^r \times \mathbb{R},\ 0 \times \mathbb{R} \to \mathbb{R}, 0$, such that

(1) $\phi^{t_o} = 1$ (so $\bar{\phi}^{t_o} = 1$), and $\varepsilon^{t_o} = 0$, and $\forall$ t in a neighborhood of t_o,
(2) $\phi^t|\mathbb{R}^n \times 0 = 1$; ϕ^t, $\bar{\phi}^t$ commute with $\pi: \mathbb{R}^{n+r} \to \mathbb{R}^r$, and (3) $E^t\phi^t + \varepsilon^t\pi = E^{t_o}$.
(i.e. $E(x',y',t) + \varepsilon(y,t) = E(x,y,t_o)$, where $\phi^t(x,y) = (x',y')$.

Lemma 4 $\Rightarrow$ Lemma 3 because the set of diffeomorphisms is open in the space of maps. (See proof of 2.9)

Lemma 5. We can replace (3) by

(4) $\sum_i \frac{\partial E}{\partial x_i'}(x',y',t)\,\frac{\partial x_i'}{\partial t}(x,y,t) + \sum_j \frac{\partial E}{\partial y_j'}(x',y',t)\,\frac{\partial y_j'}{\partial t}(y,t) + \frac{\partial E}{\partial t}(x',y',t) + \frac{\partial \varepsilon}{\partial t}(y,t) = 0.$

Differentiation of (3) with respect to t gives (4). Integration with respect to t from t_o to t of (4) gives (3). (See 2.9)

Lemma 6. $\exists$ a germ X at $(0,t_o)$ of a map $\mathbb{R}^{n+r} \times \mathbb{R},\ \mathbb{R}^n \times 0 \times \mathbb{R} \to \mathbb{R}^n,\ 0$,
" " " Y " " " " " $\mathbb{R}^r \times \mathbb{R},\ 0 \times \mathbb{R} \to \mathbb{R}^r,\ 0$,
" " " Z " " " " function $\mathbb{R}^r \times \mathbb{R},\ 0 \times \mathbb{R} \to \mathbb{R},\ 0$ such that

(5) $\sum_i \frac{\partial E}{\partial x_i}(x,y,t)X_i(x,y,t) + \sum_j \frac{\partial E}{\partial y_j}(x,y,t)Y_j(y,t) + \frac{\partial E}{\partial t}(x,y,t) + Z(y,t) = 0$, $\forall$ (x,y,t) in a neighborhood of $(0,t_o)$. $(\frac{\partial E}{\partial x}.X + \frac{\partial E}{\partial y}.Y + \frac{\partial E}{\partial t} + Z = 0)$.

<u>Proof that Lemma 6 $\Rightarrow$ Lemma 5.</u>

Let $(x',y') = \phi(x,y,t)$ be the unique solution of $\begin{cases} \dot{x}' = X(x',y',t), & x' = x \text{ at } t = t_o \\ \dot{y}' = Y(y',t), & y' = y \text{ at } t = t_o. \end{cases}$

Let $y' = \bar{\phi}(y,t)$ " " " " "

Let $\varepsilon(y,t) = \int_{t_o}^{t} Z(\bar{\phi}(y,\tau),\tau)\,d\tau$. So $\frac{\partial \varepsilon}{\partial t}(y,t) = Z(y',t)$. Substitute x', y', t for variables x, y, t in (5) and get (4).

$\phi^t|\mathbb{R}^n \times 0 = 1$ since $(x', y') = (x,0)$ is a constant solution of

$$\begin{cases} X(\mathbb{R}^n \times 0 \times \mathbb{R}) = 0 = \dot{x}', \\ Y(0 \times \mathbb{R}) \qquad = 0 = \dot{y}'. \end{cases}$$

We now choose a mixture. Let A be a free E_{r+1}-module on $(r + 1)$ variables (finitely generated), each $a = (Y_1,\dots,Y_r,Z)$, some Y_j, $Z \in E_{r+1}$. Let B be a free E_{n+r+1}-module on n variables, each $b = X = (X_1,\dots,X_n)$, some $X_i \in E_{n+r+1}$. Let C be E_{n+r+1} (finitely generated).

$\alpha: A \to C$ is given by $\alpha a = \frac{\partial E}{\partial y}.Y + Z$; it is over π^* because it is linear in Y, Z. (π is projection $\mathbb{R}^{n+r+1} \to \mathbb{R}^{r+1}$)

$\beta: B \to C$ is given by $\beta X = \frac{\partial E}{\partial x}.X$. (Recall mixture of Chapter 5).

$$\begin{tikzcd} & B \arrow[d, "\beta"] \\ A \arrow[r, "\alpha"] \arrow[d, dash] & C \arrow[d, dash] \\ E_{r+1} \arrow[r, "\pi^*"] & E_{n+r+1} \end{tikzcd}$$

<u>Lemma 7.</u> $C = \alpha A + \beta B + (\pi^* m_{r+1})C$.

<u>Proof that Lemma 7 $\Rightarrow$ Lemma 6.</u> Apply Corollary 5.6 (to the Preparation Theorem) to give $C = \alpha A + \beta B$. Then $m_r C = \alpha(m_r A) + \beta(m_r B)$, where the E_r-module structures on C, A, B are induced by projection onto $\mathbb{R}^r$.

Now $\frac{\partial E}{\partial t} = g - f$. And $f|\mathbb{R}^n \times 0 = \eta = g|\mathbb{R}^n \times 0$ $(\forall t)$. So $\frac{\partial E}{\partial t}$ vanishes on $\mathbb{R}^n \times 0 \times \mathbb{R}$ in $\mathbb{R}^{n+r+1}$. By Lemma 2 $\frac{\partial E}{\partial t} \in m_r C$, and so $\frac{\partial E}{\partial t} \in \alpha(m_r A) + \beta(m_r B)$, i.e. $\exists$ germs $X \in m_r B$, Y and $Z \in m_r A$ such that $-\frac{\partial E}{\partial t} = \frac{\partial E}{\partial x}.X + \frac{\partial E}{\partial y}.Y + Z$, as germs. Lemma 6 follows applying Lemma 2 a few times.

<u>Proof of Lemma 7.</u> (And hence of Lemma 6.7) As E^t is k-transversal $\forall t$, by (6.8) $m_n = \Delta + V_{E^{t_o}}$. So $E_n = \Delta + V_{E^{t_o}} + \mathbb{R}$. Let $\xi \in C$, and $\underline{\xi}(x) = \xi(x,0,t_o) \in E_n$. Then $\underline{\xi}(x) = \sum_i \frac{\partial \eta}{\partial x_i}.\underline{X}_i + \sum_j \partial_j E^{t_o}.\underline{Y}_j + s$, where $\underline{X}_i \in E_n$, $\underline{Y}_j \in \mathbb{R}$ and $s \in \mathbb{R}$.

