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Modern Developments in the Theory and Applications of Moving Frames

Peter J. Olver

Abstract

This article surveys recent advances in the equivariant approach to the method of moving frames, concentrating on finite-dimensional Lie group actions. A sampling from the many current applications — to geometry, invariant theory, and image processing — will be presented.

1. *Introduction.*

According to Akivis, [**2**], the method of *repères mobiles*, which was translated into English as *moving frames*†, can be traced back to the moving trihedrons introduced by the Estonian mathematician Martin Bartels (1769–1836), a teacher of both Gauß and Lobachevsky. The apotheosis of the classical development can be found in the seminal advances of Élie Cartan, [**25**, **26**], who forged earlier contributions by Frenet, Serret, Darboux, Cotton, and others into a powerful tool for analyzing the geometric properties of submanifolds and their invariants under the action of transformation groups. An excellent English language treatment of the Cartan approach can be found in the book by Guggenheimer, [**49**].

The 1970's saw the first attempts, cf. [**29**, **45**, **46**, **64**], to place Cartan's constructions on a firm theoretical foundation. However, the method remained constrained within classical geometries and homogeneous spaces, e.g. Euclidean, equi-affine, or projective, [**35**]. In the late 1990's, I began to investigate how moving frames and all their remarkable consequences might be adapted to more general, non-geometrically-based group actions that arise in a broad range of applications. The crucial conceptual leap was to decouple the moving frame theory from reliance on any form of frame bundle. Indeed, a careful reading of Cartan's analysis of moving frames for curves in the projective plane, [**25**], in which he calls a certain 3×3 unimodular matrix the "repère mobile", provided the crucial conceptual breakthrough, leading to a general, and universally applicable, definition of a moving frame as an equivariant map from the manifold back to the transformation group, thereby circumventing the complications inherent in the frame bundle approach. Building on this basic idea, and armed with the powerful tool of the variational bicomplex, [**6**, **151**], Mark Fels and I, [**36**, **37**], were able to formulate a new, powerful, constructive equivariant moving frame theory that can be systematically applied to general transformation groups. All classical moving frames can be reinterpreted in the equivariant framework, but the latter approach immediately applies in far broader generality. Indeed, in later work with Pohjanpelto, [**122**, **126**, **127**, **128**], the equivariant approach were successfully extended to the vastly more complicated arena of infinite-dimensional Lie pseudo-groups, [**79**, **80**, **143**].

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†According to my *Petit Larousse*, [**132**], the word "repère" refers to a temporary mark made during building or interior design, and so a more faithful English translation might have been "movable landmarks".

Cartan's normalization process underlying the construction of the moving frame relies on the choice of a cross-section to the group orbits. This in turn induces a powerful invariantization process that associates to each standard object (function, differential form, differential operator, tensor, variational problem, conservation law, numerical algorithm, etc.) a canonical invariant counterpart. Invariantization of the associated variational bicomplex, [**37**, **74**], produces the powerful recurrence relations, that enable one to determine the structure of the algebra of differential invariants, as well as the invariant differential forms, invariant variational bicomplex, etc., using only linear differential algebra, and, crucially, *without having to know the explicit formulas for either the invariants or the moving frame itself*! It is worth emphasizing that all of the required constructions can be implemented systematically and algorithmically, and thus readily programmed in symbolic computer packages such as MATHEMATICA and MAPLE. Mansfield's recent text, [**84**], on what she calls the "symbolic invariant calculus", provides a basic introduction to the key ideas (albeit avoiding differential forms), and some of the important applications.

In this survey, we will concentrate on prolonged group actions on jet bundles, leading to differential invariants and differential invariant signatures. Applying the moving frame algorithms to Cartesian product actions produces joint invariants and joint differential invariants, along with their associated signatures, [**37**, **115**, **13**], establishing a geometric counterpart of what Weyl, [**162**], in the algebraic framework, calls the First Main Theorem for the transformation group. Subsequently, an amalgamation of jet and Cartesian product actions, named *multi-space*, was proposed in [**116**] to serve as the basis for the geometric analysis of numerical approximations, and, via the application of the moving frame method, the systematic construction of symmetry-preserving numerical approximations and integration algorithms, [**12**, **24**, **23**, **30**, **69**, **70**, **71**, **106**, **134**, **161**].

With the basic moving frame machinery in hand, a plethora of new, unexpected, and compelling applications soon began appearing. In [**23**, **12**, **5**, **7**, **139**], the characterization of submanifolds via their differential invariant signatures was applied to the problem of object recognition and symmetry detection in digital images. The general problem in the calculus of variations of directly constructing the invariant Euler-Lagrange equations from their invariant Lagrangian was solved in [**74**], and then applied, [**118**, **66**, **8**, **155**], to the analysis of the evolution of differential invariants under invariant submanifold flows, leading to integrable soliton equations and the equations governing signature evolution. In [**9**, **72**, **73**, **113**], the theory was applied to produce new algorithms for solving the basic symmetry and equivalence problems of polynomials that form the foundation of classical invariant theory. The all-important recurrence formulae provide a complete characterization of the differential invariant algebra of group actions, and lead to new results on minimal generating invariants, even in very classical geometries, [**117**, **56**, **55**, **119**, **60**].

Further significant applications include the computation of symmetry groups and classification of partial differential equations, [**83**, **101**]; geometry and dynamics of curves and surfaces in homogeneous spaces, with applications to integrable systems, Poisson geometry, and evolution of spinors, [**86**, **87**, **88**, **89**, **91**, **133**]; construction of integral invariant signatures for object recognition in 2D and 3D images, [**38**]; solving the object-image correspondence problem for curves under projections, [**21**, **22**, **75**]; recovering structure of three-dimensional objects from motion, [**7**]; classification of projective curves in visual recognition, [**51**]; recognition of DNA supercoils, [**138**]; distinguishing malignant from benign breast cancer tumors, [**48**], as well as melanomas from moles, [**145**]; determination of invariants and covariants of Killing tensors and orthogonal webs, with applications to general relativity, separation of variables, and Hamiltonian systems, [**31**, **33**, **94**, **95**]; the Noether correspondence between symmetries and invariant conservation laws, [**42**, **43**]; symmetry reduction of dynamical systems, [**59**, **142**]; symmetry and equivalence of polygons and point configurations, [**14**, **65**]; computation

of Casimir invariants of Lie algebras and the classification of subalgebras, with applications in quantum mechanics, [**15**, **16**]; and the cohomology of the variational bicomplex, [**62**, **63**, **147**].

Applications to Lie pseudo-groups, [**122**, **126**, **127**, **128**], include infinite-dimensional symmetry groups of partial differential equations and algorithms for directly determining their structure, [**27**, **28**, **102**, **153**]; climate and turbulence modeling, [**10**], leading to new symmetry-preserving numerical schemes for systems of nonlinear partial differential equations possessing infinite-dimensional symmetry groups, [**135**]; partial differential equations arising in control theory, [**154**]; classification of Laplace invariants and factorization of linear partial differential operators, [**140**, **141**]; construction of coverings and Bäcklund transformations, [**103**]; and the method of group foliation, [**158**, **130**], for finding invariant, partially invariant, and non-invariant explicit solutions to partial differential equations, [**146**, **148**]. In [**98**, **154**, **156**] the moving frame calculus is shown to provide a new and very promising alternative to the Cartan method for solving general equivalence problems based on exterior differential systems, [**41**, **111**]. Finally, recent development of a theory of discrete equivariant moving frames has been applied to integrable differential-difference systems, [**85**]; invariant evolutions of projective polygons, [**92**], that generalize the remarkable integrable pentagram maps, [**67**, **131**]; as well as extensions of the aforementioned group foliation method to construct explicit solutions to symmetric finite difference equations, [**149**].

2. *Equivalence and Signature.*

A primary motivating application of moving frames is the equivalence and symmetry of geometric objects. In general, two objects are said to be *equivalent* if one can map one to the other by a suitable transformation. A *symmetry* of a geometric object is merely a self-equivalence, that is a transformation that maps the object back to itself. Thus a solution to the equivalence problem for objects includes a classification of their symmetries. The solution to any equivalence problem can be viewed as a description of the associated *moduli space* which, in this particular instance, represents the equivalence classes of objects (of a specified type) under the allowed transformations. Of course, equivalences come in many guises — topological, smooth, algebraic, etc. Our focus will be when the equivalence maps belong to a prescribed transformation group and the objects under consideration are submanifolds of the space upon which the group acts. For simplicity, we will restrict our attention here to the smooth — meaning C^∞ — category, and to finite-dimensional (local) Lie group actions, although the methods extend, with additional work, to the actions of infinite-dimensional Lie pseudo-groups.

In this context, Élie Cartan found a complete solution to the local submanifold equivalence problem, which relies on the associated differential invariants. In general, a *differential invariant* is a scalar-valued function that depends on the submanifold and its "derivatives". If one explicitly parametrizes the submanifold, then the differential invariant will be a combination of the parametrizing functions and their derivatives up to some finite order which is unaffected by the induced action of the transformation group and, moreover, is *intrinsic*, that is, independent of the underlying parametrization. More rigorously, [**111**], a differential invariant is a scalar-valued function defined on an open subset of the submanifold jet bundle that is invariant under the prolonged transformation group action.

A familiar example from elementary differential geometry is the equivalence problem for plane curves $C \subset \mathbb{R}^2$ under rigid motions, i.e., the action of the *special Euclidean group* $\mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$, the semi-direct product of the special orthogonal group of rotations and the two-dimensional abelian group of translations. The basic differential invariant is the curvature κ. However, κ is just the first of an infinite collection of independent differential invariants. Indeed, differentiating any differential invariant of order n with respect to the Euclidean-invariant arc

length element ds produces a differential invariant of order $n+1$. In this manner, we produce an infinite collection of independent differential invariants, namely, $\kappa, \kappa_s, \kappa_{ss}, \dots$. Moreover, it can be shown that these form a complete system, in the sense that any other differential invariant can (locally) be written as a function of a finite number of them: $I = F(\kappa, \kappa_s, \kappa_{ss}, \dots, \kappa_{n-2})$ whenever I is a differential invariant of order n.

Similarly, under the action of the equi-affine group $\mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$, consisting of unimodular linear transformations and translations, on plane curves, there is a well-known equi-affine curvature invariant[†] κ, which is of order 4, and an equi-affine arc length element ds, such that the complete system of differential invariants consists of $\kappa, \kappa_s, \kappa_{ss}, \dots$. Under the projective group $\mathrm{PSL}(3)$ acting by projective (linear fractional) transformations, the complete system of differential invariants is provided by the seventh order projective curvature invariant and its successive derivatives with respect to projective arc length element, [**25**, **72**, **111**]. Indeed, a completely analogous statement holds for almost all transitive planar Lie group actions. Every ordinary[‡] Lie group action on plane curves admits a unique, up to functions thereof, differential invariant of lowest order, denoted by κ, identified as the group-invariant curvature, and a unique, up to constant multiple, invariant[§] one-form $\omega = ds$, viewed as the group-invariant arc length element. Moreover, a complete system of differential invariants is provided by the curvature and its successive derivatives with respect to the arc length: $\kappa, \kappa_s, \kappa_{ss}, \dots$. See [**111**] for complete details, including the corresponding statements in the intransitive and non-ordinary cases.

Turning to the equivalence of space curves $C \subset \mathbb{R}^3$ under the action of the Euclidean group $\mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$, there are two basic differential invariants: the curvature κ, which is of second order, and its torsion τ, of third order. Moreover, they and their successive derivatives with respect to arc length form a complete system of differential invariants: $\kappa, \tau, \kappa_s, \tau_s, \kappa_{ss}, \tau_{ss}, \dots$. Analogous results hold for space curves under any ordinary group action on $\mathbb{R}^3$, [**49**, **111**]. Further, the Euclidean action on two-dimensional surfaces $S \subset \mathbb{R}^3$ has two familiar second order differential invariants: the Gauss curvature K and the mean curvature H. Again, one can produce an infinite collection of higher order differential invariants by invariantly differentiating the Gauss and mean curvature. Specifically, at a non-umbilic point, there exist two (non-commuting) invariant differential operators $\mathcal{D}_1, \mathcal{D}_2$, that effectively differentiate in the direction of the orthonormal Darboux frame; a complete system of differential invariants consists of $K, H, \mathcal{D}_1 K, \mathcal{D}_2 K, \mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1^2 K, \mathcal{D}_1 \mathcal{D}_2 K, \mathcal{D}_2 \mathcal{D}_1 K, \mathcal{D}_2^2 K, \mathcal{D}_1^2 H, \dots$, [**49**, **111**, **119**]. However, as we will prove below, for suitably generic surfaces, the mean curvature alone can be employed to generate the entire algebra of differential invariants! Further results on the differential invariants of surfaces in three-dimensional space under various geometrical group actions can be found in Theorem 19 below.

All of the preceding examples can be viewed as particular cases of the *Fundamental Basis Theorem*, which states that, for any Lie group action, the entire algebra of differential invariants can be generated from a *finite number* of low order invariants by repeated invariant differentiation. In differential invariant theory, this result assumes the role played by the algebraic Hilbert Basis Theorem for polynomial ideals, [**32**]. Bear in mind that here we distinguish differential invariants that are *functionally independent*, and not merely algebraically independent.

[†]We employ a common notation, keeping in mind that the curvature and arc length invariants will depend on the underlying group action.

[‡]A Lie group is said to act *ordinarily*, [**111**], if it acts transitively on M, and the maximal dimension of the orbits of its successive prolongations strictly increase until the action becomes locally free, as defined below; or, in other words, its prolongations do not "pseudo-stabilize", [**112**]. Almost all transitive Lie group actions are ordinary.

[§]Or, to be completely correct, "contact-invariant"; see below for the explanation.

THEOREM 1. *Let G be a finite-dimensional Lie group acting on p-dimensional submanifolds $S \subset M$. Then, locally, there exist a finite collection of* generating differential invariants $\mathcal{I} = \{I^1, \dots, I^\ell\}$, *along with exactly p* invariant differential operators $\mathcal{D}_1, \dots, \mathcal{D}_p$, *such that every differential invariant can be expressed as a function of the generating invariants and their invariant derivatives $I^\nu_{,J} = \mathcal{D}_{j_1}\mathcal{D}_{j_2}\cdots\mathcal{D}_{j_k} I^\nu$.*

The Basis Theorem was first formulated by Lie, [**82**, p. 760]. Modern proofs of Lie's result can be found in [**111**, **130**], while a fully constructive moving frame-based proof appears in [**37**]. Under certain technical hypotheses, the Basis Theorem also holds as stated for rather general infinite-dimensional Lie pseudo-group actions; a version first appears in the work of Tresse, [**150**]. A rigorous result, based on the machinery of Spencer cohomology, was established by Kumpera, [**79**]. A global version for algebraic pseudo-group actions, including an extension to actions on differential equations (subvarieties of jet space) can be found in [**77**], while [**105**] introduces yet another approach, based on Weil algebras. The first constructive proof of the pseudo-group Basis Theorem, based on the equivariant moving frame machinery, appears in [**128**]. While many structural questions remain as yet incompletely answered, the equivariant moving frame calculus provides a complete, systematic, algorithmic suite of computational tools, eminently suited to implementation on standard computer algebra packages, for analyzing the associated differential invariant algebra, its generators, relations (syzygies), and so on.

