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A Lagrangian Klein bottle

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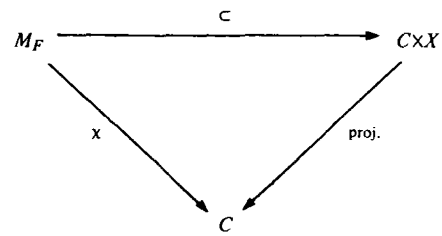
Lagrangian theory (2) and elementary catastrophe theory (5) coincide locally but differ globally. Locally the singularities that occur in the two theories are the same, but globally the manifolds that occur are different. We shall show that Lagrangian manifolds are more general.

By a catastrophe manifold M over a parameter space C we mean a component of the critical set of a family F of functions parametrized by C. It is well-known (e.g. (6), (7)) that, given such M, C, F, then F induces an immersion of M as a Lagrangian submanifold of the cotangent bundle T^*C of C. Thus every catastrophe manifold can be immersed as a Lagrangian submanifold. It is also well-known that not every Lagrangian immersion can be induced in this way; indeed the Maslov index (1) is an obstruction. More subtle is the fact that there exist manifolds M, C such that M can be immersed as a Lagrangian submanifold of T^*C but cannot occur as a catastrophe manifold over C for any F.

We prove this by constructing an example; it is useful to have an explicit low-dimensional example because surprisingly few Lagrangian manifolds are known (other than those induced from functions). The lowest dimension in which an example can occur is 2, and so we choose $C = \mathbb{R}^2$. Then the simplest candidate for M must be the Klein bottle K, due to the constraints on the Stiefel–Whitney classes imposed by the obstruction theories of Hayden (4). The non-orientability of K prevents it from occurring as a catastrophe manifold over $\mathbb{R}^2$. Therefore the main task of the paper is to construct an explicit immersion of K as a Lagrangian submanifold of $T^*\mathbb{R}^2$.

Elementary catastrophe theory. Let $F: C \times X \to \mathbb{R}$ be a C^∞-function on a manifold X, parametrized by a manifold C. We assume (generically) that 0 is a regular value of $\partial F/\partial x$, and hence the critical set of F given by $\partial F/\partial x = 0$ is a smooth submanifold of

$C \times X$ of the same dimension as C. Let M_F be a component of this critical set; we call M_F a *catastrophe manifold*. Let $\chi: M_F \to C$ be induced by projection.

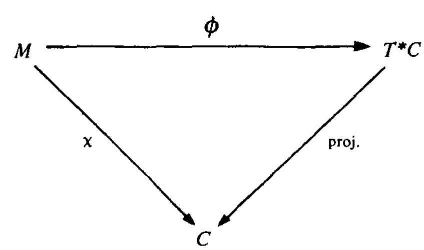


Let $W = \Sigma w_i$ denote the total Stiefel-Whitney class.

THEOREM 1. (*Hayden* (4)). $W(M_F) = \chi^* W(C)$.

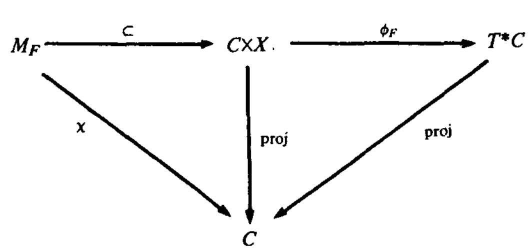
COROLLARY. *If C is orientable so is M_F. For C orientable $w_1(C) = 0 \Rightarrow w_1(M_F) = 0 \Rightarrow M_F$ orientable. Hence the Klein bottle K cannot occur as a catastrophe manifold over $\mathbb{R}^2$.*

Lagrangian theory. The cotangent bundle T^*C has a natural symplectic structure ω. If M is a manifold of the same dimension as C, then an immersion of $\phi: M \to T^*C$ is called *Lagrangian* if ω vanishes on the tangent planes of ϕM. Let $\chi: M \to C$ be induced by projection.