Let $\zeta(x,y,t) = \sum_i \frac{\partial E}{\partial y_i}(x,y,t)X_i(x,y,t) + \sum_j \frac{\partial E}{\partial y_j}(x,y,t)Y_j(y,t) -$

$$\underbrace{\sum_j \frac{\partial E}{\partial y_j}(0,0,t_o)Y_j(0,t_o) + s}_{Z\,\in\,\mathbb{R}}. \quad \text{So} \quad \zeta = \frac{\partial E}{\partial x}\cdot X + \frac{\partial E}{\partial y}\cdot Y + Z \in \beta B + \alpha A.$$

Now $\zeta(x,0,t_o) = \xi(x)$ because $E^{t_o}|\mathbb{R}^n \times 0 = \eta$ and also $\partial_j E^{t_o} = \frac{\partial E^{t_o}}{\partial y_j}|\mathbb{R}^n \times 0 - \frac{\partial E^{t_o}}{\partial y_j}(0)$. So $\xi - \zeta$ vanishes on the fibre $\mathbb{R}^n \times 0 \times t_o$. By Lemma 2 $\xi - \zeta \in (\pi^* m_{r+1})C$. Hence $\xi \in \alpha A + \beta B + (\pi^* m_{r+1})C$, proving Lemma 7.

Given an unfolding of η, (r,f), $f\colon \mathbb{R}^{n+r}, 0 \to \mathbb{R}, 0$, we introduce d disconnected controls as follows. Let g be the composition,

$\mathbb{R}^{n+r+d} = \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^d \to \mathbb{R}^{n+r} \to \mathbb{R}$

$(x,y,w) \mapsto (x,y) \mapsto f(x,y) = g(x,y,w).$

We say $(r+d,g)$ is (r,f) with d <u>disconnected controls</u>. Using the morphisms $(1\times\pi,\pi,0)\colon (r+d,g) \to (r,f)$ and $(1\times\iota,\iota,0)\colon (r,f) \to (r+d,g)$, where ι is the injection map, we see that (r,f) is universal $\Leftrightarrow$ $(r+d,g)$ is universal. Clearly also if (r,f) is k-transversal so is $(r+d,g)$.

<u>Theorem 6.9</u>. If η has finite determinacy, and has (r,f) and (r,g) as universal unfoldings, then they are isomorphic.

<u>Proof</u>. By Lemma 6.6, (r,f) and (r,g) are both k-transversal, $\forall\, k > 0$. Choose some k such that η is k-determinate. Then Lemma 6.7 provides an isomorphism.

<u>Theorem 6.10</u>. If η is k-determinate, then an unfolding (r,f) is universal $\Leftrightarrow$ it is k-transversal.

<u>Proof</u>. $\Rightarrow$ is Lemma 6.6.

Given a k-transversal unfolding (r,f) we must show that for any unfolding (s,g) (also of η), $\exists$ a morphism $(s,g) \to (r,f)$. If $c = \operatorname{cod} \eta$, choose $u_1, \ldots, u_c$ spanning m/Δ as in Corollary 6.5. Let h be the map

$\mathbb{R}^n \times \mathbb{R}^{s+c} \to \mathbb{R}$

$(x,y,v) \mapsto g(x,y) + \sum_{j=1}^{c} v_j u_j(x)$

so that $(s+c,h)$ is a k-transversal unfolding of η by Corollary 6.5.

Let $s + c + d = r + d'$, i.e. choose such integers d, d' (one can be zero). Let $(s+c+d,h')$ be $(s+c,h)$ with d disconnected controls, and $(r+d',f')$ be (r,f) with d' disconnected controls. Both will be k-transversal (as noted above), and we can apply Lemma 6.7 to show the existence of an isomorphism $(\phi,\bar{\phi},\varepsilon)$. We now have, $(s,g) \xrightarrow{1\times j_1, j_1, 0} (s+c,h) \xrightarrow{1\times j_2, j_2, 0} (s+c+d,h') \xrightarrow{\phi,\bar{\phi},\varepsilon}$ $(r+d',f') \xrightarrow{1\times\pi_r,\pi_r,0} (r,f)$, with j_1, j_2 obvious injections, π_r a projection. This is the required morphism.

Theorem 6.11. If η has finite determinacy, it has a universal unfolding (c,f) where $c = \operatorname{cod}\eta$, and moreover c is the minimum dimension of any universal unfolding of η.

Proof. By Corollary 6.5 a k-transversal unfolding (c,f) exists with $k \geq \det \eta$. (c,f) is universal by Theorem 6.10. Now use Lemma 6.6. for minimality.

CHAPTER 7. CATASTROPHE GERMS.

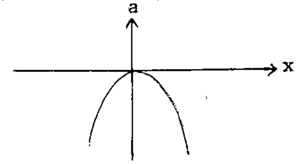
Let $\eta \in m^2$, and suppose η has an unfolding $f\colon \mathbb{R}^{n+r},0 \to \mathbb{R},0$. Represent f by a function $\tilde{f}\colon \mathbb{R}^{n+r},0 \to \mathbb{R},0$ and define M_f to be the subset of $\mathbb{R}^{n+r}$ on which $\frac{\partial\tilde{f}}{\partial x_1} = \ldots = \frac{\partial\tilde{f}}{\partial x_n} = 0$. Let the function $\tilde{X}_f$ be the composition $M_f \subset \mathbb{R}^{n+r} \xrightarrow{\pi_r} \mathbb{R}^r$. Observe that $0 \in M_f$ because $\eta \in m^2$. So we can define X_f to be the germ at 0 of $\tilde{X}_f$. X_f is called the catastrophe germ of f.

Lemma 7.1. Let $\eta \in m^3$ and $\operatorname{cod}\eta = c$. Then $\exists$ a universal unfolding (c,f) such that M_f is diffeomorphic to $\mathbb{R}^c$. Then X_f is a germ at 0 of a map $\mathbb{R}^c,0 \to \mathbb{R}^c,0$.

Proof. $\eta \in m^3 \Rightarrow \Delta \subset m^2$. And so when choosing a base $u_1, \ldots, u_c$ for m/Δ, we can demand that $u_j(x) = \begin{cases} x_j & \text{if } j \leq n \\ \text{a monomial of degree} \geq 2, & \text{if } n < j \leq c. \end{cases}$

Let $f(x,y) = \eta(x) + \sum_{j=1}^{c} y_j u_j(x)$; (c,f) is k-transversal $\forall\ k > 0$, and so is universal using Theorem 6.10 with $k \geq \det \eta$. $\frac{\partial f}{\partial x_i} = \frac{\partial \eta}{\partial x_i} + y_i + \sum_{j=n+1}^{c} y_j \frac{\partial u_j}{\partial x_i} =$ $0 \equiv M_f$, i.e. M_f is the subset of $\mathbb{R}^{n+c}$ where $y_i = \psi_i(x_1,\ldots,x_n,y_{n+1},\ldots,y_c)$ $\forall\ i = 1,\ldots,n$. So ψ is a map $\mathbb{R}^n_x \times \mathbb{R}^{c-n}_y \to \mathbb{R}^n_y$. The graph of such a polynomial map is diffeomorphic to its source, and M_f = graph of $\psi \subset \mathbb{R}^n_x \times \mathbb{R}^c_y = \mathbb{R}^n_x \times \mathbb{R}^{c-n}_y \times \mathbb{R}^n_y$, so $M_f \simeq \mathbb{R}^c$.