Knowing the differential invariants, we return to the equivalence problem. Clearly, any two equivalent submanifolds must have the same differential invariants at points corresponding under the equivalence transformation. If a differential invariant is constant, then it must necessarily assume the same constant value on any equivalent submanifold. For example, if a plane curve has Euclidean curvature $\kappa = 2$, it must be a circular arc of radius $\frac{1}{2}$. Any rigidly equivalent curve must also be a circular arc of the same radius, and hence have the same curvature. On the other hand, if a differential invariant is not constant, then this, in and of itself, does not provide much information, because its expression will depend upon the parametrization of the underlying submanifold, and hence direct comparison of two non-constant differential invariants may be problematic. Instead, Cartan tells us to look at the functional inter-relationships among the differential invariants, which *are* intrinsic. These functional relationships are also known as *syzygies*, again in analogy with the algebraic Hilbert Syzygy Theorem, [**32**], although, as above, we do not restrict to polynomial relations but allow arbitrary smooth functions. For example, if a plane curve satisfies the syzygy $\kappa_s = e^\kappa - 1$ between its two lowest order differential invariants, then so must any equivalent curve.

Remark: There are two distinct kinds of syzygy. *Universal syzygies* are satisfied by all submanifolds. A celebrated example is the Gauss-Codazzi relation among the differential invariants of Euclidean surfaces $S \subset \mathbb{R}^3$, [**49**, **119**]. The second kind are particular to the individual submanifolds, and, as we will see, serve to prescribe their local equivalence and symmetry properties.

Cartan's solution to the equivalence problem states, roughly, that two suitably nondegenerate submanifolds are *locally equivalent*[†] if and only if they have *identical* syzygies among *all* their differential invariants. Cartan's proof relies on his "Technique of the Graph", in which the graph $\Gamma_g \subset M \times M$ of the equivalence transformation $g\colon M \to M$ is realized as a suitable

[†]For example, any two circular arcs having the same radius are locally equivalent under the Euclidean group irrespective of their overall length, which is a global property. Global equivalence issues are also very interesting, and in need of significant further investigation. In this vein, see [**125**], which employs the language of groupoids, [**160**], to better understand the inherent local versus global structure of symmetries and equivalences.

solution (integral submanifold) of an overdetermined system of partial differential equations on $M \times M$, the Cartesian product of the underlying manifold with itself. In the proof, the first order of business is to establish involutivity of this overdetermined system, which then implies, via Frobenius' Theorem or, in the pseudo-group case, assuming analyticity, the Cartan–Kähler Theorem, the existence of a suitable integral submanifold which represents the graph of the desired equivalence map. See [**111**] for a detailed development and complete proofs.

Of course, as we have just seen, there are an infinite number of differential invariants, and hence an infinite number of syzygies since, locally, on any p-dimensional submanifold there can be at most p independent functions. However, one finds that, in general, one can generate all the higher order syzygies from only a finite number of low order ones. To see why this might be the case consider the case of a plane curve under a prescribed ordinary transformation group, e.g., Euclidean, equi-affine, projective, etc. Temporarily leaving aside the case when the curvature invariant κ is constant, there is, when restricted to the one-dimensional curve C, but one functionally independent differential invariant, which we may as well take to be κ. At a point where $\kappa_s \neq 0$, we can locally write any other differential invariant as a function of κ, and hence the syzygies are all consequences of

$$\kappa_s = H_1(\kappa), \qquad \kappa_{ss} = H_2(\kappa), \qquad \kappa_{sss} = H_3(\kappa), \qquad \dots .$$

However, the first of these completely determines the rest. Indeed, by the chain rule,

$$\kappa_{ss} = \frac{d\kappa_s}{ds} = \frac{d}{ds} H_1(\kappa) = H_1'(\kappa)\, \kappa_s = H_1'(\kappa)\, H_1(\kappa), \quad \text{hence} \quad H_2(\kappa) = H_1'(\kappa)\, H_1(\kappa). \tag{2.1}$$

Iterating this computation enables one to explicitly determine all the higher order syzygy functions $H_2(\kappa), H_3(\kappa), \dots$, in terms of $H_1(\kappa)$ and its derivatives. We conclude that, generically, *the local equivalence of plane curves under an ordinary transformation group is entirely determined by the functional relationship among its two lowest order differential invariants*:

$$\kappa_s = H_1(\kappa). \tag{2.2}$$

The syzygy (2.2) relies on the assumption that $\kappa_s \neq 0$. Moreover, the explicit determination of the function $H_1(\kappa)$ may be problematic. As I observed in [**111**], both objections can be overcome by instead regarding the differential invariants (κ, κ_s) as parametrizing a plane curve $\Sigma \subset \mathbb{R}^2$, known as the differential invariant *signature curve*. In the special case when κ is constant, and hence $\kappa_s \equiv 0$, the signature curve degenerates to a single point.

More generally, as a consequence of the Fundamental Basis Theorem, one can prove that, when restricted to any suitable submanifold, there always exists a finite number of low order differential invariants, say $J^1, \dots, J^k$ with the property that all the higher order differential invariant syzygies can be generated from the syzygies among the J^α's via invariant differentiation. These typically include the generating differential invariants $I^1, \dots, I^\ell$ as well as a certain finite collection of their invariant derivatives $I^\nu_{,J}$. These differential invariants serve to define a *signature map* $\sigma\colon S \to \Sigma \subset \mathbb{R}^N$ whose image is a *differential invariant signature* of the original submanifold S. Under certain regularity assumptions, the signature solves the equivalence problem: two p-dimensional submanifolds are locally equivalent under the transformation group if and only if they have identical signatures. The precise determination of the differential invariants required to form a signature is facilitated through the use of the moving frame calculus to be presented in the final section.

Remark: In my earlier work, [**37**, **111**], the differential invariant signature was called the *classifying manifold*. The more compelling term *signature* was adopted in light of significant applications in image processing, [**23**, **53**], and is now consistently used in the literature. In [**111**], an alternative approach to the construction of the differential invariant signature is founded on the Cartan calculus of exterior differential systems, [**19**, **41**].

Remark: The reader may be familiar with the classical result, [**49**], that a Euclidean curve is uniquely determined up to rigid motion by its curvature function, expressed in terms of arc length $\kappa(s)$. This solution to the equivalence problem has several practical shortcomings in comparison with the differential invariant signature. First, the arc length is ambiguously defined, since it depends on the choice of an initial point on the curve. Hence, one must identify two curvature functions that differ by a translation, $\kappa(s+c) \simeq \kappa(s)$. On the other hand, the differential invariants parametrizing the signature curve are entirely local. This is important in practical applications, particularly when occlusions are present, and so part of the image curve is missing, [**17**, **18**]. The effect on the signature curve is minimal, being only the omission of a (hopefully) small part; on the other hand, it is not even possible to reconstruct the arc length relating two disconnected pieces of an occluded contour. Finally, and most importantly of all, there are, in general, *no* canonical invariant parameters that can assume the role of arc length in the case of surfaces and higher dimensional submanifolds, whereas the differential invariant signature method applies in complete generality.

In this manner, we have effectively reduced the equivalence problem of submanifolds under a transformation group to the problem of recognizing when their signatures are identical. In the restricted case when the submanifolds (signatures) are rationally parametrized, the latter problem can be rigorously solved by Gröbner basis techniques, [**20**]. In practical applications, one introduces a measure of closeness of the signatures, keeping in mind that noise and other artifacts may prevent their being exactly the same. Quite a few measures have been proposed, such as Hausdorff distance, [**61**], metrics based on Monge–Kantorovich optimal transport, [**50**, **159**], and Gromov–Hausdorff and Gromov–Wasserstein metrics, [**96**, **97**]. A comparison of the advantages and disadvantages of several proposed shape metrics can be found in [**11**, **104**]. In the applications to jigsaw puzzle assembly, discussed below, our preferred measure of closeness comes from viewing the two signature curves as wires that have opposite electrical charges, and then computing their electrostatic attraction, cf. [**39**, **163**], or, equivalently, their gravitational attraction, suitably renormalized. Statistical techniques based on latent semantic analysis have been successfully applied in [**5**, **139**], while in [**47**, **48**], the skewness measure of the cumulative distance and polar/spherical angle magnitudes was employed.

Since symmetries are merely self-equivalences, the signature also determines the (local) symmetries of the submanifold. In particular, the dimension of the signature equals the codimension of the symmetry group. More specifically, if a suitably nondegenerate, connected, p-dimensional submanifold $S \subset M$ has signature Σ of dimension $0 \leqslant t \leqslant p$, then the connected component of its local symmetry group G_S containing the identity is an $(r-t)$–dimensional local Lie subgroup of G. In particular, the signature of connected submanifold degenerates to a single point if and only if all its differential invariants are constant. Such *maximally symmetric* submanifolds, [**120**], can, in fact, be characterized algebraically.

THEOREM 2. *A connected nondegenerate p-dimensional submanifold S has 0-dimensional signature if and only if its local symmetry group is a p-dimensional subgroup $H \subset G$ and hence S is an open submanifold of an H–orbit: $S \subset H \cdot z_0$.*

Remark: So-called *totally singular* submanifolds may admit even larger symmetry groups. For example, in three-dimensional Euclidean geometry, the maximally symmetric curves are arcs of circles, whose local symmetry group is contained in a one-parameter rotation subgroup, and segments of circular helices, with a one-parameter local symmetry group of screw motions. On the other hand, straight lines are totally singular curves, possessing a two-dimensional symmetry group, consisting of translations in its direction and rotations around it. Similarly, the maximally symmetric surfaces are open submanifolds of circular cylinders, whose local

symmetry group consists of translations in the direction of the axis of the cylinder and rotations around it. In contrast, both planes and spheres are totally umbilic, and hence totally singular, each possessing a three-dimensional symmetry group. A complete Lie algebraic characterization of totally singular submanifolds for general Lie group actions can be found in [**114**].

At the other extreme, if a nondegenerate p-dimensional submanifold has p-dimensional signature, it only admits a discrete symmetry group. The number of local symmetries is determined by its *index*, which is defined as the number of points in S map to a single generic point of the signature:

$$\operatorname{ind} S = \min \left\{ \# \, \sigma^{-1}\{\zeta\} \;\middle|\; \zeta \in \Sigma \right\}. \tag{2.3}$$

To illustrate, Figure 1 displays the Euclidean signatures of two images of a hardware nut, computed using the invariant numerical approximations we developed in [**24**, **23**]; the horizontal axis in the signature graph is κ while the vertical axis is κ_s. The evident four-fold (approximate) rotational symmetry is represented by the fact that the signature graph is approximately retraced four times. (Folding the graph, by plotting $|\,\kappa_s\,|$ instead of κ_s on the vertical axis, would reveal the 8-fold reflection and rotation symmetry group.) The indicated measure of closeness of the two signatures is based on their (pseudo-)electrostatic repulsion.

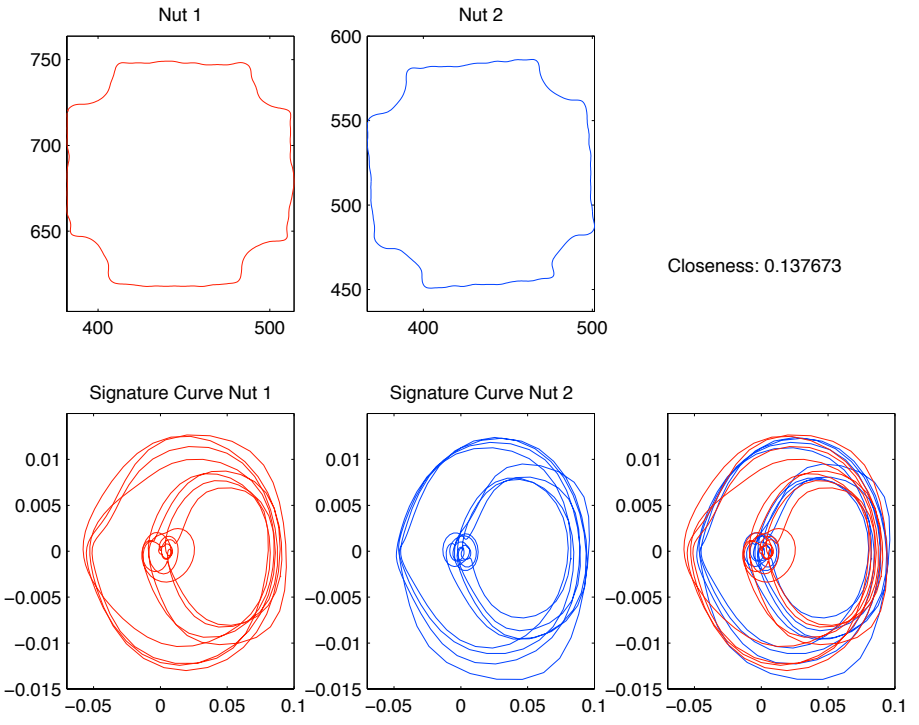

FIGURE 1. *Signatures of Two Images of a Nut*

The following subsections contain brief descriptions of some novel applications of signature curves.

An Initial Investigation into Medical Imaging:

The following is taken from [**23**] as a "proof-of-concept" illustration of the potential of signature curve-based methods in practical image processing, concentrating on a particular

medical image. In Figure 4 at the end of the paper, we display our starting point: a 70×70, 8-bit gray-scale image of a cross section of a canine heart, obtained from an MRI scan.

The first step in geometric object recognition in digital images is to extract the boundary of the object in question, an operation that is known as *segmentation*. A variety of techniques have been developed to accomplish this, one of the most powerful being based on the method of *active contours*, also known as *snakes*, which nowadays are included as a standard tool in many basic image processing software packages. The aim is, starting with a more or less arbitrary contour that encircles the object, to actively shrink the contour so that it converges to the desired boundary. A variety of methods that realize this goal have been developed, many based on nonlinear geometric partial differential equations, [**68**, **137**, **165**]. The one used here starts with the celebrated Euclidean-invariant curve shortening flow that was studied by Gage, Hamilton, and Grayson, [**40**, **44**], as a precursor to the deep analysis of the Ricci flow on higher dimensional manifolds that led to Perelman's celebrated solution to the Poincaré conjecture, [**100**]. Here, one evolves the curve by moving each point in its normal direction in proportion to curvature; their theorem is that any smooth Jordan curve remains a simple closed curve throughout the evolution, ultimately becoming asymptotically circular before shrinking down to a point in finite time. Now, in order to capture the boundary of an object in a digital image with the shrinking curve, one modifies the underlying Euclidean metric by a conformal factor that highlights[†] object boundaries, e.g., points where the gradient of the gray-scale image is large.