THEOREM 2. (*Hayden* (4)). $W^2(M) = \chi^* W^2(C)$.

The relationship between the two theorems is as follows. Given F, let $\phi_F: C \times X \to T^*C$ be given by $\phi_F(c, x) = (c, \partial F/\partial c)$. Then $\phi_F \mid M_F$ is a Lagrangian immersion (6, p. 716).



Hence in this special case Theorem 2 is a corollary of Theorem 1. However, in general Theorem 2 cannot be sharpened to Theorem 1. To prove this we choose C with $W(C)$ trivial, and seek M such that $W^2(M)$ is trivial but $W(M)$ is non-trivial. Hence M cannot be 1-dimensional, and if it is compact and 2-dimensional then it must be non-orientable with even genus. Hence the Klein bottle K is the simplest candidate. We now set about constructing a Lagrangian immersion of K in $T^*\mathbb{R}^2$, thus proving that the two theories differ.

Construction of the immersion. We shall use an immersion in the form of a figure-eight multiplied by an interval, with the ends glued together by a half-rotation

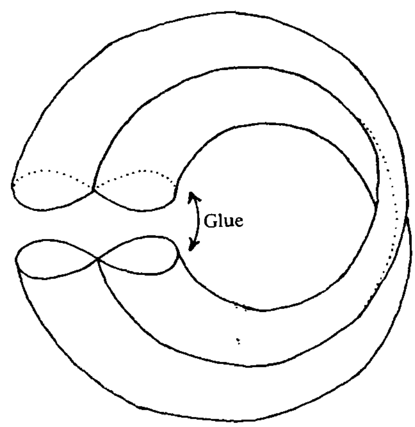


Fig. 1.

(Fig. 1). Notice that a half-rotation of a figure-eight reverses orientation, and so this is indeed an immersed Klein bottle. Analytic immersions of this type do exist in $\mathbb{R}^3$ (3), but we shall need the extra room of the fourth dimension in order to satisfy the Lagrangian condition.

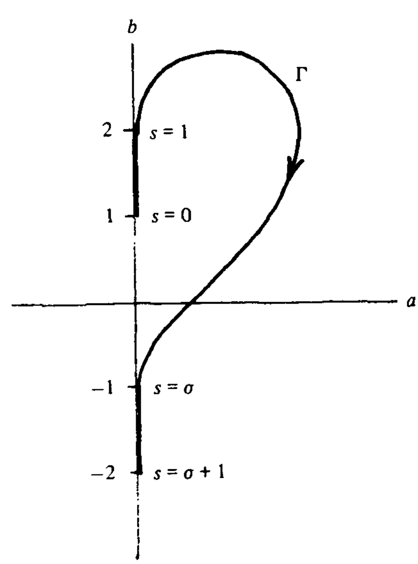


Fig. 2.

Let (a,b) be coordinates for $\mathbb{R}^2$, inducing coordinates (a,b,c,d) for $T^*\mathbb{R}^2$. The symplectic structure is then given by

$$\omega(u,v) = \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \begin{vmatrix} u_2 & u_4 \\ v_2 & v_4 \end{vmatrix},$$

where $u = (u_1, u_2, u_3, u_4)$, $v = (v_1, v_2, v_3, v_4)$ are two tangents at a point of $T^*\mathbb{R}^2$.

Let Γ be a C^∞-curve in $\mathbb{R}^2$ that starts at $(0,1)$, runs up the b-axis to $(0,2)$, bends smoothly round to rejoin the b-axis at $(0,-1)$, before finally descending down the b-axis to $(0,-2)$, as shown in Fig. 2. Let $\sigma+1$ be the length of Γ. Parametrize Γ by arc-length s, $0 \leqslant s \leqslant \sigma+1$, and let $(\alpha(s), \beta(s))$ be the coordinates of the point s. In particular, denoting the derivative with respect to s by a dash, we have:

$$(*) \quad \begin{aligned} 0 \leqslant s \leqslant 1 &\Rightarrow \alpha = \alpha' = 0, \quad \beta = 1+s, \quad \beta' = 1. \\ \sigma \leqslant s \leqslant \sigma+1 &\Rightarrow \alpha = \alpha' = 0, \quad \beta = \sigma - s - 1, \quad \beta' = -1. \end{aligned}$$