We remark that M_f is not a manifold in general. E.g. $\eta = x^5$, $f = \frac{x^5}{5} + \frac{ax^3}{3}$. $\frac{\partial f}{\partial x} = x^4 + ax^2$, and for $(x,a) \in \mathbb{R}^2$, M_f looks like:



<u>Lemma 7.2</u>. Suppose η has finite determinacy, and $\eta = q + p$, where $q = x_1^2 + \ldots - x_\rho^2$ and p is a polynomial in $x_{+1}, \ldots, x_n$ only, consisting of monomials of degree ≥ 3. Suppose (r,f) is a universal unfolding of p. Then if $g = q + f$, (r,g) is a universal unfolding of η and $\mathcal{X}_f = \mathcal{X}_g$.

<u>Proof</u>. By Lemma 6.6 (r,f) is k-transversal $\forall\ k > 0$, and in particular for $k \geq \det p = \det \eta$, Lemma 6.4 gives $m_\lambda = \Delta(p) + V_f + m^{k+1}$ which, with $m_\lambda^{k+1} \subset \Delta(p)$ (Theorem 2.9) gives $m_\lambda = \Delta(p) + V_f$. Here $\lambda = n - \rho$, and m_λ is the ideal of $\mathcal{E}_\lambda$ generated by $x_{\rho+1}, \ldots, x_n$. Similarly m_ρ is the ideal of $\mathcal{E}_\rho$ generated by $x_1, \ldots, x_\rho$. m and $\mathcal{E}$ denote m_n and $\mathcal{E}_n$. Then $m_\rho\mathcal{E} + m_\lambda\mathcal{E} = m_\rho\mathcal{E} + \Delta(p)\mathcal{E} + V_f$.

Now $m = m_\rho\mathcal{E} + m_\lambda\mathcal{E}$ and $V_f = V_g$. Also $\Delta(\eta) = (x_1,\ldots,x_\rho,\frac{\partial f}{\partial x_{\rho+1}},\ldots,\frac{\partial f}{\partial x_n})$

$$= m_\rho\mathcal{E} + \Delta(p)\mathcal{E}.$$

So $m = \Delta(\eta) + V_g = \Delta(\eta) + V_g + m^{k+1}$ for $k \geq \det \eta$ and so by Lemma 6.4 and Theorem 6.10, (r,g) is universal.

$$\left.\begin{array}{l} \text{If } i \leq \rho,\ \frac{\partial g}{\partial x_i} = 2x_i \quad (= 0 \text{ for } M_g) \\ \text{If } i > \rho,\ \frac{\partial g}{\partial x_i} = \frac{\partial f}{\partial x_i} \quad (= 0 \text{ for } M_g) \end{array}\right\} \Rightarrow M_g = 0 \times M_f.$$

We have
$$\left.\begin{array}{l} X_f : M_f \subset 0 \times \mathbb{R}^{r+\lambda} \xrightarrow{\pi_r} \mathbb{R}^r \\ X_g : M_g \subset \mathbb{R}^{\rho} \times \mathbb{R}^{r+\lambda} \xrightarrow{\pi_r} \mathbb{R}^r \end{array}\right\} \Rightarrow X_f = X_g .$$

Lemma 7.3. Suppose (r,f) and (s,g) are 2 unfoldings of η, and $\exists$ a morphism $(\phi,\bar{\phi},\varepsilon)$: $(s,g) \to (r,f)$. Then $M_g = \phi^{-1}M_f$, and X_g is the pullback of X_f under ϕ, $\bar{\phi}$.

Proof. We have
$$\begin{tikzcd} \mathbb{R}^n x y \arrow[r, "\phi^y"] \arrow[d, "g^y"] & \mathbb{R}^n x \bar{\phi} y \arrow[d, "f^{\bar{\phi} y}"] \\ \mathbb{R} \arrow[r, "-\varepsilon y", "\text{sheer}"'] & \mathbb{R} \end{tikzcd}$$
commuting.

Giving,
$$\begin{tikzcd} T_x(\mathbb{R}^n x y) \arrow[r, "T_x(\phi^y)", "\text{iso}"'] \arrow[d, "T_x(g^y)"] & T_{\phi^y x}(\mathbb{R}^n x \bar{\phi} y) \arrow[d, "T_{\phi^y x}(f^{\bar{\phi} y})"] \\ \mathbb{R} \arrow[r, "1"] & \mathbb{R} \end{tikzcd}$$
commuting.

$\phi^0 = 1$, so ϕ^y is a diffeomorphism for small y, and $T_x(\phi^y)$ is an isomorphism for small y. $(x,y) \in M_g \Leftrightarrow T_x(g^y) = 0$ (definition of M_g)

$\Leftrightarrow T_{\phi^y x}(f^{\bar{\phi} y}) = 0$ (diagram commutes)

$\Leftrightarrow (\phi^y x, \bar{\phi} y) \in M_f$ (definition of M_f)

$\Leftrightarrow \phi(x,y) \in M_f$, i.e. $M_g = \phi^{-1}M_f$.

We have that
$$\begin{tikzcd} \phi^{-1} M_f \arrow[r, "\phi"] \arrow[d, "X_g"] & M_f \arrow[d, "X_f"] \\ \mathbb{R}^s \arrow[r, "\bar{\phi}"'] & \mathbb{R}^r \end{tikzcd}$$
commutes, completing the lemma.

Recall that if θ_i is a germ $M_i,p_i \to M_i',p_i'$ where M_i, M_i' are C^∞ manifolds, $i = 1, 2$, then $\theta_1 \sim \theta_2 \Leftrightarrow \exists$ diffeomorphism-germs δ_1, δ_2 such that
$$\begin{tikzcd} M_1,p_1 \arrow[r, "\theta_1"] \arrow[d, "\delta_1"] & M_1',p_1' \arrow[d, "\delta_2"] \\ M_2,p_2 \arrow[r, "\theta_2"] & M_2',p_2' \end{tikzcd}$$
commutes.

Corollary 7.4. If $(\phi,\bar{\phi},\varepsilon)$ is an isomorphism, $X_g \sim X_f$.

Proof. ϕ, $\bar{\phi}$ will be diffeomorphism-germs; the requisite diagram is at the end of Lemma 7.3.

Lemma 7.5. If (r,g) and (r,f) are universal unfoldings of an η, of finite determinacy, then $X_f \sim X_g$.

Proof. This follows from Theorem 6.9 and Corollary 7.4.

Lemma 7.6. If η has finite determinacy and (s,g), (r,f) are universal unfoldings of η with $s > r$, then $X_g \sim X_f \times 1^{s-r}$.