Next, to illustrate robustness of the signature curve under smoothing/denoising, the resulting segmented ventricle boundary curve is then further smoothed by application of the unmodified curve shortening flow. The corresponding Euclidean signatures are computed using the invariant numerical approximations introduced in [**23**], and then smoothly spline-interpolated. Observe that, as the evolving curves approach circularity, their signatures exhibit less variation in curvature and appear to be winding more and more tightly around a point on the κ axis, which eventually runs off to ∞ as the asymptotic circle shrinks down to a single point. Despite the rather extensive smoothing, except for an overall shrinkage as the contour approaches circularity, the basic qualitative features of the different signature curves, and particularly their winding behavior, appear to be remarkably robust. See [**66**] for a theoretical justification of these observations, through use of the maximum principle for the induced parabolic flow of the signature curve, which in turn is based on the moving frame-based analysis of the evolution of differential invariants under invariant submanifold flows, [**118**].

Jigsaw Puzzle Assembly:

In [**54**], the Euclidean-invariant signature was applied to design a MATLAB program that automatically assembles apictorial jigsaw puzzles. The term "apictorial" means that the algorithm uses only the shapes of the pieces and not any superimposed picture or design. An example, the *Baffler Nonagon*, [**164**], appears in Figure 7; assembly takes under an hour on a standard Macintosh laptop. It is important to point out that, unlike most automatic puzzle-solvers in the literature, the algorithm is not restricted to puzzles with "traditionally shaped" pieces situated on a rectangular grid, nor does it depend upon knowledge of the outer boundary of the puzzle. Indeed, it tends to prefer the more exotically shaped pieces, and thus assembles the puzzle from the inside out. The algorithm succeeds even when several pieces are missing, as it is not affected by any holes that might show up in the final assembly.

In detail, the first step is to digitize the individual puzzle pieces, which are photographed at random orientations, and then segment their boundaries, again using a standard active contour package included within MATLAB. The next step is to smooth the resulting curves. It was found

[†]Or, more accurately, is small near regions of interest, in this case potential boundaries of objects.

that the preceding smoothing process based on the curve shortening flow was not suitable since it tends to blur important features such as arcs of high curvature or corners. Instead, a naïve smoothing technique based on iterated spline interpolation and respacing was employed.

Assembly of individual puzzle pieces requires only comparing certain a priori unknown parts of their boundaries. The method, based on the extended signature introduced in [**53**] in response to [**107**], is to split up the boundary curves into *bivertex arcs*, meaning sub-arcs on which $\kappa_s \neq 0$ except at the endpoints. The signatures of the individual bivertex arcs are compared, using the electrostatic-based measure of closeness, in order to locate potential matches. Once a sufficient number of bivertex arcs contained in the boundaries of two pieces are deemed to be equivalent under the *same* Euclidean transformation, a second procedure, called *piece locking* and based on minimizing idealized forces and torques between the edges, then refines the match. The resulting algorithm is surprisingly effective, producing correct matches in such a fashion that it is able to completely assemble several commercially available puzzles.

A subsequent project, the "Humpty Dumpty problem", [**47**], looks at reassembly of three-dimensional jigsaw puzzles obtained by decomposing a curved surface, e.g., a broken eggshell. Here the boundaries of the pieces are space curves, whose Euclidean signatures are parametrized by the curvature and torsion invariants κ, κ_s, τ. An argument similar to that in (2.1) demonstrates that the syzygies among these three basic differential invariants determine all the higher order ones, including τ_s. The resulting signature-based algorithm works quite well on synthetically generated surface puzzles, even in the presence of noise, and already has had some success in treating real-world data. It is worth pointing out that the algorithm works only with the (digitized) pieces and does not require any a priori knowledge of the overall shape of the assembled surface. Further potential applications, especially after combining our approach with algorithms based on picture, design, or texture, include the assembly of broken archaeological artifacts such as ceramics or pottery shards [**136**, **152**].

The extension to broken three-dimensional solid objects, e.g., statues, bones, etc., requires matching their bounding surfaces. While the theoretical underpinnings of the differential invariant signature solution to the surface equivalence problem, based on the mean and Gauss curvatures and their low order invariant derivatives, are known, [**111**, **119**], a number of practical issues remain to be resolved, including the identification of suitable "signature codons" that will play the role of the bivertex arcs, as well as the construction of suitably robust invariant numerical approximations to the required signature invariants.

Cancer Detection:

In a paper of Grim and Shakiban, [**48**], Euclidean signature curves are used to distinguish benign from malignant breast tumors in two-dimensional X-ray photos. (An analogous analysis of melanomas and moles can be found in [**145**].) The guiding principle is that the outline of cancerous tumors will display a higher degree of geometric complexity, and this will be reflected in the overall structure of their associated signature curve. One method of measuring curvature complexity is through the range and frequency of points at which the signature curve crosses the κ and κ_s axes. Indeed, it was found that malignant signature curves exhibit a wider range and larger number of axis crossing points than benign contours. A second measurement distinguishes local from global symmetry of the signatures. Here "symmetry" means a simple bilateral reflectional symmetry of the signature curve across the two axes. "Global symmetry" refers to the entire signature, while "local symmetry" refers to individual sub-arcs. Malignant tumors tend to exhibit a higher degree of local symmetry due to increased spiculation of their outline. On the other hand, the higher degree of global symmetry seen in benign tumor signatures can be viewed as a manifestation of a higher degree of cellular functionality. The above methods of signature comparisons were applied to a data base consisting of 150 breast tumors, and the resulting classification into malignant and benign proved to be statistically

significant. The proposed method thus has potential as a preliminary diagnostic tool enabling one to sort through large numbers of such images.

Classical Invariant Theory:

In a more mathematical direction, we refer the reader to Example 9 below and also [**9**, **72**, **73**, **113**] for the construction of other types of differential invariant signatures in the context of the basic problems of classical invariant theory: the equivalence and symmetry properties of binary and ternary forms.

3. *Equivariant Moving Frames.*

In this section, we develop the basics of the equivariant method of moving frames. To keep the exposition as simple as possible, we only consider global finite-dimensional Lie group actions. Extensions to local Lie group actions are reasonably straightforward, while infinite-dimensional Lie pseudo-groups are more technically demanding, and, for the latter, we refer the interested reader to the survey paper [**122**] for an introduction.

Example 3. Let us begin on familiar ground. Consider the usual action of the special Euclidean group $\mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$ on space curves $C \subset \mathbb{R}^3$. In this situation, as one learns in any basic differential geometry course, [**34**, **49**], the moving frame contains three distinguished orthonormal vectors along the curve: its *unit tangent* $\mathbf{t}$, *unit normal* $\mathbf{n}$, and *unit binormal* $\mathbf{b}$. In coordinates, if one parametrizes the curve by arc length, $z(s) \in \mathbb{R}^3$, then

$$\mathbf{t} = z_s, \qquad \mathbf{n} = \frac{z_{ss}}{\| z_{ss} \|}, \qquad \mathbf{b} = \mathbf{t} \times \mathbf{n}. \tag{3.1}$$

The basic curvature κ and torsion τ differential invariants then arise through the classical *Frénet–Serret equations*

$$\frac{d\mathbf{t}}{ds} = \kappa\, \mathbf{n}, \qquad \frac{d\mathbf{n}}{ds} = -\kappa\, \mathbf{t} + \tau\, \mathbf{b}, \qquad \frac{d\mathbf{b}}{ds} = -\tau\, \mathbf{n}. \tag{3.2}$$

However, Cartan emphasizes that there is, in fact, one further constituent to the moving frame: the point on the curve $z = z(s)$, which he calls the "moving frame of order 0", [**25**]. The moving frame of order 1 includes the unit tangent $\mathbf{t}$, while the entire moving frame, which consists of the point on the curve z along with the orthonormal frame vectors $\mathbf{t}, \mathbf{n}, \mathbf{b}$ based there, is of order 2 since it depends upon second order derivatives. The curvature and torsion invariants have order 2 and 3, respectively.

Let us also look briefly at the equi-affine group $\mathrm{SA}(3) = \mathrm{SL}(3) \ltimes \mathbb{R}^3$, consisting of volume–preserving affine transformations $z \mapsto A\, z + b$, $\det A = 1$, acting on space curves $C \subset \mathbb{R}^3$. The moving frame, now of order 4, consists of a point on the curve, a tangent vector $\mathbf{t}$, no longer of unit length (indeed, there is no intrinsic notion of length in equi-affine geometry) along with two vectors $\mathbf{n}, \mathbf{b}$ transverse to the curve, with the property that the three vectors form a unimodular frame: $\mathbf{t} \cdot \mathbf{n} \times \mathbf{b} = 1$. Again, Cartan clearly states that the point on the curve z at which the frame vectors are based is an essential component of the moving frame. The two independent differential invariants resulting from the associated Frénet–Serret equations are both of order 5, [**49**].

For the equivariant approach, the starting point is an arbitrary r-dimensional Lie group G acting smoothly on an m-dimensional manifold M. The general definition of an equivariant moving frame proposed in [**37**] is as follows:

DEFINITION 4. A *moving frame* is a smooth, G-equivariant map[†] $\rho : M \to G$.

There are two principal types of equivariance:

$$\rho(g \cdot z) = \begin{cases} g \cdot \rho(z) : & \text{left moving frame,} \\ \rho(z) \cdot g^{-1} : & \text{right moving frame.} \end{cases} \tag{3.3}$$

In classical geometries, as in [**49**], one can always reinterpret the frame-based moving frames as left-equivariant maps. For example, in the standard Euclidean moving frame for a space curve, if one views the orthonormal frame vectors (3.1) as the columns of an orthogonal matrix and their base point on the curve z as a translation vector, this effectively defines a map from the curve[†] to the Euclidean group $\mathrm{E}(3) = \mathrm{O}(3) \ltimes \mathbb{R}^3$, which is readily seen to be left-equivariant, and hence satisfies the requirement of Definition 4. A similar interpretation holds for the equi-affine moving frame described above — now the frame vectors form the columns of a unimodular matrix, and the point on the curve continues to serve as a translation vector, thus defining a left-equivariant map from the curve to the equi-affine group, that now depends upon fourth order derivatives. On the other hand, right-equivariant moving frames are at times easier to compute, and will be the primary focus here. Bear in mind that if $\rho(z)$ is a right-equivariant moving frame, then application of the inversion map on G produces a left-equivariant counterpart: $\widetilde{\rho}(z) = \rho(z)^{-1}$.

With this definition in place, it is not difficult to establish the basic requirements for the existence of an equivariant moving frame. To this end, recall that the group G is said to act *freely* if the *isotropy subgroup*

$$G_z = \{\, g \in G \mid \ g \cdot z = z \,\} \tag{3.4}$$

of each point $z \in M$ is trivial: $G_z = \{e\}$. Slightly weaker is the notion of *local freeness*, which requires that the isotropy subgroups G_z be discrete, or, equivalently, that the group orbits all have the same dimension, r, as G itself. On the other hand, *regularity* requires that, in addition, the orbits form a regular foliation, but this is a global condition that plays no role in practical applications and hence can be safely ignored.

THEOREM 5. *A moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.*

The explicit construction of an equivariant moving frame map is based on Cartan's normalization procedure. This relies on the choice of a (local) *cross-section* to the group orbits, meaning an $(m-r)$–dimensional submanifold $\mathcal{K} \subset M$ that intersects each orbit at most once, and transversally, meaning that the orbit and the cross-section have no non-zero tangent vectors in common.

THEOREM 6. *Let G act freely and regularly on M, and let $\mathcal{K} \subset M$ be a cross-section. Given $z \in M$, let $g = \rho(z)$ be the unique group element that maps z to the cross-section: $g \cdot z = \rho(z) \cdot z = k \in \mathcal{K}$. Then $\rho : M \to G$ is a right moving frame.*

[†]Throughout, functions, maps, etc., may only be defined on an open subset of their indicated source space, so that $\operatorname{dom} \rho \subseteq M$.

[†]Or, more accurately, the second order jet of the curve, since the frame vectors depend upon second order derivatives of the curve's parametrization.

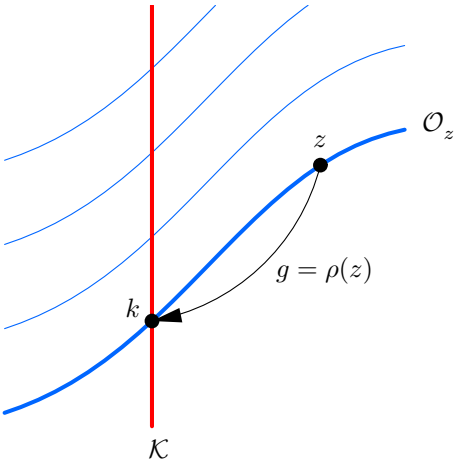

FIGURE 2. *Moving Construction Based on Cross–Section*

The normalization construction of the moving frame is illustrated in Figure 2. The curves represent group orbits, with $\mathcal{O}_z$ denoting the orbit through the point $z \in M$. The unique point in the intersection, namely $k = \rho(z) \cdot z \in \mathcal{O}_z \cap \mathcal{K}$, can be viewed as the *canonical form* or *normal form* of the point z, as prescribed by the cross-section $\mathcal{K}$. In practice, cross-sections are local, and the resulting moving frame defined on a certain open subset of the entire manifold. Further, if the action is locally free, the resulting (local) moving frame will be locally equivariant in the evident manner.

Introducing local coordinates $z = (z_1, \dots, z_m)$ on M, the cross-section $\mathcal{K}$ will be defined by r equations

$$W_1(z) = c_1, \qquad \dots \qquad W_r(z) = c_r, \tag{3.5}$$

where $W_1, \dots, W_r$ are scalar-valued functions, while $c_1, \dots, c_r$ are suitably chosen constants. In the vast majority of applications, the W_ν are merely a subset of the coordinate functions $z_1, \dots, z_m$, in which case they are said to define a *coordinate cross-section*. (Indeed, Figure 2 is drawn as if $\mathcal{K}$ is a coordinate cross-section.) The associated right moving frame $g = \rho(z)$ is thus obtained by solving the *normalization equations*

$$W_1(g \cdot z) = c_1, \qquad \dots \qquad W_r(g \cdot z) = c_r, \tag{3.6}$$

for the group parameters $g = (g_1, \dots, g_r)$ in terms of the coordinates $z = (z_1, \dots, z_m)$. Transversality of the cross-section combined with the Implicit Function Theorem ensures the existence of a local solution $g = \rho(z)$ to the normalization equations (3.6), whose equivariance is assured by Theorem 6. In practical applications, the art of the method is to select a well-adapted cross-section meaning, typically, one that simplifies the calculations as much as possible. More prosaically, this usually means choosing a simple coordinate cross-section and setting as many of the normalization constants $c_\nu = 0$ as possible, keeping in mind the requirement that the resulting equations define a valid cross-section. The method is self-correcting, in that an invalid choice will lead to a system of equations that is not uniquely and smoothly soluble for the group parameters.