Choose ϵ, $0 < \epsilon < (\max\{|\alpha''|^2 + |\beta''|^2\})^{-\frac{1}{2}}$. This ensures that ϵ is less than the radius of curvature everywhere, and so the ϵ-disks normal to Γ in $\mathbb{R}^2$ are locally disjoint; by further choosing ϵ sufficiently small we can ensure that they are also globally disjoint from one another.

Let $\zeta(a,b)$ denote the distance from (a,b) to Γ in $\mathbb{R}^2$. Let $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$, with coordinates (a,b,x), and identify $\mathbb{R}^2 = \mathbb{R}^2 \times 0$. Let $F\colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ be given by

$$F(a,b,x) = \frac{x^3}{3} + x(\zeta^2 - \epsilon^2).$$

Then F has catastrophe manifold M_F given by

$$\frac{\partial F}{\partial x} = x^2 + \zeta^2 - \epsilon^2 = 0.$$

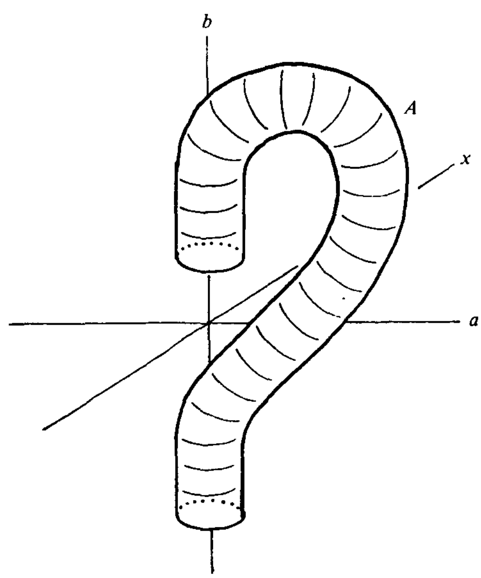


Fig. 3.

Since M_F is the boundary of the ϵ-neighbourhood of Γ in $\mathbb{R}^3$, we can think of M as a tubular cylinder (Fig. 3) of radius ϵ running along Γ, together with two hemispheres at the ends. We shall in fact only use the cylinder part, which we call A. The cylinder is the union of the circles bounding the ϵ-disks normal to Γ in $\mathbb{R}^3$, which are disjoint by our choice of ϵ. Each circle can be parametrized by θ, such that

$$\zeta = |\epsilon \cos \theta|, \quad x = \epsilon \sin \theta.$$

Hence A is parametrized by $(s, \theta), 0 \leqslant s \leqslant \sigma + 1, 0 \leqslant \theta < 2\pi$, and the coordinates of the point (s, θ) in $\mathbb{R}^3$ are given by

$$(a, b, x) = (\alpha + \epsilon\beta' \cos\theta, \beta - \epsilon\alpha' \cos\theta, \epsilon \sin\theta).$$

Now F induces a Lagrangian immersion of M, and hence of A, in $T^*\mathbb{R}^2$, given by

$$(a, b, x) \mapsto \left(a, b, \frac{\partial F}{\partial a}, \frac{\partial F}{\partial b}\right)$$

Since s measures arc-length,

$$\frac{\partial \zeta}{\partial a} = \beta', \frac{\partial \zeta}{\partial b} = -\alpha'.$$

Therefore at (s, θ),

$$\frac{\partial F}{\partial a} = 2x\zeta \frac{\partial a}{\partial a} = 2(\epsilon \sin\theta)(\epsilon \cos\theta)\beta' = \epsilon^2 \beta' \sin 2\theta,$$

$$\frac{\partial F}{\partial b} = -\epsilon^2 \alpha' \sin 2\theta.$$

Therefore the Lagrangian immersion $\phi_A \to T^*\mathbb{R}^2$ is given by

$$\phi_A(s, \theta) = (\alpha + \epsilon\beta' \cos\theta, \beta - \epsilon\alpha' \cos\theta, \epsilon^2\beta' \sin 2\theta, -\epsilon^2\alpha' \sin 2\theta).$$