Proof. Let (s,f') be (r,f) with s−r disconnected controls. Then (s,f') is universal, so that $X_{f'} \sim X_g$ by Lemma 7.5. Also

$$\begin{tikzcd} M_{f'} \arrow[d, "X_{f'}"] & = & M_f \arrow[d, "X_f"] & \times & \mathbb{R}^{s-r} \arrow[d, "1^{s-r}"] \\ \mathbb{R}^s & = & \mathbb{R}^r & \times & \mathbb{R}^{s-r} \end{tikzcd},$$

i.e. $X_{f'} = X_f \times 1^{s-r}$.

Lemma 7.7. If η has finite determinacy and is right equivalent to η', and if (r,f) and (r,f') are respective universal unfoldings, then $X_f \sim X_{f'}$.

Proof. We have $\eta' = \eta\gamma$ where $\gamma \in G$. Let $g = f(\gamma\times 1)$:

$$\begin{tikzcd} \mathbb{R}^{n+r} \arrow[r, "\gamma\times 1"] \arrow[d, "\pi_r"] & \mathbb{R}^{n+r} \arrow[r, "f"] \arrow[d, "\pi_r"] & \mathbb{R} \\ \mathbb{R}^r \arrow[r, "1"] & \mathbb{R}^r & \end{tikzcd}$$

This induces

$$\begin{tikzcd} M_g \arrow[r, "*"] \arrow[d, "X_g"] & M_f \arrow[d, "X_f"] \\ \mathbb{R}^r \arrow[r] & \mathbb{R}^r \end{tikzcd}$$

$*$ is a diffeomorphism because γ is. And so $X_f \sim X_g$.

Now $g|\mathbb{R}^n \times 0 = f\gamma|\mathbb{R}^n \times 0 = \eta\gamma|\mathbb{R}^n \times 0 = \eta'|\mathbb{R}^n \times 0$. So (r,g) unfolds η', and (r,g) is a universal unfolding because (r,f) is, clearly. By Lemma 7.5, $X_g \sim X_{f'}$. Hence $X_f \sim X_{f'}$.

Theorem 7.8. If $\eta \in m^2$ of finite determinacy has a catastrophe germ X_f, then the equivalence class of X_f depends only upon the equivalence class of η. Moreover it is uniquely determined by the essential coordinates of η.

Proof. Denote the equivalence class of X_f by $[X_f]$. $[X_f]$ is independent of the choices of: n by Lemma 7.2, universal unfolding f by Lemma 7.5, r by Lemma 7.6, and of η by Lemma 7.7. Lemma 7.2 shows that $[X_f]$ is uniquely determined by the essential coordinates (of η).

Corollary 7.9. $\exists$ only 11 catastrophe germs if we restrict to those η of codimension ≤ 5.

Proof. If there are more than 2 essential coordinates of η, i.e. rank $\eta \leq n - 3$, then Lemma 4.11 shows $\operatorname{cod} \eta > 5$. So restrict to $n \leq 2$. $+\eta$ and $-\eta$ give the same M_f and hence the same X_f. So the (distinct) essential coordinates giving distinct $[X_f]$'s are: x^3, x^4, x^5. x^6, x^7, $x^3 + xy^2$, $x^3 - xy^2$, $x^2y + y^4$, $x^3 + y^4$, $x^2y + y^5$, $x^2y - y^5$. These are the 11.

Definition. If $[X_f]$ is one of the 11 of Corollary 7.9 then $[X_f]$ is called an elementary catastrophe.

Corollary 7.10. If η has finite determinacy and (r,f) is a universal unfolding of η, where $r \leq 5$, then $[X_f]$ is an elementary catastrophe.

Proof. By Corollary 4.7 and the Reduction Lemma 4.9, $\eta \sim q + p$ and $p \in m^3$. Also Lemma 6.6 tells us that $r \geq c = \operatorname{cod} \eta$, so that $c \leq 5$ and p is one of the germs written out in the proof of Corollary 7.9 ($\operatorname{cod} p \leq 5$ and consult Diagram 4.1). By Lemma 7.1 applied to p $\exists$ a standard universal unfolding (c,g) of p such that X_g is a germ $\mathbb{R}^c, 0 \to \mathbb{R}^c, 0$. Now use Lemma 7.2 to provide a universal unfolding (c,f') of η such that $X_{f'} = X_g$. By Lemma 7.6 $X_f \sim X_{f'} \times 1^{r-c} = X_g \times 1^{r-c}: \mathbb{R}^r,0 \to \mathbb{R}^r,0$. Now $[X_g]$ is an elementary catastrophe by choice, and so in a certain obvious sense $[X_f]$ is an elementary catastrophe too. This is the same sense in which we said that "$[X_f]$ is independent of the choice of r by Lemma 7.6" in Theorem 7.8.

CHAPTER 8. GLOBALISATION.

We shall first define the Whitney C^∞ topology on the space of C^∞ functions $\mathbb{R}^{n+r} \to \mathbb{R}$, denoted by F.

Given $f: \mathbb{R}^{n+r} \to \mathbb{R}$ define a map $f^k: \mathbb{R}^{n+r} \to J^k_{n+r}$ (where, recall, $J^k_n = E_n/m_n^{k+1}$) which sends $p \in \mathbb{R}^{n+r}$ to the k-jet at 0 of the function $R^{n+r} \to \mathbb{R}$

$w \mapsto f(p+w)$.

Then given a function $\mu: \mathbb{R}^{n+r} \to \mathbb{R}_+$ we define a basic neighborhood of 0 as $V^k_\mu = \{f \in F: \forall\ p \in \mathbb{R}^{n+r}, |f^k p| < \mu p\}$. For $f \in F$, $V^k_\mu(f) = \{g \in F: \forall\ p \in \mathbb{R}^{n+r}, |f^k p - g^k p| < \mu p\}$ is a basic open neighborhood of f. These form a base for a topology, called the Whitney C^k-topology. The topology with a base of all such $V^k_\mu(f)$, $\forall\ k \geq 0$, is called the Whitney C^∞ topology. F will be assumed to have this topology.

Theorem 8.1. If $r \leq 5$, then $\exists$ an open dense set $F_* \subset F$ such that if $f \in F_*$, then $\tilde{X}_f$ has only elementary catastrophes as singularities (and these are already classified), and M_f is an r-manifold.

We shall need several lemmas to prove the theorem.

Given $f \in F$, $\varepsilon > 0$, and $X \subset \mathbb{R}^{n+r}$, define an open set, $V^k_{\varepsilon,X}(f)$ as $\{g \in F: \forall\ p \in X, |f^k p - g^k p| < \varepsilon\}$, so that ε controls all partial derivatives of order $\leq k$ on X. It is open because it is the union of all $V^k_\mu(f)$ for $\mu: \mathbb{R}^{n+r}, X \to R_+, (0,\varepsilon)$.

Definition. Let J be a manifold. A stratification Q of J is a decomposition into a finite number of submanifolds $\{Q_i\}$ such that,

(1) $\partial Q_i = \overline{Q}_i - Q_i$ = the union of Q_j of lower dimension.