With the equivariant moving frame in hand, the next step is to determine the *invariants*, that is, (locally defined) functions $I \colon M \to R$ that are unchanged by the group action: $I(g \cdot z) = I(z)$ for all $z \in \operatorname{dom} I$ and all $g \in G$ such that $g \cdot z \in \operatorname{dom} I$. Equivalently, a function is invariant if and only if it is constant on the orbits. Since any orbit that intersects the cross-section meets it in a unique point, the value of an invariant on those orbits is uniquely determined by its

value on the cross-section. This serves to define a process (depending upon the cross-section) that converts functions to invariants.

DEFINITION 7. The *invariantization* $I = \iota(F)$ of a function $F\colon M \to \mathbb{R}$ is the unique invariant function that coincides with F on the cross-section: $I \mid \mathcal{K} = F \mid \mathcal{K}$.

In particular, if I is any invariant, then clearly $\iota(I) = I$. Thus, invariantization can be viewed as a projection from the space of functions to the space of invariants. Moreover, by construction, invariantization preserves all algebraic operations on functions. Invariantization (and its many consequences) constitutes a key advantage of the equivariant approach over classical frame-based methods.

Computationally, a function $F(z)$ is invariantized by first transforming it according to the group action, producing $F(g \cdot z)$, and then replacing the group parameters by their moving frame formulae $g = \rho(z)$, so that

$$\iota[\,F(z)\,] = F(\rho(z) \cdot z). \tag{3.7}$$

Invariantization of the coordinate functions yields the *fundamental invariants*:

$$I_1(z) = \iota(z_1), \qquad \dots \qquad I_m(z) = \iota(z_m). \tag{3.8}$$

With these in hand, the invariantization of a general function $F(z)$ is simply obtained by replacing each variable z_j in its local coordinate expression by the corresponding fundamental invariant I_j:

$$\iota\big[\,F(z_1, \dots, z_m)\,\big] = F(I_1(z), \dots, I_m(z)). \tag{3.9}$$

In particular, the functions defining the cross-section (3.5) have constant invariantization, $\iota(W_\nu(z)) = c_\nu$, and are known as the *phantom invariants*. One can then select precisely $m - r$ functionally independent *basic invariants* from among the invariantized coordinate functions (3.8), in accordance with Frobenius' Theorem, [**110**]. For a coordinate cross-section given by setting the first r, say, coordinates to constants: $z_1 = c_1, \ \dots \ , z_r = c_r$, then the remaining $m - r$ non-phantom fundamental invariants $I_{r+1}(z) = \iota(z_{r+1}), \ \dots \ , I_m(z) = \iota(z_m)$ are the functionally independent basic invariants.

The fact that invariantization does not affect invariants implies the elegant and powerful *Replacement Rule*, that enables one to immediately rewrite any invariant $J(z_1, \dots, z_m)$ in terms of the basic invariants:

$$J(z_1, \dots, z_m) = J(I_1(z), \dots, I_m(z)). \tag{3.10}$$

In symbolic analysis, (3.10) is known as a *rewrite rule*, [**57**, **58**], and underscores the power of the moving frame approach over rival invariant-theoretic constructions, including Hilbert and Gröbner bases, [**32**].

According to Theorem 5, for the constructions presented above to succeed, the key requirement is that the group act freely or, at the very least, locally freely. Of course, most interesting group actions are *not* free — indeed, typically, the dimension of G is strictly greater than the dimension of M, as is always the case when $M = G/H$ is a nontrivial homogeneous space — and therefore do not per se admit moving frames in the sense of Definition 4. Thus, for example, the dimension of the three-dimensional Euclidean group SE(3) is 6, which is greater than the dimension of the space it acts upon, namely 3, and so the action cannot be free; indeed, the isotropy group of a point $z \in \mathbb{R}^3$ consists of all rotations aroud that point $G_z \simeq \mathrm{SO}(3)$.

There are two classical methods that (usually) convert a non-free action into a free action. The first is the Cartesian product action of G on several copies of M; application of the moving frame normalization construction and invariantization produces joint invariants, [**115**]. The

second is to prolong the group action to jet space, which is the natural setting for the traditional moving frame theory, and leads to differential invariants, [**37**]. Combining the two methods of jet prolongation and Cartesian product results in joint differential invariants, [**115**], also known in the computer vision literature as semi-differential invariants, [**99**, **157**]. In applications of symmetry methods in numerical analysis, one requires an amalgamation of all these actions into a common framework, called *multi-space*, introduced in [**116**] — although the complete construction is so far only known for curves. (However, a very recent preprint of Marí Beffa and Mansfield, [**90**], makes an initial foray into the multivariate realm.) In this paper we will deal only with on the jet space version of prolongation, and refer the interested reader to [**124**] for a more complete overview.

4. *Moving Frames on Jet Space and Differential Invariants.*

Given an action of the Lie group G on the manifold M, our goal is to understand its induced action on (embedded) submanifolds $S \subset M$ of a prescribed dimension $1 \leqslant p < m = \dim M$. We begin by prolonging the group action to the n-th order (extended) *jet bundle* $\mathrm{J}^n = \mathrm{J}^n(M,p)$, which is defined as the set of equivalence classes of p-dimensional submanifolds under the equivalence relation of n-th order contact at a single point; see [**108**, **111**] for details. Since G maps submanifolds to submanifolds while preserving the contact equivalence relation, it induces an action on the jet space J^n, known as its n-th order *prolongation* and denoted here by $z^{(n)} \longmapsto g \cdot z^{(n)}$ for $g \in G$ and $z^{(n)} \in \mathrm{J}^n$. In local coordinates — see below for details — the formulas for the prolonged group action are straightforwardly found by implicit differentiation, the disadvantage being that the resulting expressions can rapidly become extremely unwieldy.

We assume, without significant loss of generality, that G acts *effectively on open subsets* of M, meaning that the only group element that fixes *every* point in any given open $U \subset M$ is the identity element: $\bigcap_{z \in U} G_z = \{e\}$. This implies, [**114**], that the prolonged action is locally free on a dense open subset $\mathcal{V}^n \subset \mathrm{J}^n$ for $n \gg 0$ sufficiently large, whose points $z^{(n)} \in \mathcal{V}^n$ are known as *regular jets*. In all known examples that arise in applications, the prolonged action is, in fact, free on such an open subset $\widetilde{\mathcal{V}}^n \subset \mathrm{J}^n$ for suitably large n. However, recently, Scot Adams, [**1**], constructed rather intricate examples of smooth Lie group actions that do not become eventually free on any open subset of the jet space. Indeed, Adams proves that if the group has compact center, the prolonged actions always become eventually free on an open subset of jet space, whereas any connected Lie group with non-compact center admits actions that do not become eventually free. In practice, one is often content to work with locally free prolonged actions, producing locally equivariant moving frames, keeping in mind that certain algebraic ambiguities arising from the normalization construction, e.g., branches of square roots, must be handled with some care.

A real-valued function on jet space, $F \colon \mathrm{J}^n \to \mathbb{R}$ is known as a *differential function*[†]. A *differential invariant* is a differential function $I \colon \mathrm{J}^n \to \mathbb{R}$ that is an invariant for the prolonged group transformations, so $I(g \cdot z^{(n)}) = I(z^{(n)})$ for all $z^{(n)} \in \mathrm{J}^n$ and all $g \in G$ such that both $z^{(n)}$ and $g \cdot z^{(n)}$ lie in the domain of I. Clearly, any algebraic combination of differential invariants is a differential invariant (on their common domain of definition) and thus we speak, somewhat loosely, of the *algebra of differential invariants* associated with the action of the transformation group on submanifolds of a specified dimension. Since differential invariants are often only

[†]As noted above, functions, maps, etc., may only be defined on an open subset of their indicated source space: $\operatorname{dom} F \subset \mathrm{J}^n$. Also, we identify F with its pull-backs, $F \circ \pi^k_n$, under the standard jet projections $\pi^k_n \colon \mathrm{J}^k \to \mathrm{J}^n$ for any $k \geqslant n$. Similar remarks apply to differential forms on jet space.

locally defined[‡], to be fully rigorous, we should introduce the category of *sheaves of differential invariants*, [**78**, **79**]. However, since here we concentrate entirely on local results, this extra level of abstraction is unnecessary, and so we will leave the sheaf-theoretic reformulation of the theory as a translational exercise for the experts.

As before, the normalization construction based on a choice of local cross-section $\mathcal{K}^n \subset \mathcal{V}^n \subset \mathrm{J}^n$ to the prolonged group orbits can be used to produce an *n-th order equivariant moving frame* $\rho : \mathrm{J}^n \to G$ in a neighborhood of any regular jet. The cross-section $\mathcal{K}^n$ is prescribed by setting a collection of $r = \dim G$ independent n-th order differential functions to suitably chosen constants

$$W_1(z^{(n)}) = c_1, \qquad \dots \qquad W_r(z^{(n)}) = c_r. \tag{4.1}$$

The associated right moving frame $g = \rho(z^{(n)})$ is then obtained by solving the corresponding *normalization equations*

$$W_1(g \cdot z^{(n)}) = c_1, \qquad \dots \qquad W_r(g \cdot z^{(n)}) = c_r, \tag{4.2}$$

for the group parameters $g = (g_1, \dots, g_r)$ in terms of the jet coordinates $z^{(n)}$. Once the moving frame is established, the induced invariantization process will map general differential functions $F(z^{(k)})$, of any order k, to differential invariants $I = \iota(F)$, which are obtained by first transforming them by the prolonged group action and then substituting the moving frame formulas for the group parameters:

$$I(z^{(l)}) = \iota\big[\, F(z^{(k)}) \,\big] = F(\rho(z^{(n)}) \cdot z^{(k)}), \qquad l = \max\{\, k, n \,\}. \tag{4.3}$$

Invariantization preserves differential invariants, $\iota(I) = I$, and hence defines a canonical projection (depending on the moving frame) from the algebra of differential functions to the algebra of differential invariants that preserves all algebraic operations.

Remark: Although essential for theoretical progress, one practical disadvantage of the normalization procedure described above is that it requires one to first prolong the group action to a sufficiently high order in order that it become free. The intervening formulae, obtained by implicit differentiation, may become unwieldy, making the symbolic implementation of the algorithm on a computer impractical due to excessive expression swell. To circumvent this difficulty, a recursive version of the moving frame construction, that successively normalizes the group parameters at each jet space order before prolonging the resulting reduced action to the next higher order can be found in [**123**]. See also [**129**] for a recent extension of the recursive algorithm to Lie pseudo-group actions.

For calculations, we introduce local coordinates $z = (x, u) = (x^1, \dots, x^p, u^1, \dots, u^q)$ on M, considering the first p components $x = (x^1, \dots, x^p)$ as independent variables, and the latter $q = m - p$ components $u = (u^1, \dots, u^q)$ as dependent variables. Submanifolds that are transverse to the vertical fibers $\{\, x = \text{constant} \,\}$ can thus be locally identified as the graphs of functions $u = f(x)$. This splitting into independent and dependent variables induces corresponding local coordinates $z^{(n)} = (x, u^{(n)}) = (\ \dots\ x^i\ \dots\ u^\alpha_J\ \dots\ )$ on J^n, whose components u^α_J, with $1 \leqslant \alpha \leqslant q$, and $J = (j_1, \dots, j_k)$, with $1 \leqslant j_\nu \leqslant p$, a symmetric multi-index of order $0 \leqslant k = \#J \leqslant n$, represent the partial derivatives, $\partial^k u^\alpha / \partial x^{j_1} \cdots \partial x^{j_k}$, of the dependent variables with respect to the independent variables, cf. [**110**, **111**]. Equivalently, we can identify the jet $(x, u^{(n)})$ with the n-th order Taylor polynomial of the function at the point x — or, when $n = \infty$, its Taylor series.

[‡]On the other hand, in practical examples, differential invariants turn out to be algebraic functions defined on Zariski open subsets of jet space, and so reformulating the theory in a more algebro-geometric framework would be a worthwhile endeavor; see, for instance, [**58**, **78**].

The *fundamental differential invariants* are obtained by invariantization of the individual jet coordinate functions, in accordance with (4.3):

$$H^i = \iota(x^i), \qquad I^\alpha_J = \iota(u^\alpha_J), \qquad \alpha = 1, \dots, q, \qquad \#J \geqslant 0. \tag{4.4}$$

We abbreviate those obtained from all the jet coordinates of order $\leqslant k$ by $(H, I^{(k)}) = \iota(x, u^{(k)})$. Keep in mind that the invariant I^α_J has order $\leqslant \max\{\, \#J, n \,\}$, where n is the order of the moving frame, while H^i has order $\leqslant n$. The fundamental differential invariants (4.4) are of two types. The $r = \dim G$ combinations defining the cross-section (4.1) will be constant, and are known as the *phantom differential invariants*. (In particular, if G acts transitively on M and the moving frame is of minimal order, as in [**117**], then all the H^i and I^α are constant.) For $k \geqslant n$, the remaining *basic differential invariants* provide a complete system of functionally independent differential invariants of order $\leqslant k$.

According to (3.9), the invariantization of a differential function $F(x, u^{(k)})$ can be immediately found by replacing each jet coordinate by the corresponding fundamental differential invariant (4.4):

$$\iota\big[\, F(x, u^{(k)}) \,\big] = F(H, I^{(k)}). \tag{4.5}$$

In particular, the Replacement Rule (3.10) allows one to immediately rewrite *any* differential invariant $J(x, u^{(k)})$ in terms the basic differential invariants:

$$J(x, u^{(k)}) = J(H, I^{(k)}), \tag{4.6}$$

which thereby trivially establishes their completeness.