Notice that each of the circles s = constant is immersed as a figure-eight by ϕ_A; for example when $s = 0$, by (*),

$$\phi_A(0, \theta) = (\epsilon \cos\theta, 1, \epsilon^2 \sin 2\theta, 0).$$

This is a figure-eight as shown in Fig. 4. Increasing s has the effect of isotoping this figure-eight in $T^*\mathbb{R}^2$ so that the self-intersection point runs along Γ, $\subset \mathbb{R}^2 \subset T^*\mathbb{R}^2$, and as it goes along the 2-plane containing the figure-eight rotates in the 3-planes

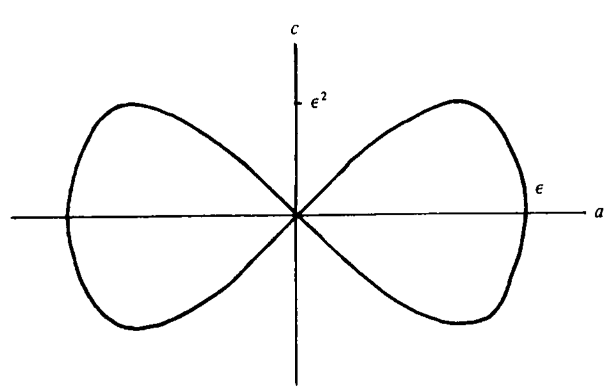


Fig. 4.

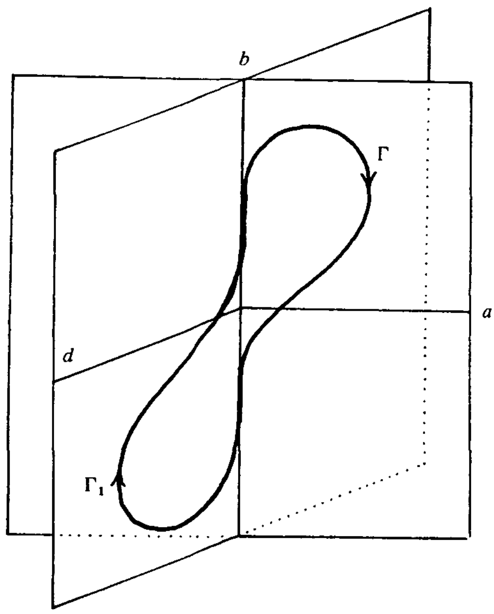


Fig. 5.

normal to Γ. This rotation is just what is necessary to satisfy the Lagrangian condition.

We shall now construct the Klein bottle by immersing another cylinder B and glueing it onto A; only this time we have to use a different process for manufacturing B in order to avoid constructing a torus by mistake. Then the whole immersed Klein bottle will consist of a figure-eight running along the curve $\Gamma \cup \Gamma_1$ shown by Fig. 5.

Any 1-dimensional manifold immersed in $T^*\mathbb{R}$ is trivially Lagrangian, and the cartesian product of two Lagrangian manifolds is another; hence we can manufacture B by taking the cartesian product of the figure-eight shown in Fig. 4 with a reflected copy Γ_1 of Γ in the (b, d)-plane, as shown in Fig. 5. More precisely let S^1 be the circle parametrized by θ, $0 \leqslant \theta < 2\pi$, and let B be the cylinder

$$B = \{(s, \theta); \sigma \leqslant s \leqslant 2\sigma + 1, 0 \leqslant \theta < 2\pi\}, \quad \subset \mathbb{R} \times S^1.$$

Let $\phi_B: B \to T^*\mathbb{R}^2$ be the Lagrangian immersion given by

$$\phi_B(s, \theta) = (-\epsilon \cos\theta,\ -\beta(s-\sigma),\ -\epsilon^2 \sin 2\theta, \alpha(s-\sigma)).$$

Note that we could have just written down the formulae for ϕ_A, ϕ_B and verified directly that they were Lagrangian immersions, by checking that the vectors $u = \partial\phi/\partial s, v = \partial\phi/\partial\theta$ were linearly independent and satisfied $\omega(u, v) = 0$. However this approach would have been less intuitive.