(2) If $z \in Q_j \subset \partial Q_i$ and a submanifold S of J is transverse to Q_j at z, then S is transverse to Q_i in a neighborhood of z. (8.2)

Following the construction of the k-jet prolongation of an unfolding (r,f) in Chapter 6, given $f \in F$ we let F be the induced map

$\mathbb{R}^{n+r} \to J^k$

$p = (x,y) \mapsto$ k-jet at 0 of the function $\mathbb{R}^n, 0 \to \mathbb{R}, 0$

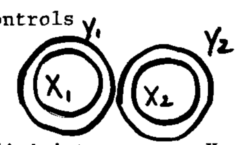
$$x' \mapsto f(x+x',y) - f(x,y).$$

Given $X \subset \mathbb{R}^{n+r}$ we let $F^X = \{f \in F\colon \forall\, p \in X,\ F$ is transversal to Q at $p\}$, where Q is either a submanifold or a stratification of J^k.

<u>Open Lemma 1</u>. (OL1) If $X \subset \mathbb{R}^{n+r}$ is compact and $f \in F^X$, then $\exists$ a neighborhood $V^{k+1}_{\varepsilon,X}(f) \subset F^X$. (i.e. F^X is C^{k+1}-open.)

<u>Proof</u>. Given $p \in X$, F is transversal to Q at p. By continuity and (8.2) (if appropriate), F is transversal to Q in a neighborhood of p, in particular in a compact neighborhood N of p. This remains true for all sufficiently small changes of F and TF on N, and so for all sufficiently small changes in f^{k+1} on N. Because N is compact, $\exists\ \varepsilon > 0$ such that $V^{k+1}_{\varepsilon,N}(f) \subset F^N$. Cover compact X by a finite number of such N_i, and let $\varepsilon = \min \varepsilon_i$. Then

$$\begin{aligned} V^{k+1}_{\varepsilon,X}(f) &= \bigcap_i V^{k+1}_{\varepsilon,N_i}(f) && (\textstyle\bigcup_i N_i = X) \\ &\subset \bigcap_i V^{k+1}_{\varepsilon_i,N_i}(f) && \text{relaxing controls} \\ &\subset \bigcap_i F^{N_i} = F^X. \end{aligned}$$



<u>Open Lemma 2</u>. (OL2) Let $X = \bigcup X_i$, a countable union of disjoint compact X_i with neighborhoods Y_i. Then F^X is C^{k+1}-open.

<u>Proof</u>. Choose a C^∞ bump function $\beta_i\colon \mathbb{R}^{n+r} \to [0,1]$, which takes values 1 on X_i and 0 outside Y_i, for each i. Let $\beta_o = 1 - \sum_{i=1}^{\infty} \beta_i$. Given $f \in F^X$, then $f \in F^{X_i}$. So $\exists\ \varepsilon_i > 0$ such that $V^{k+1}_{\varepsilon_i,X_i}(f) \subset F^{X_i}$ (OL1)

Let $\mu = \beta_o + \sum_{i=1}^{\infty} \varepsilon_i \beta_i$. Then

$$\begin{aligned} V^{k+1}_{\mu}(f) &\subset \bigcap_{i=1}^{\infty} V^{k+1}_{\varepsilon_i,X_i}(f) && (\mu = \varepsilon_i \text{ on } X_i) \\ &\subset \bigcap F^{X_i} = F^X. \end{aligned}$$

<u>Density Lemma 3</u>. (DL3) $\forall\, p \in \mathbb{R}^{n+r}$ and $\forall\, f \in F$, $\exists$ a compact neighborhood N of p in $\mathbb{R}^{n+r}$ and $\exists$ a neighborhood V of $f \in F$ such that F^N is C^∞-dense in V.

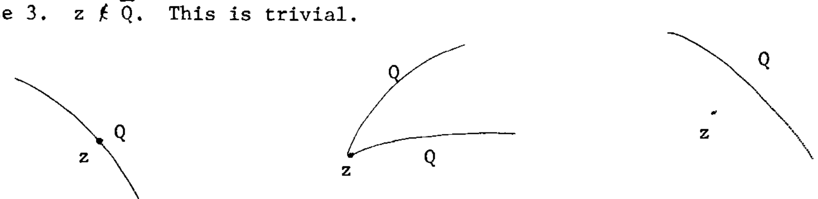
<u>Proof</u>. Having chosen N and V we must show that $\forall\, g \in V$, $\exists$ an arbitrarily

C^∞-close $h \in \mathcal{F}^N$. Now $\mathcal{F}^N = \{f \in \mathcal{F}: F$ is transversal to Q in $N\}$, where Q is (first) a submanifold of J^k. Given f let $z = F(0,0)$ and w.l.o.g. $p = (0,0)$.

Case 1. $z \in Q$. This is hard.

Case 2. $z \in \overline{Q} - Q$. This does not occur if Q is closed, but we need this case where Q is one stratum of a stratification.

Case 3. $z \notin \overline{Q}$. This is trivial.



Case 1 Case 2 Case 3

Case 3. Pick N such that $FN \pitchfork Q$, and V such that $\forall\, g \in V$, $GN \pitchfork Q$. Then $g \in \mathcal{F}^N$, trivially. So $V \subset \mathcal{F}^N$, and $h = g$ will do.

Case 1. Let q be the codimension of Q in J^k. Choose a product neighborhood B of z in J^k and a projection $\theta: B \to \mathbb{R}^q$ such that $\theta^{-1}0 = B \cap Q$. Now J^k is spanned by monomials in $x_1, \ldots, x_n$. Of these choose $u_1, \ldots, u_q$ spanning the q-plane transverse to Q at z. Let e_w be the function $\mathbb{R}^n \to \mathbb{R}$

$$x \mapsto \sum_{i=1}^{q} w_i u_i(x), \quad \text{where } w_i \in \mathbb{R} \text{ form } w \in \mathbb{R}^q,$$

and so $e: \mathbb{R}^q \times \mathbb{R}^n \to \mathbb{R}$. As usual e induces $E: \mathbb{R}^q \times \mathbb{R}^n, 0 \to J^k, 0$

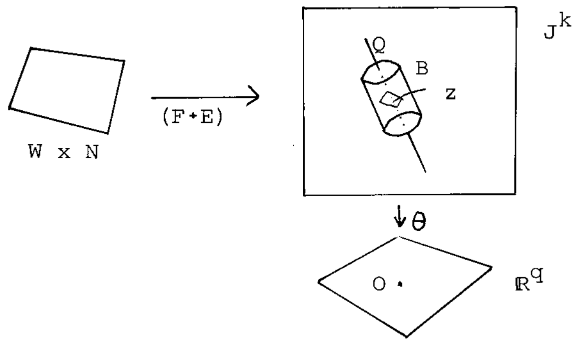
$(w,x) \mapsto$ k-jet of the function $\mathbb{R}^n, 0 \to \mathbb{R}, 0$.

$$x' \mapsto e_w(x+x') - e_w(x).$$

Then $(F+E): \mathbb{R}^q \times \mathbb{R}^{n+r}, 0 \to J^k, z$

$$(w,x,y) \mapsto F(x,y) + E(w,x),$$

is convenient notation. Now choose a compact neighborhood $W \times N$ of 0 in $\mathbb{R}^q \times \mathbb{R}^{n+r}$ such that $(F+E)(W \times N) \subset B$.