The specification of independent and dependent variables on M further splits the differential one-forms on the infinite order† jet bundle J^∞ into *horizontal one-forms*, spanned by $dx^1, \dots, dx^p$, and *contact one-forms*, spanned by the *basic contact one-forms*

$$\theta^\alpha_J = du^\alpha_J - \sum_{i=1}^p u^\alpha_{J,i}\, dx^i, \qquad \alpha = 1, \dots, q, \qquad 0 \leqslant \#J. \tag{4.7}$$

In general, a differential one-form θ on J^n is called a *contact form* if and only if it is annihilated by all jets, so $\theta \mid \mathrm{j}_n S = 0$ for all p-dimensional submanifolds $S \subset M$. Every contact one-form is a linear combination of the basic contact one-forms (4.7). This splitting induces a bigrading of the space of differential forms on J^∞ where the differential decomposes into horizontal and vertical components: $d = d_H + d_V$, with d_H increasing the horizontal degree and d_V the vertical (contact) degree. Clearly, closure, $d \circ d = 0$, implies that $d_H \circ d_H = 0 = d_V \circ d_V$, while $d_H \circ d_V = -\, d_V \circ d_H$. The resulting structure is known as the *variational bicomplex*, and lies at the heart of the geometric/topological approach to differential equations, variational problems, symmetries and conservation laws, characteristic classes, etc., bringing powerful cohomological tools such as spectral sequences, [**93**], to bear on analytical and geometrical problems. A complete development plus a broad range of applications can be found in [**6**, **151**].

The invariantization process induced by a moving frame can also be applied to differential forms on jet space. Thus, given a differential form ω on J^k, its *invariantization* $\iota(\omega)$ is the unique invariant differential form that agrees with ω when pulled back to the cross-section. As with differential functions, the invariantized form is found by first transforming (pulling back) the form by the prolonged group action, and then replacing the group parameters by their moving frame formulae. An invariantized contact form remains a contact form, while an invariantized horizontal form is, in general, a combination of horizontal and contact forms. The complete collection of invariantized differential forms serves to define the *invariant variational bicomplex*, studied in detail in [**74**, **147**].

†The splitting only works at infinite order, [**6**, **111**].

For the purposes of analyzing the differential invariants, we can ignore the contact forms. (They do, however, play an important role in other applications, including invariant variational problems, [**74**], submanifold flows, [**118**], and cohomology classes, [**62**, **63**, **147**].) We let π_H denote the projection that maps a one-form onto its horizontal component. The horizontal components of the invariantized basis horizontal one-forms

$$\omega^i = \pi_H(\varpi^i), \qquad \text{where} \qquad \varpi^i = \iota(dx^i), \qquad i = 1, \dots, p, \tag{4.8}$$

form, in the language of [**111**], a *contact-invariant coframe*, meaning that each ω^i is invariant modulo contact forms under the prolonged group action. The corresponding dual *invariant differential operators* $\mathcal{D}_1, \dots, \mathcal{D}_p$ are defined by

$$\sum_{i=1}^{p} (D_i F)\, dx^i = d_H F = \sum_{i=1}^{p} (\mathcal{D}_i F)\, \omega^i, \tag{4.9}$$

for any differential function F, where

$$D_i = \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{J} u^{\alpha}_{J,i} \frac{\partial}{\partial u^{\alpha}_{J}}, \qquad i = 1, \dots, p, \tag{4.10}$$

are the usual total derivative operators, [**110**, **111**], and the initial equality in (4.9) follows directly from the definition of d_H. In practice, the invariant differential operator $\mathcal{D}_i$ can be obtained by substituting the moving frame formulas for the group parameters into the corresponding implicit differentiation operators used to produce the prolonged group actions. As usual, the invariant differential operators map differential invariants to differential invariants, and hence can be iteratively applied to generate the higher order differential invariants.

Example 8. The paradigmatic example is the action of the special Euclidean group SE(2), consisting of orientation-preserving rigid motions — translations and rotations — on plane curves $C \subset M = \mathbb{R}^2$. The group transformation $g = (\varphi, a, b) \in \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$ maps the point $z = (x, u)$ to the point $w = (y, v) = g \cdot z$, given by

$$y = x \cos\varphi - u \sin\varphi + a, \qquad v = x \sin\varphi + u \cos\varphi + b. \tag{4.11}$$

If the curve C is given as the graph of a function $u = f(x)$, the equations (4.11) for the transformed curve $\widetilde{C} = g \cdot C$ implicitly define the graph of a function $v = h(y)$, at least away from points with vertical tangents. The derivatives of v with respect to y are then obtained by successively applying the implicit differentiation operator

$$D_y = \frac{1}{\cos\varphi - u_x \sin\varphi}\, D_x, \tag{4.12}$$

producing

$$\begin{aligned} v_y &= D_y v = \frac{\sin\varphi + u_x \cos\varphi}{\cos\varphi - u_x \sin\varphi}, \qquad v_{yy} = D_y^2 v = \frac{u_{xx}}{(\cos\varphi - u_x \sin\varphi)^3}, \\ v_{yyy} &= D_y^3 v = \frac{(\cos\varphi - u_x \sin\varphi) u_{xxx} + 3\, u_{xx}^2 \sin\varphi}{(\cos\varphi - u_x \sin\varphi)^5}, \qquad \dots, \end{aligned} \tag{4.13}$$

which serve to define the successive prolonged actions of SE(2). The only group elements that fix a given first order jet (x, u, u_x) are the identity, $\varphi = a = b = 0$, and rotation by 180°, with $\varphi = \pi$, $a = b = 0$. (This reflects the fact that a 180° around a point on a curve preserves its tangent line.) We conclude that the prolonged action is locally free on the entire first order jet space, and so $\mathcal{V}^1 = \mathrm{J}^1$.

The classical Euclidean moving frame is based on the cross-section

$$\mathcal{K}^1 = \{\, x = u = u_x = 0 \,\}. \tag{4.14}$$

The corresponding normalization equations (4.2) are

$$y = v = v_y = 0 \tag{4.15}$$

as prescribed by (4.11), (4.13). Solving the normalization equations for the group parameters produces the right moving frame

$$\varphi = -\tan^{-1} u_x\,, \qquad a = -\frac{x + u u_x}{\sqrt{1+u_x^2}}\,, \qquad b = \frac{x u_x - u}{\sqrt{1+u_x^2}}\,, \tag{4.16}$$

which defines a locally right-equivariant map from J^1 to $\mathrm{SE}(2)$, the ambiguity in the inverse tangent indicative of the above-mentioned local freeness of the prolonged action. The classical left-equivariant Frenet frame, [**49**], is obtained by inverting the Euclidean group element (4.16), with resulting group parameters

$$\widetilde{\varphi} = \tan^{-1} u_x\,, \qquad \widetilde{a} = x, \qquad \widetilde{b} = u. \tag{4.17}$$

Observe that the translation component $(\widetilde{a}, \widetilde{b}) = (x, u) = z$ can be identified with the point on the curve (Cartan's moving frame of order 0), while the columns of the corresponding rotation matrix

$$R = \begin{pmatrix} \cos\widetilde{\varphi} & -\sin\widetilde{\varphi} \\ \sin\widetilde{\varphi} & \cos\widetilde{\varphi} \end{pmatrix} = \frac{1}{\sqrt{1+u_x^2}} \begin{pmatrix} 1 & -u_x \\ u_x & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{t}, & \mathbf{n} \end{pmatrix}$$

are precisely the orthonormal frame vectors $\mathbf{t}, \mathbf{n}$ based at $z \in C$, thereby identifying the left moving frame (4.17) with the classical construction, [**49**].

Invariantization of the jet coordinate functions is accomplished by substituting the moving frame formulae (4.16) into the prolonged group transformations (4.13), producing the fundamental differential invariants:

$$\begin{aligned} &H = \iota(x) = 0, \qquad I_0 = \iota(u) = 0, \qquad I_1 = \iota(u_x) = 0, \\ &I_2 = \iota(u_{xx}) = \frac{u_{xx}}{(1+u_x^2)^{3/2}}\,, \qquad I_3 = \iota(u_{xxx}) = \frac{(1+u_x^2)u_{xxx} - 3\,u_x u_{xx}^2}{(1+u_x^2)^3}\,, \end{aligned} \tag{4.18}$$

and so on. The first three, corresponding to the functions defining the cross-section (4.14), are the *phantom invariants*. The lowest order basic differential invariant is the Euclidean curvature: $I_2 = \kappa$. The higher order differential invariants $I_3, I_4, \dots$ will be identified below.

Similarly, to invariantize the horizontal form dx, we first apply a Euclidean transformation:

$$dy = \cos\varphi\, dx - \sin\varphi\, du = (\cos\varphi - u_x \sin\varphi)\, dx - (\sin\varphi)\, \theta, \tag{4.19}$$

where $\theta = du - u_x\, dx$ is the order zero basic contact form. Note that its horizontal component

$$d_H y = \pi_H(dy) = (\cos\varphi - u_x \sin\varphi)\, dx = (D_x y)\, dx$$

serves to define the dual implicit differentiation operator D_y given in (4.12), since

$$d_H F = (D_y F)\, d_H y = (D_x F)\, dx$$

for any differential function F. Substituting the moving frame formulae (4.16) into (4.19) produces the invariant one-form

$$\varpi = \iota(dx) = \sqrt{1+u_x^2}\; dx + \frac{u_x}{\sqrt{1+u_x^2}}\; \theta. \tag{4.20}$$

Its horizontal component

$$\omega = \pi_H(\varpi) = \sqrt{1+u_x^2}\; dx = ds \tag{4.21}$$

is the usual Euclidean arc length element, and is itself contact-invariant. The dual invariant differential operator, cf. (4.9), is the arc length derivative

$$\mathcal{D} = \frac{1}{\sqrt{1+u_x^2}}\, D_x = D_s, \tag{4.22}$$

which can also be directly obtained by substituting the moving frame formulae (4.16) into the implicit differentiation operator (4.12). As we will see, the higher order differential invariants can all be found by successively differentiating the basic curvature invariant with respect to arc length.

Example 9. Let us next consider a non-geometrically-based, but very classical example. Let $n \geqslant 2$ be an integer. In classical invariant theory, the planar actions

$$y = \frac{\alpha\, x + \beta}{\gamma\, x + \delta}\,, \qquad\qquad v = (\gamma\, x + \delta)^{-n} u, \tag{4.23}$$

of the general linear group $G = \mathrm{GL}(2)$ govern the equivalence and symmetry properties of *binary forms*, meaning polynomial functions $u = q(x)$ of degree $\leqslant n$ under the action of the projective group, [**9**, **52**, **109**, **113**], although the results below apply equally well to the equivalence of general smooth functions. The graph of $u = q(x)$ is viewed as a plane curve, and the equivariant moving frame method is applied to determine the differential invariants and associated differential invariant signature.

Since

$$dy = d_H y = \frac{\Delta}{\sigma^2}\, dx, \qquad \text{where} \qquad \sigma = \gamma\, x + \delta, \qquad \Delta = \alpha\, \delta - \beta\, \gamma,$$

the prolonged action, relating the derivatives of a binary form or function and its transformed counterpart, is computed by successively applying the dual implicit differentiation operator

$$D_y = \frac{\sigma^2}{\Delta}\, D_x \tag{4.24}$$

to v, producing

$$\begin{aligned} v_y &= \frac{\sigma\, u_x - n\gamma\, u}{\Delta\, \sigma^{n-1}}\,, \qquad\qquad v_{yy} = \frac{\sigma^2\, u_{xx} - 2\,(n-1)\,\gamma\,\sigma\, u_x + n(n-1)\,\gamma^2 u}{\Delta^2\, \sigma^{n-2}}\,, \\ v_{yyy} &= \frac{\sigma^3 u_{xxx} - 3\,(n-2)\gamma\,\sigma^2 u_{xx} + 3\,(n-1)(n-2)\gamma^2 \sigma u_x - n(n-1)(n-2)\gamma^3 u}{\Delta^3\, \sigma^{n-3}}\,, \end{aligned} \tag{4.25}$$

and so on. It is not hard to show[†] that the prolonged action is locally free on the regular subdomain

$$\mathcal{V}^2 = \{uH \neq 0\} \subset \mathrm{J}^2, \qquad \text{where} \qquad H = u\, u_{xx} - \frac{n-1}{n}\, u_x^2$$

is the classical *Hessian covariant* of u, cf. [**52**, **113**]. Let us choose the cross-section defined by the normalizations

$$y = 0, \qquad\qquad v = 1, \qquad\qquad v_y = 0, \qquad\qquad v_{yy} = 1.$$

Substituting (4.23), (4.25), and then solving the resulting algebraic equations for the group parameters produces

$$\begin{aligned} \alpha &= u^{(1-n)/n} \sqrt{H}, & \beta &= -x\, u^{(1-n)/n} \sqrt{H}, \\ \gamma &= \frac{1}{n}\, u^{(1-n)/n} u_x, & \delta &= u^{1/n} - \frac{1}{n}\, x\, u^{(1-n)/n} u_x, \end{aligned} \tag{4.26}$$

[†]The simplest way to accomplish this is to show that the prolonged infinitesimal generators are linearly independent at each point of $\mathcal{V}^2$; see below for details.

which serve to define a locally[‡] right-equivariant moving frame map $\rho\colon \mathcal{V}^2 \to \mathrm{GL}(2)$. Substituting the moving frame formulae (4.26) into the higher order transformation rules yields the desired differential invariants, the first two of which are

$$v_{yyy} \;\longmapsto\; J = \frac{T}{H^{3/2}}, \qquad v_{yyyy} \;\longmapsto\; K = \frac{V}{H^2}, \tag{4.27}$$

where the differential polynomials

$$T = u^2 u_{xxx} - 3\,\frac{n-2}{n}\,u\,u_x u_{xx} + 2\,\frac{(n-1)(n-2)}{n^2}\,u_x^3,$$
$$V = u^3 u_{xxxx} - 4\,\frac{n-3}{n}\,u^2 u_x u_{xxx} + 6\,\frac{(n-2)(n-3)}{n^2}\,u\,u_x^2 u_{xx} - 3\frac{(n-1)(n-2)(n-3)}{n^3}\,u_x^4,$$

can be identified with classical covariants of the binary form $u = q(x)$ obtained through the transvection process, cf. [**52**, **113**]. Using $J^2 = T^2/H^3$ as the fundamental differential invariant of lowest order will remove the ambiguity caused by the square root. As in the Euclidean case, the higher order differential invariants can be written in terms of the basic "curvature invariant" J and its successive invariant derivatives with respect to the invariant differential operator

$$\mathcal{D} = u\,H^{-1/2} D_x, \tag{4.28}$$

which is itself obtained by substituting the moving frame formulae (4.26) into the implicit differentiation operator (4.24).

We can now produce a signature-based solution to the equivalence and symmetry problems for binary forms. The signature curve $\Sigma = \Sigma_q$ of a polynomial $u = q(x)$ — or, indeed, of any smooth function — is parametrized by the covariants J^2 and K, given in (4.27). In this manner, we have established a strikingly simple solution to the equivalence problem for complex-valued binary forms that, surprisingly, does not appear in any of the classical literature on the subject. Extensions of this result to real forms can be found in [**109**, **113**].