To glue the images of A, B together we shall define K abstractly, cover it with two charts, map the two charts by ϕ_A, ϕ_B, and verify that the maps agree on the overlap as follows. Define $\psi: \mathbb{R} \times S^1 \to \mathbb{R} \times S^1$ to be the diffeomorphism given by

$$\psi(s, \theta) = (2\sigma + s, \pi - \theta).$$

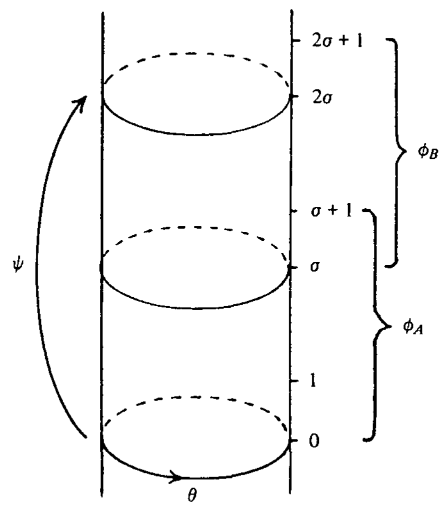


Fig. 6.

Let $K = \mathbb{R} \times S^1/\mathbb{Z}$, where $\mathbb{Z}$ is the integer group action generated by ψ. Then K is a Klein bottle because ψ translates $\mathbb{R}$ and reflects S^1.

Define $\phi: K \to T^*\mathbb{R}^2$ by

$$\phi = \begin{cases} \phi_A, & 0 < s < \sigma + 1, \\ \phi_B, & \sigma < s < 2\sigma + 1. \end{cases}$$

Then charts overlap when $\sigma < s < \sigma + 1$ and when $0 < s < 1$ (or equivalently $2\sigma < s < 2\sigma + 1$ due to the identification under ψ), (Fig. 6) and so we have to verify that the definition is unambiguous in these cases. First, if $\sigma < s < \sigma + 1$ then

$$\begin{aligned} \phi_B(s, \theta) &= (-\epsilon \cos\theta, -\beta(s-\sigma), -\epsilon^2 \sin 2\theta, \alpha(s-\sigma)) \\ &= (-\epsilon\cos\theta, -(s-\sigma+1), -\epsilon^2 \sin 2\theta, 0), \quad \text{by } (*) \\ &= (-\epsilon\cos\theta, \sigma - s - 1, -\epsilon^2 \sin 2\theta, 0) \\ &= \phi_A(s, \theta), \quad \text{by } (*). \end{aligned}$$

Next, if $0 < s < 1$ then

$$\begin{aligned} \phi_B \psi(s, \theta) &= \phi_B(2\sigma + s, \pi - \theta) \\ &= (-\epsilon\cos(\pi - \theta), -\beta(\sigma + s), -\epsilon^2 \sin 2(\pi - \theta), \alpha(\sigma + s)) \\ &= (\epsilon \cos\theta, -(\sigma - (\sigma + s) - 1), \epsilon^2 \sin 2\theta, 0) \\ &= (\epsilon\cos\theta, s + 1, \epsilon^2 \sin 2\theta, 0) \\ &= \phi_A(s, \theta). \end{aligned}$$

Hence $\phi_B \psi = \phi_A$, and the definition is unambiguous as required. Finally ϕ is a C^∞ Lagrangian immersion since it has these properties on each open chart.

Remark. We do not know if the immersion can be made analytic or made into an embedding. Possibly w_1 is an obstruction to a Lagrangian embedding, as it is to an embedding in $\mathbb{R}^3$.

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