Choose a neighborhood V of f in F such that $\forall\ g \in V$, $(G+E)(W\times N) \subset B$. This is possible because $W \times N$ is compact and B is open.

Sublemma 1. The matrix of partial derivatives with respect to W at 0 of the composite map $W \times N, 0 \xrightarrow{(F+E)} B, z \longrightarrow \mathbb{R}^q, 0$ is a nonsingular matrix.

Proof. $(F+E)(w,0,0) = F(0) + E(w,0) = z + E(w,0)$.

$E(w,0)$ is the k-jet at 0 of $\mathbb{R}^n \to \mathbb{R}$

$$x' \to e_w(x') - \overset{0}{\overset{\shortparallel}{\cancel{e_w(0)}}} = \sum_{i=1}^{q} w_i u_i(x').$$

So $(F+E)(w,0,0) = z + \sum_{i=1}^{q} w_i u_i$, which is in the q-plane transverse to Q at z by construction. Hence $\theta(F+E)$ is transversal to 0 in $\mathbb{R}^q$.

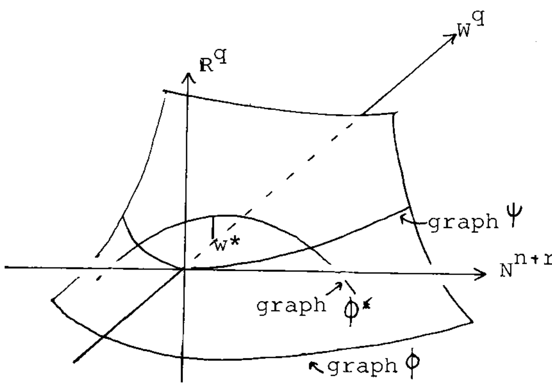
Corollary. By choosing W, N, V sufficiently small, the matrix of partial derivatives of the composition map $\phi\colon W \times N \xrightarrow{(G+E)} B \longrightarrow \mathbb{R}^q$ with respect to W is nonsingular at $(w,p)\ \forall\ (w,p) \in W \times N,\ \forall\ g \in V$.

Proof. By continuity from Sublemma 1.

Sublemma 2. (Implicit Function Theorem) Given $W^q \times N^{n+r} \to \mathbb{R}^q$ with the matrix of partial derivatives of ϕ with respect to W nonsingular $\forall\ (w,p) \in W \times N$, then $\exists$ a unique C^∞ map $\psi\colon N^{n+r} \to W^q$ such that $\phi^{-1}0 = \text{graph}\ \psi$.

By Sard's Theorem choose a regular value w^* of ψ, arbitrarily small. Let ϕ^* be the map: $N^{n+r} \to \mathbb{R}^q$

$$p \mapsto \phi(w^*,p).$$

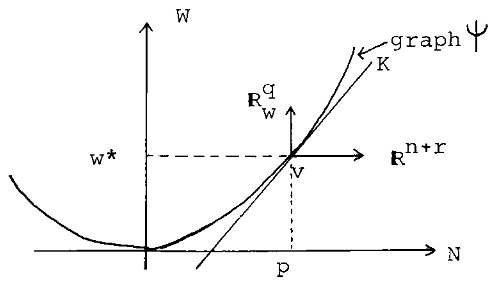


If N = east, W = north, $\mathbb{R}^q$ = vertical, ϕ = height above sea level, and ψ = coastline, then Sublemma 2 says that $\exists$ a coastline.

Sublemma 3. ϕ^* is transversal to 0.

Proof. Suppose $\phi^* p = 0$. Let $v = (w^*, p)$, $\in$ graph $\psi \subset W \times N$ as $\phi v = 0$. Consider $\underset{\mathbb{R}^q_w \times \mathbb{R}^{n+r}}{T_v(W\ N)} \xrightarrow{T_v\phi} \underset{\mathbb{R}^q}{T_0\,\mathbb{R}^q}$. $T_v\phi$ is surjective by the Corollary to Sublemma 1.

Let K be the kernel of $T_v\phi$, $K = (T_v\phi)^{-1}0$. Dim $K = (q+n+r) - q = n+r$, by surjectivity.



Because w^* is a regular value of ψ, the map $K^{n+r} \subseteq \mathbb{R}^q_w \times \mathbb{R}^{n+r} \xrightarrow{\pi_q} \mathbb{R}^q_w$ is surjective. So K^{n+r} meets $\mathbb{R}^{n+r}$ transversely; dim $K^{n+r} \cap \mathbb{R}^{n+r} =$ $(n+r) + (n+r) - (n+r+q) = n + r - q$. Consider $\underset{\mathbb{R}^{n+r}}{T_p(N^{n+r})} \xrightarrow{T_p\phi^*} \underset{\mathbb{R}^q}{T_0(\mathbb{R}^q)}$.

Kernel of $T_p\phi^*$ = kernel of $T_v\phi \cap \mathbb{R}^{n+r} = K^{n+r} \cap \mathbb{R}^{n+r}$, and so is of dimension $n + r - q$. Hence $T_p\phi^*$ is surjective, and p is a regular point of ϕ^*. Thus ϕ^* is transversal to 0.

We have chosen N and V. Choose now a bump function $\beta: \mathbb{R}^{n+r} \to [0,1]$ such that $\beta = 1$ on N, $\beta = 0$ outside a compact neighborhood of N. Given $g \in V$, choose w^* (dependent upon g), a regular value of ψ, w^* arbitrarily small. Define $h: \mathbb{R}^{n+r} \to \mathbb{R}$ by $h(x,y) = g(x,y) + \sum_{i=1}^{q} w_i^* u_i(x)\beta(x,y)$. Then by Sublemma 3, $\theta H = \phi^*$ is transversal to 0 on N. So H is transversal to Q, and $h \in F^N$. Given an arbitrary C^∞-neighborhood $V_\mu^\ell(g)$, we can reduce the partial derivatives of $w^*u\beta$ of order $\le \ell$, below μ, on a compact neighborhood of N by making w^* sufficiently small. So $h \in V_\mu^\ell(g)$. h is arbitrarily C^∞-close to g.

This completes Case 1 of DL3.

Case 2. $z \in Q \subset \partial Q' = \overline{Q'} - Q'$ where $\{Q\}$ form a stratification. Given $g \in V$ we must show $\exists$ h such that H is transversal to both Q and Q' (and any other incident strata) at the same time, on N. Given g, find h as in Case 1 arbitrarily C^∞-close such that H is transversal to Q on N. Automatically by (8.2) H is transversal to Q' at all points in a compact neighborhood L of z in B.