THEOREM 10. *Two nondegenerate complex-valued binary forms $q(x)$ and $\widetilde{q}(x)$ are equivalent if and only if their signature curves are identical*: $\Sigma_q = \Sigma_{\tilde{q}}$.

Thus, the equivalence and symmetry properties of binary forms are entirely encoded by the functional relation between two particular absolute rational covariants, namely, J^2 and K. Moreover, any equivalence map $\widetilde{x} = \psi(x)$ must satisfy the pair of rational equations

$$J(x)^2 = \widetilde{J}(\widetilde{x})^2, \qquad K(x) = \widetilde{K}(\widetilde{x}). \tag{4.29}$$

Indeed, the theory guarantees that any solution to this system is necessarily a linear fractional transformation! Specializing to the case when $\widetilde{q} = q$, the symmetries of a nonsingular binary form can be explicitly determined by solving the rational equations (4.29) with $\widetilde{J} = J$ and $\widetilde{K} = K$. See [**9**] for a MAPLE package, based on this method, that automatically computes discrete symmetries of univariate polynomials.

As a consequence of Theorem 2 and (2.3), we are led to a complete characterization of the symmetry groups of binary forms. (The totally singular case (*a*) is established by a separate calculation.)

[‡]See [**9**] for a detailed discussion of how to systematically resolve the square root ambiguities caused by local freeness.

THEOREM 11. *The symmetry group of a binary form $q(x) \not\equiv 0$ of degree n is*:

a) *A two-parameter group if and only if it Hessian $H \equiv 0$ if and only if $q(x)$ is equivalent to a constant.*
b) *A one-parameter group if and only if $H \not\equiv 0$ and T^2 is a constant multiple of H^3 if and only if $q(x)$ is complex-equivalent to a monomial x^k, with $k \neq 0, n$. In this case the signature Σ_q is just a single point, and the graph of q coincides with the orbit of the connected component of its one-parameter symmetry subgroup of* GL(2).
c) *A finite group in all other cases. The cardinality of the group equals the index of the signature curve Σ_q.*

In her thesis, Kogan, [**72**], extends these results to forms in several variables. In particular, the resulting signature for ternary forms, including elliptic curves, leads to a practical algorithm for computing their discrete symmetries, [**73**].

5. *Recurrence and the Algebra of Differential Invariants.*

While the invariantization process respects all algebraic operations on functions and differential forms, it does *not* commute with differentiation. A *recurrence relation* expresses a differentiated invariant in terms of the basic differential invariants — or, more generally, a differentiated invariant differential form in terms of the normalized invariant differential forms. The recurrence relations are the master key that unlocks the entire structure of the algebra of differential invariants, including the specification of generators, the classification of syzygies and, as a result, the general specification of differential invariant signatures. Remarkably, the recurrence relations can be explicitly determined even in the absence of explicit formulas for the differential invariants, or the invariant differential operators, or even the moving frame itself! The only necessities are the well-known and relatively simple formulas for the infinitesimal generators of the group action and their jet space prolongations, combined with the choice of cross-section normalizations.

A basis for the infinitesimal generators of our effectively acting r-dimensional transformation group G is provided by linearly independent vector fields on M taking the local coordinate form

$$\mathbf{v}_\sigma = \sum_{i=1}^{p} \xi_\sigma^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi_\sigma^\alpha(x,u)\frac{\partial}{\partial u^\alpha}, \qquad \sigma = 1,\dots,r, \tag{5.1}$$

which we identify with a basis of its Lie algebra $\mathfrak{g}$. Their associated flows $\exp(t\,\mathbf{v}_\sigma)$ form one-parameter subgroups that serve to generate the action of the (connected component containing the identity of) the transformation group. The corresponding prolonged infinitesimal generator

$$\operatorname{pr}\mathbf{v}_\sigma = \sum_{i=1}^{p} \xi_\sigma^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{k=\#J\geqslant 0} \varphi_{J,\sigma}^\alpha(x,u^{(k)})\frac{\partial}{\partial u_J^\alpha}, \qquad \sigma = 1,\dots,r, \tag{5.2}$$

generates the prolongation of the associated one-parameter subgroup acting on jet bundles. The higher order coefficients

$$\varphi_{J,\sigma}^\alpha = \operatorname{pr}\mathbf{v}_\sigma(u_J^\alpha), \qquad \#J \geqslant 1,$$

are calculated using the *prolongation formula*, [**110**], first written in the following explicit, non-recursive form in [**108**]:

$$\varphi_{J,\sigma}^\alpha = D_J\left(\varphi_\sigma^\alpha - \sum_{i=1}^{p} \xi_\sigma^i\, u_i^\alpha\right) + \sum_{i=1}^{p} \xi_\sigma^i u_{J,i}^\alpha. \tag{5.3}$$

Here $D_J = D_{j_1} \cdots D_{j_k}$ are iterated total derivative operators, cf. (4.10), and $u^\alpha_i = D_i u^\alpha$ represents $\partial u^\alpha / \partial x^i$.

Given an equivariant moving frame on jet space, the *universal recurrence relation* for differentiated invariants can now be stated. As in (4.9), $\mathcal{D}_i = \iota(D_i)$ will denote the associated invariant differential operators.

THEOREM 12. *Let $F(x, u^{(k)})$ be a differential function and $\iota(F)$ its moving frame invariantization. Then*

$$\mathcal{D}_i\big[\,\iota(F)\,\big] = \iota\big[\,D_i(F)\,\big] + \sum_{\sigma=1}^{r} R^\sigma_i\, \iota\big[\,\mathrm{pr}\,\mathbf{v}_\sigma(F)\,\big], \tag{5.4}$$

where R^σ_i, $i = 1, \dots, p$, $\sigma = 1, \dots, r$, are called the Maurer–Cartan differential invariants.

The Maurer–Cartan differential invariants R^σ_i can, in fact, be characterized as the coefficients of the horizontal components of the pull-backs of the Maurer–Cartan forms on G via the moving frame map $\rho : \mathrm{J}^n \to G$, [**37**]. But in practical calculations, one, in fact, does not need to know where the Maurer–Cartan invariants come from, or even what a Maurer–Cartan form is, since the R^σ_i can be directly determined from the recurrence relations for the phantom differential invariants, as prescribed by the cross-section (4.1). Namely, since $\iota(W_\nu) = c_\nu$ is constant, for each $1 \leqslant i \leqslant p$, the *phantom recurrence relations*

$$0 = \iota\big[\,D_i(W_\nu)\,\big] + \sum_{\sigma=1}^{r} R^\sigma_i\, \iota\big[\,\mathrm{pr}\,\mathbf{v}_\sigma(W_\nu)\,\big], \qquad \nu = 1, \dots, r, \tag{5.5}$$

form a system of r linear equations that, as a consequence of the transversality of the cross-section, can be uniquely solved for the r Maurer–Cartan invariants $R^1_i, \dots, R^r_i$. Substituting the resulting expressions back into the remaining non-phantom recurrence formulae (5.4) produces the complete system of differential identities satisfied by the basic differential invariants, which in turn fully characterizes the structure of the differential invariant algebra, [**37**, **117**, **128**].

EXAMPLE 13. Using (5.2), (5.3), the prolonged infinitesimal generators of the planar Euclidean group action on curve jets, as described in Example 8, are

$$\mathrm{pr}\,\mathbf{v}_1 = \partial_x, \qquad \mathrm{pr}\,\mathbf{v}_2 = \partial_u,$$
$$\mathrm{pr}\,\mathbf{v}_3 = -u\,\partial_x + x\,\partial_u + (1 + u_x^2)\,\partial_{u_x} + 3\,u_x u_{xx}\,\partial_{u_{xx}} + (4\,u_x u_{xxx} + 3\,u_{xx}^2)\,\partial_{u_{xxx}} + \cdots,$$

where $\mathbf{v}_1, \mathbf{v}_2$ generate translations, while $\mathbf{v}_3$ generates rotations. According to (5.4), the invariant arc length derivative $\mathcal{D} = \iota(D_x)$ of any differential invariant $I = \iota(F)$ obtained by invariantizing a differential function F is specified by the recurrence relation

$$\mathcal{D}I = \mathcal{D}\,\iota(F) = \iota(D_x F) + R^1\,\iota\big(\mathrm{pr}\,\mathbf{v}_1(F)\big) + R^2\,\iota\big(\mathrm{pr}\,\mathbf{v}_2(F)\big) + R^3\,\iota\big(\mathrm{pr}\,\mathbf{v}_3(F)\big), \tag{5.6}$$

where R^1, R^2, R^3 are the three Maurer–Cartan invariants. To determine their formulas, we write out (5.6) for the three phantom invariants which come from the cross-section variables x, u, u_x, cf. (4.14):

$$0 = \mathcal{D}\,\iota(x) = \iota(1) + R^1\,\iota(\mathrm{pr}\,\mathbf{v}_1(x)) + R^2\,\iota(\mathrm{pr}\,\mathbf{v}_2(x)) + R^3\,\iota(\mathrm{pr}\,\mathbf{v}_3(x)) = 1 + R^1,$$
$$0 = \mathcal{D}\,\iota(u) = \iota(u_x) + R^1\,\iota(\mathrm{pr}\,\mathbf{v}_1(u)) + R^2\,\iota(\mathrm{pr}\,\mathbf{v}_2(u)) + R^3\,\iota(\mathrm{pr}\,\mathbf{v}_3(u)) = R^2,$$
$$0 = \mathcal{D}\,\iota(u_x) = \iota(u_{xx}) + R^1\,\iota(\mathrm{pr}\,\mathbf{v}_1(u_x)) + R^2\,\iota(\mathrm{pr}\,\mathbf{v}_2(u_x)) + R^3\,\iota(\mathrm{pr}\,\mathbf{v}_3(u_x)) = \kappa + R^3.$$

Solving the resulting linear system of equations yields

$$R^1 = -1, \qquad R^2 = 0, \qquad R^3 = -\kappa = -I_2. \tag{5.7}$$

Thus, the general recurrence relation (5.6) becomes

$$\mathcal{D}\,\iota(F) = \iota(D_x F) - \iota\big(\operatorname{pr}\mathbf{v}_1(F)\big) - \kappa\,\iota\big(\operatorname{pr}\mathbf{v}_3(F)\big). \tag{5.8}$$

In particular, the first few — obtained by successively setting $F = u_{xx}, u_{xxx}, u_{xxxx}, u_{xxxxx}$ in (5.8), and letting $I_k = \iota(u_k)$ denote the normalized differential invariants corresponding to $u_k = D_x^k u$ — are

$$\begin{aligned} \kappa &= I_2, & \kappa_s &= \mathcal{D}I_2 = I_3, & \mathcal{D}I_4 &= I_5 - 10\,I_2^2 I_3, \\ & & \kappa_{ss} &= \mathcal{D}I_3 = I_4 - 3\,I_2^3, & \mathcal{D}I_5 &= I_6 - 15\,I_2^2\,I_4 - 10\,I_2\,I_3^2. \end{aligned} \tag{5.9}$$

These can be iteratively solved to produce the explicit formulae

$$\begin{aligned} \kappa &= I_2, & I_2 &= \kappa, \\ \kappa_s &= I_3, & I_3 &= \kappa_s, \\ \kappa_{ss} &= I_4 - 3\,I_2^3, & I_4 &= \kappa_{ss} + 3\,\kappa^3, \\ \kappa_{sss} &= I_5 - 19\,I_2^2\,I_3, & I_5 &= \kappa_{sss} + 19\,\kappa^2\kappa_s, \\ \kappa_{ssss} &= I_6 - 34\,I_2^2\,I_4 - 48\,I_2\,I_3^2 + 57\,I_2^5, & I_6 &= \kappa_{ssss} + 34\,\kappa^2\kappa_{ss} + 48\,\kappa\,\kappa_s^2 + 45\,\kappa^5, \end{aligned} \tag{5.10}$$

and so on, relating the normalized and differentiated curvature invariants. The skeptical reader is invited to verify these identities by substituting the explicit formulae that were computed in Example 8.

The invariant differential operators $\mathcal{D}_1, \dots, \mathcal{D}_p$ given in (4.9) map differential invariants to differential invariants. Keep in mind that they do not necessarily commute, and so the order of differentiation is important. On the other hand, each commutator can be re-expressed as a linear combination

$$[\,\mathcal{D}_j, \mathcal{D}_k\,] = \mathcal{D}_j\,\mathcal{D}_k - \mathcal{D}_k\,\mathcal{D}_j = \sum_{i=1}^{p} Y_{jk}^i \mathcal{D}_i, \tag{5.11}$$

where the coefficients

$$Y_{jk}^i = -\,Y_{kj}^i = \sum_{\sigma=1}^{r}\sum_{j=1}^{p} R_k^\sigma\,\iota(D_j\xi_\sigma^i) - R_j^\sigma\,\iota(D_k\xi_\sigma^i) \tag{5.12}$$

are known as the *commutator invariants*, whose explicit formulae are a consequence of the recurrence relations adapted to differential forms; see [**37**] for a derivation of formula (5.12).

Furthermore, the differentiated invariants $\mathcal{D}_J I_\nu$ are *not* necessarily functionally independent, but may be subject to certain functional relations or *differential syzygies* of the form

$$H(\ \dots\ \mathcal{D}_J I_\nu\ \dots\ ) \equiv 0. \tag{5.13}$$

The *Syzygy Theorem*, first stated (not quite correctly) in [**37**] for finite-dimensional actions, and then rigorously formulated and proved in [**128**], states that there are, in essence, a finite number of generating differential syzygies along with those induced by the commutator equations (5.11). Again, this result can be viewed as the differential invariant algebra counterpart of the Hilbert Syzygy Theorem for polynomial ideals, [**32**].

Let us end with a synopsis of some recent results on generating sets $\mathcal{I} = \{I^1, \dots, I^l\}$ of differential invariants, satisfying the conditions of the Basis Theorem 1. The first is an immediate consequence of the recurrence formulae (5.4) and the induced construction of the Maurer–Cartan invariants.

THEOREM 14. *If the moving frame has order n, then the set of fundamental differential invariants*

$$\mathcal{I}^{(n+1)} = \{\, H_i, I^\alpha_J \mid \ i = 1, \dots, p, \ \ \alpha = 1, \dots, q, \ \ \#J \leqslant n+1 \,\}$$

of order $\leqslant n+1$ forms a generating set.

Of course, one can immediately omit any constant phantom differential invariants from this collection. Even so, the resulting set of generating invariants is typically far from minimal.