Choose a product neighborhood B' of $Q' \cap (B-L)$, and a map $\theta': B' \to \mathbb{R}^{q'}$ such that $\theta'^{-1}0 = Q' \cap (B-L)$, where q' is the codimension of Q' in J^k. Find now h' arbitrarily C^∞-close to h so that

(a) $\theta H'$ remains transversal to all points of θL, and

(b) $\theta' H'$ becomes transversal to 0 in $\mathbb{R}^{q'}$ by Case 1 for Q'.

Then H' is transversal to Q and Q' on N. By induction, $H^{(s)}$ is transversal to the stratification because there are only a finite number $(s+1$, say) of strata through z, by (8.2). Then $h^{(s)} \in F^N$ is arbitrarily C^∞-close to $g \in V$.

<u>Density Lemma 4</u>. (DL4) If $X \subset \mathbb{R}^{n+r}$ is compact, then F^X is C^∞-dense in F.

<u>Proof</u>. Given $f \in F$, cover X by a finite number of N_i given by DL3. Let $V = \bigcap_i V_i$. Then F^{N_i} is C^∞-open by OL1 (because C^{k+1}-open) and is C^∞-dense (by DL3) in V_i. So F^{N_i} is C^∞ open dense in V. Now $F^X \supset F^{\cup N_i} = \bigcap F^{N_i}$ is C^∞

open dense in V. So F^X is dense in V, i.e. $\forall f \in F$, $\exists V$ such that F^X is dense in V.

Therefore F^X is dense.

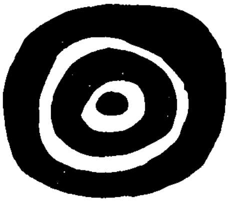
<u>Density Lemma 5</u>. (DL5) Let $X = \bigcup X_i$ as in OL2, then F^X is C^∞-dense.

<u>Proof</u>. Given $f \in F$ and given a basic C^∞-neighborhood $V^\ell_\mu(f)$, we want $g \in V^\ell_\mu(f) \cap F^X$. Let $\{\beta_i\}$ be as in OL2. For each i choose $\varepsilon_i > 0$ such that $h \in V^\ell_{\varepsilon_i, Y_i} \Rightarrow \beta_i h \in V^\ell_\mu$. (This is possible by the boundedness of the derivatives of order $\leq \ell$ of β_i on Y_i). By DL4, choose $f_i \in V^\ell_{\varepsilon_i, Y_i}(f) \cap F^{X_i}$. Define $g = \beta_o f + \sum_{i=1}^{\infty} \beta_i f_i$. Then $g = f$ outside $\bigcup Y_i$. On Y_i, $g = (1-\beta_i)f + \beta_i f_i = f + \beta_i(f_i - f)$. Now $f_i - f \in V^\ell_{\varepsilon_i, Y_i}$ by choice, so $\beta_i(f_i - f) \in V^\ell_\mu$. Meanwhile $g = f_i$ on X_i. But F_i is transversal to Q on X_i, and so G is also transversal to Q on X_i. Therefore $g \in \bigcap F^{X_i} = F^X$.

So $g \in V^\ell_\mu(f) \cap F^X$ as required.

The result of DL5 can also be proved by showing that F with the Whitney C^∞ topology is a Baire space, but the proof is longer.

<u>Lemma 6</u>. $F^{\mathbb{R}^{n+r}}$ is C^{k+1}-open and C^∞-dense in F.



<u>Proof</u>. Choose X, X' each as in OL2 such that $\mathbb{R}^{n+r} = X \cup X'$. Then $F^{\mathbb{R}^{n+r}} = F^X \cap F^{X'}$, each C^{k+1}-open and C^∞-dense, by OL2 and DL5 respectively.

<u>Proof of Theorem 8.1</u>. We describe the stratification Q of J^7 resulting from the classification of orbits in I^7 in Chapter 4.

(a) the open subspace $J^7 - I^7$,

(b) $n + 1$ orbits of jets of stable germs in m^2 of codimension 0 in I^7,

(c) the orbits of jets of germs in m^2 of codimension 1, 2, 3, 4 and 5 in I^7,

(d) the strata of the algebraic variety of jets of germs in m^2 of codimension ≥ 6 in I^7.

These come directly from Diagram 4.1.

Because Σ^7_6, i.e. (d), is of codimension $n + 6$ and we are not interested in its internal structure, we shall let Q be the stratification (a), (b), (c) of $J^7 - \Sigma^7_6$. The <u>strata</u> are the Γ^7_c for $c = 0, 1, 2, 3, 4$ and 5, together with $J^1 - 0$ (this last making $J^7 - \Sigma^7_6$, rather than $I^7 - \Sigma^7_6$).

<u>Lemma 7</u>. Q satisfies (8.2) (and hence is a stratification).

Let $F_o = \{f \in F: F$ misses $\Sigma^7_6\}$, i.e. where F is transversal to Σ^7_6 if $r \le 5$ (F maps $\mathbb{R}^{n+r}$ into J^7). By general position, F_o is C^o-open (and hence C^8-open) and C^∞-dense. Let $F_* = F_o \cap F^{\mathbb{R}^{n+r}}$, then F_* $\{f \in F:$ F is transversal to Q and $\Sigma^7_6\}$, and is C^8-open and C^∞-dense, using Lemma 6.

Suppose $f \in F_*$. Then F is transversal to $m^2/m^8 = I^7$, since I^7 is the union of strata of Q and Σ^7_6. So $F^{-1}(I^7)$ is of codimension n, and of dimension r. (I^7 is of codimension n in J^7). Now $F^{-1}(I^7)$ is the set of points (x,y) in $\mathbb{R}^{n+r}$ such that the 1-jet of $x' \mapsto f(x+x',y) - f(x,y)$ is zero, i.e. such that $\frac{\partial f}{\partial x_1}(x,y) = \dots = \frac{\partial f}{\partial x_n}(x,y) = 0$. So $F^{-1}(I^7)$ is precisely M_f and M_f is an r-manifold. Suppose that $\tilde{X}_f: M_f \to \mathbb{R}^r$ has a singularity at (x,y). Let η be the germ at (x,y) of $f|\mathbb{R}^n \times y$. W.l.o.g. $(x,y) = (0,0)$, so $\eta \in m^2$. The germ of f at $(0,0)$ is a 7-transversal unfolding of η, because $f \in F_*$ and so F is transversal to the orbit $(j^7\eta)G^7$, contained in some stratum.

<u>Lemma 8</u>. If (r,f) is a 7-transversal unfolding of $\eta \in m^2$, and $r \le 5$, then (r,f) is a universal unfolding.

<u>Proof</u>. By Lemma 6.4, $m = \Delta + V_f + m^8$. ($\Delta = \Delta(\eta)$). So $\dim m/(\Delta + m^8) \le \dim V_f \le r \le 5$, using (6.3). In the notation of Theorem 3.3, $\tau(j^8\eta) \le 5$. But $\operatorname{cod} \eta = \tau(j^8\eta) \le 5$, by (3.5), and so by Lemma 3.1, $\det \eta \le 7$, and we can apply Theorem 6.10 to show that (r,f) is universal.

By Corollary 7.10 we now know that if X_f is the germ at $(0,0)$ of $\tilde{X}_f$, then $[X_f]$ is an elementary catastrophe.

So the only singularities of $\tilde{X}_f$ are elementary catastrophes.

<u>Proof of Lemma 7</u>. (Which we have used to complete Theorem 8.1). Q has a finite number of strata, each of which is a submanifold by Corollary 4.3. (There are in fact 7 strata.) Condition (1) of (8.2) follows from Corollary 3.6 since each Σ^7_c is closed (Theorem 3.3). Note that $\overline{\Gamma}^7_c$ now refers to the closure

in $J^7 - \Sigma_6^7$.

Condition (2): Let Q_1, Q_2 be strata, $z_1 \in Q_1 \subset \partial Q_2$, and S a submanifold of $J^7 - \Sigma_6^7$ transverse to Q_1 at z_1. Then S is transverse to $z_1 G^7$ at z_1.

Write α for the C^∞ map $J^7 \to C^\infty(G^7, J^7)$. $\alpha(z_1)$ is

$z \to$ the map taking γ to $z \circ \gamma$.

now transversal to S in a neighborhood U of the identity e. Spanning, and hence transversality, is an open property, so $\exists$ an open neighborhood V of $\alpha(z_1)$ in $C\,(G^7, J^7)$ and a neighborhood U_1 of e (perhaps smaller than U) so that $\beta \in V$ implies β is transversal to S in U_1. $\alpha^{-1}(V)$ is open and contains z_1, and if $z \in \alpha^{-1}(V)$, $\alpha(z)$ is transversal to S in U_1; in particular zG^7 is transverse to S at z. But Q_2 is the finite union of such orbits zG^7. Hence S is transverse to Q_2 in $\alpha^{-1}(V)$, a neighborhood of z_1.

Thus condition (2) is satisfied, completing the proof of Lemma 7.

CHAPTER 9. STABILITY.

Given $f \in F_*$, let $X_f\colon M_f \to \mathbb{R}^r$ be induced by projection. (See Chapter 1) We have to show that X_f is locally stable at all points of M_f.

<u>Definition</u>. X_f is locally stable at $(x_o, y_o) \in M_f$ if given a neighborhood N of (x_o, y_o) in $\mathbb{R}^{n+r}$, $\exists$ a neighborhood V of f in F_*, such that given $g \in V$, $\exists$ (x_1, y_1) in $N \cap M_g$ such that X_f at (x_o, y_o) is locally equivalent to X_g at (x_1, y_1).

Let $\hat{f}$, $X_{\hat{f}}$ denote the germs of f, X_f at (x_o, y_o) and $\hat{g}$, $X_{\hat{g}}$ the germs of g, X_g at (x_1, y_1). Then $X_{\hat{f}}$, $X_{\hat{g}}$ agrees with the notation in Chapter 7, and we also have that

(9.1) $X_{\hat{f}} \sim X_{\hat{g}} \Leftrightarrow X_f$ at (x_o, y_o) is locally equivalent to X_g at (x_1, y_1).

<u>Theorem 9.2</u>. If $r \leq 5$ and $f \in F_*$, then X_f is locally stable at each point of M_f.

Proof. f induces $F: \mathbb{R}^{n+r} \to J^7$ as at the beginning of Chapter 8. Let (x_o,y_o) be in M_f, and $F(x_o,y_o) = z_o$. We suppose we are given a neighborhood N of (x_o,y_o). Since $f \in F_*$, F is transversal to z_oG^7 at z_o; hence we can choose a disc D^q with centre (x_o,y_o) contained in N, where q is the codimension of z_oG^7 in J^7, whose image under F intersects z_oG^7 transversely at z_o, and so that $F|_{D^q}$ is an embedding. $F(D^q)$ will then have intersection number 1 with z_oG^7. If F is perturbed slightly to G, $G(D^q)$ will still be a q-disc whose intersection number with z_oG^7 is still 1. I.e. $\exists$ an open neighborhood V_o of f in F with this property for $g \in V_o$. Write $V = V_o \cap F_*$. Given $g \in V$, G is transversal to z_oG^7 and we may choose $(x_1,y_1) \in D^q$ such that $G(x_1,y_1) = z_1 = G(D^q) \cap z_oG^7$. Then z_1 and z_o are in the same orbit and are right equivalent as germs $\mathbb{R}^n,0 \to \mathbb{R},0$.

Let $f_o(x,y) = f(x_o+x,y_o+y) - f(x_o,y_o)$ and $g_1(x,y) = g(x_1+x,y_1+y)-g(x_1,y_1)$ define f_o and $g_1: \mathbb{R}^{n+r},0 \to \mathbb{R},0$. Then $z_o = j^7(f_o|\mathbb{R}^n \times 0)$ and $z_1 = j^7(g_1|\mathbb{R}^n \times 0)$. Note that $F(\mathbb{R}^{n+r})$ is the same point-set as $F_o(\mathbb{R}^{n+r})$ and so F_o is transversal to z_oG^7 and $(r,\hat{f}_o)$ is a k-transversal unfolding of the germ z_o: so we can apply Lemma 8 in Chapter 8 (similarly for $\hat{g}_1$). As $r \le 5$ the proof of this lemma gives that z_o (and so also z_1) is finitely determined as a germ. The result of the same lemma tells us that $\hat{f}_o$ and $\hat{g}_1$ are also universal unfoldings of germs z_o, z_1 respectively. Now apply Lemma 7.7 which says $X_{\hat{f}_o} \sim X_{\hat{g}_1}$ (germs at $(0,0)$ of X_{f_o}, X_{g_1}).

Now M_f is merely a translate of M_{f_o} : $M_f = M_{f_o} + (x_o,y_o)$. And so

$$X_f(x,y) = X_{f_o}(x-x_o,y-y_o) + y_o.$$

Then

$$\begin{tikzcd} M_f,(x_o,y_o) \arrow[r, "-(x_o,y_o)"] \arrow[d, "X_f"] & M_{f_o},(0,0) \arrow[d, "X_{f_o}"] \\ \mathbb{R}^r,y_o \arrow[r, "-y_o"] & \mathbb{R}^r,0 \end{tikzcd}$$

commutes, so that $X_{\hat{f}} \sim X_{\hat{f}_o}$ (by (9.1)).

Similarly $X_{\hat{g}} \sim X_{\hat{g}_1}$.

Hence $X_{\hat{f}} \sim X_{\hat{f}_o} \sim X_{\hat{g}_1} \sim X_{\hat{g}}$. This completes Theorem 9.2.
(Observe that $(x_o,y_o) \in M_f$ and $M_f = F^{-1}(I^7)$ so that z_o and $z_o G^7 \subset I^7$. Then $z_1 \in I^7$ and $(x_1,y_1) \in M_g = G^{-1}(I^7)$, i.e. $(x_1,y_1) \in N \cap M_g$ as required.)

Remark. This is a result about local stability. It would be interesting and useful to have a similar global stability result.

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