Another interesting consequence of the recurrence formulae, first noticed by Hubert, [**55**], is that the Maurer–Cartan invariants

$$\mathcal{R} = \{\, R^\sigma_i \mid \ i = 1, \dots, p, \ \ \sigma = 1, \dots, r \,\}$$

also form a, again typically non-minimal, generating set when the action is transitive on M. More generally:

THEOREM 15. *The differential invariants $\mathcal{I}^{(0)} \cup \mathcal{R}$ form a generating set.*

Let us now discuss the problem of finding a *minimal generating set* of differential invariants. The case of curves, $p = 1$, has been well understood for some time. For an ordinary Lie group action on curves in a m-dimensional manifold, there are precisely $m - 1$ generating differential invariants, [**111**, **45**], and this is a minimal system, meaning that none of them can be expressed as a combination of the invariant arc length derivatives of the others. Moreover, there are no syzygies among their invariant derivatives. (The relatively rare non-ordinary actions are not significantly more complicated and are also well understood.) Thus, for space curves $C \subset \mathbb{R}^3$, there are two generating invariants, which are typically identified as the group-invariant curvature and torsion.

On the other hand, when dealing with submanifolds of dimension $p \geqslant 2$, i.e., functions of more than one variable, there are, as yet, *no* general results on the minimal number of generating differential invariants. Indeed, even in well-studied examples, the conventional wisdom on minimal generating sets is often mistaken.

EXAMPLE 16. Consider the action of the Euclidean group $\mathrm{E}(3) = \mathrm{O}(3) \ltimes \mathbb{R}^3$ on surfaces $S \subset \mathbb{R}^3$. In local coordinates, we can identify (transverse) surfaces with graphs of functions $u = f(x, y)$. The corresponding local coordinates on the surface jet bundle $\mathrm{J}^n = \mathrm{J}^n(\mathbb{R}^3, 2)$ are $x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \dots$, and, in general, $u_{jk} = D^j_x D^k_y u$ for $j + k \leqslant n$. The classical moving frame construction, [**49**], relies on the coordinate cross-section

$$\mathcal{K}^2 = \{\, x = y = u = u_x = u_y = u_{xy} = 0, \ \ u_{xx} \neq u_{yy} \,\}. \tag{5.14}$$

The resulting left moving frame consists of the point on the curve defining the translation component $a = z \in \mathbb{R}^3$, while the columns of the rotation matrix $R = [\, \mathbf{t}_1, \mathbf{t}_2, \mathbf{n} \,] \in \mathrm{O}(3)$ consist of the orthonormal tangent vectors $\mathbf{t}_1, \mathbf{t}_2$ forming the diagonalizing *Darboux frame*, along with the unit normal $\mathbf{n}$.

The fundamental differential invariants are denoted as $I_{jk} = \iota(u_{jk})$. In particular,

$$\kappa_1 = I_{20} = \iota(u_{xx}), \qquad\qquad \kappa_2 = I_{02} = \iota(u_{yy}),$$

are the *principal curvatures*; the moving frame is valid provided $\kappa_1 \neq \kappa_2$, meaning that we are at a non-umbilic point. Indeed, the prolonged Euclidean action is locally free on the regular subset $\mathcal{V}^2 \subset \mathrm{J}^2$ consisting of second order jets of surfaces at non-umbilic points. The *mean* and

Gaussian curvature invariants

$$H = \tfrac{1}{2}(\kappa_1 + \kappa_2), \qquad K = \kappa_1\kappa_2,$$

are often used as convenient alternatives, since they eliminate some (but not all, owing to the local freeness of the second order prolonged action) of the residual discrete ambiguities in the locally equivariant moving frame. Higher order differential invariants are obtained by differentiation with respect to the dual *Darboux coframe* $\omega^1 = \pi_H\,\iota(dx)$, $\omega^2 = \pi_H\,\iota(dy)$. We let $\mathcal{D}_1 = \iota(D_x)$, $\mathcal{D}_2 = \iota(D_y)$, denote the dual invariant differential operators, which are in the directions of the Darboux frame vectors. These are *not* the same as the operators of covariant differentiation, but are closely related, [**49**, **111**]; indeed, the latter do not map differential invariants to differential invariants.

To characterize the full differential invariant algebra, we derive the recurrence relations. A basis for the infinitesimal generators for the action on $\mathbb{R}^3$ is provided by the six vector fields

$$\begin{aligned} \mathbf{v}_1 &= -\,y\,\partial_x + x\,\partial_y, & \mathbf{v}_2 &= -\,u\,\partial_x + x\,\partial_u, & \mathbf{v}_3 &= -\,u\,\partial_y + y\,\partial_u,\\ \mathbf{w}_1 &= \partial_x, & \mathbf{w}_2 &= \partial_y, & \mathbf{w}_3 &= \partial_u, \end{aligned} \tag{5.15}$$

the first three generating the rotations and the second three the translations. The recurrence formulae (5.4) of order $\geqslant 1$ have the explicit form

$$\mathcal{D}_1 I_{jk} = I_{j+1,k} + \sum_{\sigma=1}^{3} \varphi_\sigma^{jk}(0,0,I^{(j+k)})R_1^\sigma, \quad \mathcal{D}_2 I_{jk} = I_{j,k+1} + \sum_{\sigma=1}^{3} \varphi_\sigma^{jk}(0,0,I^{(j+k)})R_2^\sigma, \tag{5.16}$$

provided $j + k \geqslant 1$. Here R_1^σ, R_2^σ are the Maurer–Cartan invariants associated with the rotational group generator $\mathbf{v}_\sigma$, while

$$\varphi_\sigma^{jk}(0,0,I^{(j+k)}) = \iota\big[\,\varphi_\sigma^{jk}(x,y,u^{(j+k)})\,\big] = \iota\big[\,\operatorname{pr}\mathbf{v}_\sigma(u_{jk})\,\big]$$

are its invariantized prolongation coefficients, as given by the standard formula (5.3). (The translational generators and associated Maurer–Cartan invariants only appear in the order 0 recurrence formulae, and so, for our purposes, can be ignored.) In particular, the phantom recurrence formulae of order > 0 are

$$\begin{aligned} 0 &= \mathcal{D}_1 I_{10} = I_{20} + R_1^2, & 0 &= \mathcal{D}_2 I_{10} = R_2^2,\\ 0 &= \mathcal{D}_1 I_{01} = R_1^3, & 0 &= \mathcal{D}_2 I_{01} = I_{02} + R_2^3,\\ 0 &= \mathcal{D}_1 I_{11} = I_{21} + (I_{20} - I_{02})R_1^1, & 0 &= \mathcal{D}_2 I_{11} = I_{12} + (I_{20} - I_{02})R_2^1. \end{aligned} \tag{5.17}$$

Solving these produces the Maurer–Cartan invariants:

$$R_1^1 = Y_2, \qquad R_1^2 = -\,\kappa_1, \qquad R_1^3 = 0, \qquad R_2^1 = -\,Y_1, \qquad R_2^2 = 0, \qquad R_2^3 = -\,\kappa_2, \tag{5.18}$$

where

$$Y_1 = \frac{I_{12}}{I_{20} - I_{02}} = \frac{\mathcal{D}_1\kappa_2}{\kappa_1 - \kappa_2}, \qquad Y_2 = \frac{I_{21}}{I_{02} - I_{20}} = \frac{\mathcal{D}_2\kappa_1}{\kappa_2 - \kappa_1}, \tag{5.19}$$

the latter expressions following from the third order recurrence formulae, obtained by substituting (5.18) into (5.16):

$$\begin{aligned} I_{30} &= \mathcal{D}_1 I_{20} = \mathcal{D}_1\kappa_1, & I_{21} &= \mathcal{D}_2 I_{20} = \mathcal{D}_2\kappa_1,\\ I_{12} &= \mathcal{D}_1 I_{02} = \mathcal{D}_1\kappa_2, & I_{03} &= \mathcal{D}_2 I_{02} = \mathcal{D}_2\kappa_2. \end{aligned} \tag{5.20}$$

The general commutator formula (5.12) implies that the Maurer–Cartan invariants (5.19) are also the *commutator invariants*:

$$\big[\,\mathcal{D}_1, \mathcal{D}_2\,\big] = \mathcal{D}_1\,\mathcal{D}_2 - \mathcal{D}_2\,\mathcal{D}_1 = Y_2\,\mathcal{D}_1 - Y_1\,\mathcal{D}_2. \tag{5.21}$$

Further, equating the two fourth order recurrence relations for $I_{22} = \iota(u_{xxyy})$, namely,

$$\mathcal{D}_2 I_{21} + \frac{I_{30}I_{12} - 2\,I_{12}^2}{\kappa_1 - \kappa_2} + \kappa_1\kappa_2^2 = I_{22} = \mathcal{D}_1 I_{12} - \frac{I_{21}I_{03} - 2\,I_{21}^2}{\kappa_1 - \kappa_2} + \kappa_1^2\kappa_2,$$

leads us to the celebrated *Codazzi syzygy*

$$\mathcal{D}_2^2\kappa_1 - \mathcal{D}_1^2\kappa_2 + \frac{\mathcal{D}_1\kappa_1\,\mathcal{D}_1\kappa_2 + \mathcal{D}_2\kappa_1\,\mathcal{D}_2\kappa_2 - 2\,(\mathcal{D}_1\kappa_2)^2 - 2\,(\mathcal{D}_2\kappa_1)^2}{\kappa_1 - \kappa_2} - \kappa_1\kappa_2(\kappa_1 - \kappa_2) = 0. \tag{5.22}$$

Using (5.19), we can, in fact, rewrite the Codazzi syzygy in the more succinct form

$$K = \kappa_1\kappa_2 = -\,(\mathcal{D}_1 + Y_1)Y_1 - (\mathcal{D}_2 + Y_2)Y_2. \tag{5.23}$$

As noted in [**49**], the right hand side of (5.23) depends only on the first fundamental form of the surface. Thus, the Codazzi syzygy (5.23) immediately implies *Gauss' Theorema Egregium*, that the Gauss curvature is an intrinsic, isometric invariant. Another direct consequence of (5.23) is the celebrated Gauss–Bonnet Theorem; see [**74**] for details.

Since we are dealing with a second order moving frame, Theorem 14 implies that the differential invariant algebra for Euclidean surfaces is generated by the basic differential invariants of order $\leqslant 3$. However, (5.20) express the third order invariants as invariant derivatives of the principal curvatures κ_1, κ_2, and hence they, or, equivalently, the Gauss and mean curvatures H, K, form a generating system of differential invariants. This is well known, [**111**]. However, surprisingly, [**119**, **125**], neither is a minimal generating set! To investigate, we begin by distinguishing a special class of surfaces.

Definition 17. A surface $S \subset \mathbb{R}^3$ is *mean curvature degenerate* if, for any non-umbilic point $z_0 \in S$, there exist scalar functions $f_1(t), f_2(t)$, such that

$$\mathcal{D}_1 H = f_1(H), \qquad \mathcal{D}_2 H = f_2(H), \tag{5.24}$$

at all points $z \in S$ in a suitable neighborhood of z_0.

Clearly any constant mean curvature surface is mean curvature degenerate, with $f_1(t) \equiv f_2(t) \equiv 0$. Surfaces with non-constant mean curvature that admit a one-parameter group of Euclidean symmetries, i.e., non-cylindrical or non-spherical surfaces of rotation, non-planar surfaces of translation, or helicoidal surfaces, obtained by, respectively, rotating, translating, or screwing a plane curve, are also mean curvature degenerate since, by the signature characterization of symmetry groups, [**37**], they have exactly one non-constant functionally independent differential invariant, namely the mean curvature H and hence any other differential invariant, including the invariant derivatives of H — as well as the Gauss curvature K — must be functionally dependent upon H. There also exist surfaces without continuous symmetries that are, nevertheless, mean curvature degenerate since it is entirely possible that (5.24) holds, but the Gauss curvature remains functionally independent of H. However, I do not know a nice geometric characterization of such surfaces, which are well deserving of further investigation.

Theorem 18. *If a surface is mean curvature nondegenerate then its algebra of Euclidean differential invariants is generated entirely by the mean curvature and its successive invariant derivatives.*

Proof: In view of the Codazzi formula (5.23), it suffices to write the commutator invariants Y_1, Y_2 in terms of the mean curvature. To this end, we note that the commutator identity (5.21) can be applied to any differential invariant. In particular,

$$\mathcal{D}_1\mathcal{D}_2 H - \mathcal{D}_2\mathcal{D}_1 H = Y_2\,\mathcal{D}_1 H - Y_1\,\mathcal{D}_2 H, \tag{5.25}$$

and, furthermore,

$$\mathcal{D}_1\mathcal{D}_2\mathcal{D}_j H - \mathcal{D}_2\mathcal{D}_1\mathcal{D}_j H = Y_2\,\mathcal{D}_1\mathcal{D}_j H - Y_1\,\mathcal{D}_2\mathcal{D}_j H, \qquad j = 1, 2. \tag{5.26}$$

Provided at least one of the nondegeneracy conditions

$$(\mathcal{D}_1 H)\,(\mathcal{D}_2\mathcal{D}_j H) \neq (\mathcal{D}_2 H)\,(\mathcal{D}_1\mathcal{D}_j H), \qquad \text{for } \ j = 1 \text{ or } 2, \tag{5.27}$$

holds, we can solve (5.25)–(5.26) to write the commutator invariants Y_1, Y_2 as explicit rational functions of invariant derivatives of H. Plugging these expressions into the right hand side of the Codazzi identity (5.23) produces an explicit formula for the Gauss curvature as a rational function of the invariant derivatives, of order $\leqslant 4$, of the mean curvature, valid for all surfaces satisfying the nondegeneracy condition (5.27).

Thus it remains to show that (5.27) is equivalent to mean curvature nondegeneracy of the surface. First, if (5.24) holds, then

$$\mathcal{D}_i\mathcal{D}_j H = \mathcal{D}_i f_j(H) = f_j'(H)\,\mathcal{D}_i H = f_j'(H)\,f_i(H), \qquad i, j = 1, 2.$$

This immediately implies that

$$(\mathcal{D}_1 H)\,(\mathcal{D}_2\mathcal{D}_j H) = (\mathcal{D}_2 H)\,(\mathcal{D}_1\mathcal{D}_j H), \qquad j = 1, 2, \tag{5.28}$$

proving mean curvature degeneracy. Vice versa, noting that, when restricted to the surface, since the contact forms all vanish, d_H reduces to the usual differential, and hence the degeneracy condition (5.28) implies that, for each $j = 1, 2$, the differentials dH and $d(\mathcal{D}_j H)$ are linearly dependent everywhere on S. The standard characterization of functional dependency, cf. [**110**, Theorem 2.16], thus implies that, locally, $\mathcal{D}_j H$ can be written as a function of H, thus establishing the mean curvature degeneracy condition (5.24). *Q.E.D.*

Similar results hold for surfaces in several other classical three-dimensional Klein geometries; see [**60**, **121**] for details.

THEOREM 19. *The differential invariant algebra of a generic surface $S \subset \mathbb{R}^3$ under the standard action of*

- *the* centro-equi-affine group SL(3) *is generated by a single second order differential invariant;*
- *the* equi-affine group $\mathrm{SA}(3) = \mathrm{SL}(3) \ltimes \mathbb{R}^3$ *is generated by a single third order differential invariant, known as the* Pick invariant, [**144**]*;*
- *the* conformal group $\mathrm{SO}(4,1)$ *is generated by a single third order differential invariant;*
- *the* projective group PSL(4) *is generated by a single fourth order differential invariant.*

Lest the reader be tempted at this juncture to make a general conjecture concerning the differential invariants of surfaces in three-dimensional space, the following elementary example shows that the number of generating invariants can be arbitrarily large.

EXAMPLE 20. Consider the abelian group action

$$z = (x, y, u) \ \longmapsto \ \big(x + a, y + b, u + p(x, y)\big), \tag{5.29}$$

where $a, b \in \mathbb{R}$, and $p(x, y)$ is an arbitrary polynomial of degree $\leqslant n$. In this case, for surfaces $u = f(x, y)$, the individual jet coordinate functions $u_{jk} = D_x^j D_y^k u$ with $j + k \geqslant n + 1$ form a complete system of independent differential invariants. The invariant differential operators are the usual total derivatives: $\mathcal{D}_1 = D_x$, $\mathcal{D}_2 = D_y$, which happen to commute. The higher order differential invariants are generated by differentiating the differential invariants u_{jk} of order

$n+1 = j+k$. Moreover, these invariants clearly form a minimal generating set, of cardinality $n+2$.

Complete local classifications of Lie group actions on plane curves and their associated differential invariant algebras are known, [**111**]. Lie, in volume 3 of his monumental treatise on transformation groups, [**81**], exhibits a large fraction of the three-dimensional classification, and claims to have completed it, but writes there is not enough space to present the full details. As far as I know, his calculations have not been found in his archived notes or personal papers. Later, Amaldi, [**3**, **4**], lists what he says is the complete classification. More recently, unaware of Amaldi's papers, Komrakov, [**76**], asserts that such a classification is not possible since one of the branches contains an intractable algebraic problem. Amaldi and Komrakov's competing claims remain to be reconciled, although I suspect that Komrakov is right. Whether or not the Lie–Amaldi classification is complete, it would, nevertheless, be a worthwhile project to systematically analyze the differential invariant algebras of space curves and, especially, surfaces under each of the transformation groups appearing in the Lie–Amaldi lists.

In conclusion, even with the powerful recurrence formulae at our disposal, the general problem of finding and characterizing a minimal set of generating differential invariants when the dimension of the submanifold is $\geqslant 2$ remains open. Indeed, I do not know of a verifiable criterion for minimality, except in the trivial case when there is a single generating invariant, let alone an algorithm that will produce a minimal generating set. The main difficulty lies in establishing a bound on the order of possible syzygies among a given set of differential invariants. It is worth pointing out that the corresponding problem for polynomial ideals — finding a minimal Hilbert basis — appears to be intractable. However, the special structure of the differential invariant algebra prescribed by the form of the recurrence relations gives some reasons for optimism that such a procedure might be possible.

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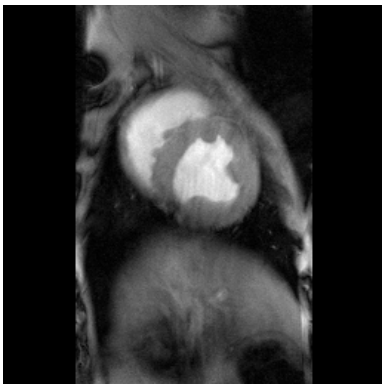
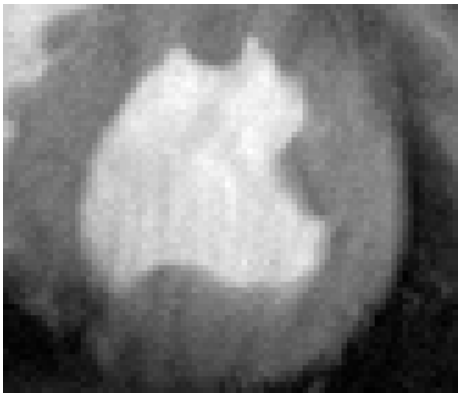
Peter J. Olver
School of Mathematics
University of Minnesota
Minneapolis, MN 55455
U.S.A.

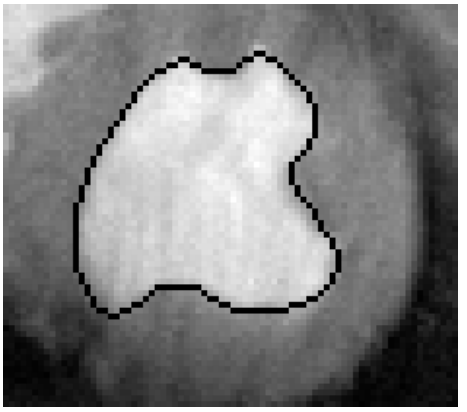
olver@umn.edu
`http://www.math.umn.edu/∼olver`



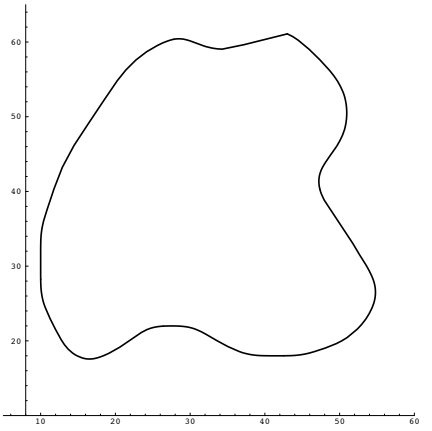
Original Canine Heart MRI Image



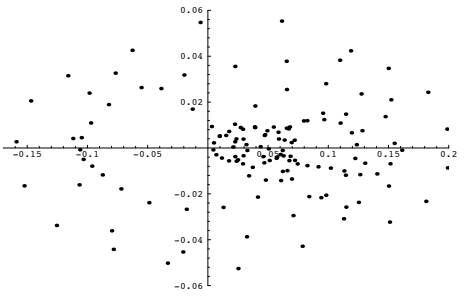
Blow Up of the Left Ventricle



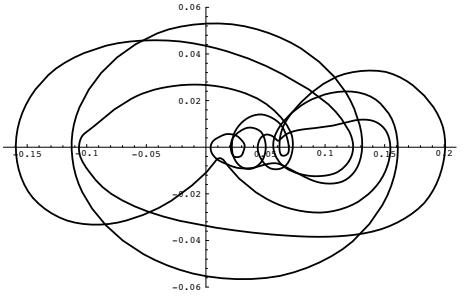
Boundary of Left Ventricle



Original Contour



Discrete Euclidean Signature



Smoothly Connected Euclidean Signature

FIGURE 4. *Canine Left Ventricle Signature*

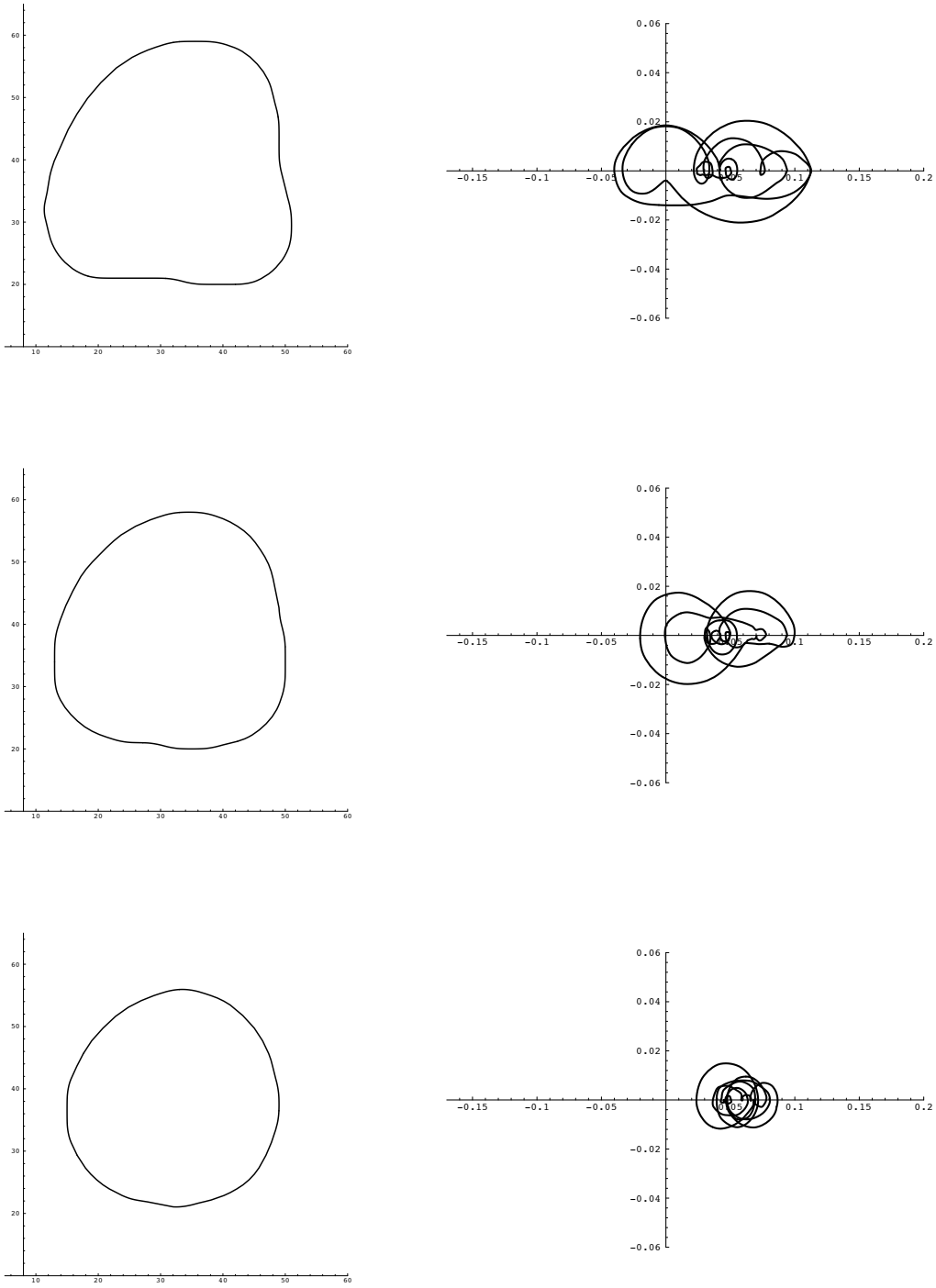


FIGURE 6. *Smoothed Canine Left Ventricle Signatures*

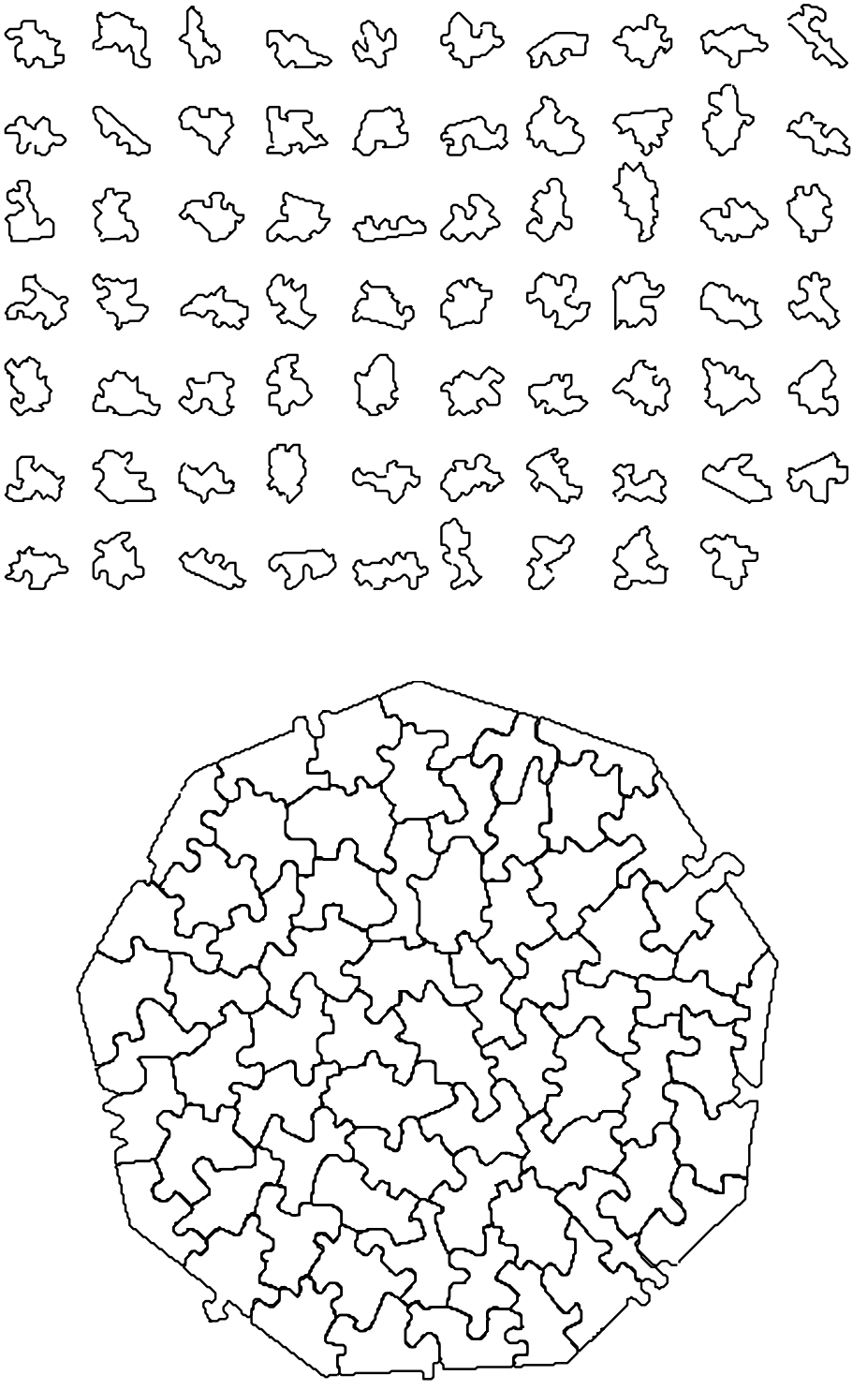


FIGURE 7. *The Baffler Jigsaw Puzzle*