Noether's Conservation Laws - 100 years on

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Relating to joint work with Tania Gonçalves,
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We have heard so far:

Lie symmetries $\Longrightarrow$ conservation laws; Noether (1918)

Why were they derived? their significance

This talk:

Combat expression swell: use invariants and moving frames

Discrete Noether's Theorem - embedding the physics via the Lie symmetry into the numerics

Noether's Theorem relates to the following problem. Suppose we have some set of smooth functions, for example,

$$\{f : \Omega \subset \mathbb{R}^4 \to \mathbb{R}^3 \,|\, f \text{ smooth}\}$$

and we consider a functional,

$$f \mapsto \int_{\Omega} L(t, \mathbf{x}, f, \nabla f)\, \mathrm{d}\mathbf{x}\mathrm{d}t \in \mathbb{R}.$$

We seek those f which minimise or maximise this quantity, usually subject to some boundary conditions.

The integrand, called a Lagrangian, quantifies something physically important, such as

- distance or time traveled by a particle,
- surface area spanned by the graph of f,
- a gravitational field subject to some given matter distribution, or
- an electromagnetic field subject to some charge distribution.

$$\{f : \Omega \subset \mathbb{R}^4 \to \mathbb{R}^p \,|\, f \text{ smooth}\}$$

$$f \mapsto \int_\Omega L(t, \mathbf{x}, f, \nabla f)\, \mathrm{d}\mathbf{x}\mathrm{d}t \in \mathbb{R}.$$

The function f extremising the functional satisfies a differential system, the Euler–Lagrange equations, $E^i(L) = 0$, $i = 1, 2, p$.

Noether: If the Lagrangian is invariant under a Lie group symmetry, there are guaranteed conservation laws;

$$\sum_i Q^i E^i(L) + \frac{\partial}{\partial t} A^0(t, \mathbf{x}, f) + \sum_1^3 \frac{\partial}{\partial x^j} A^j(t, \mathbf{x}, f) = 0.$$

The proof is constructive.

The quantity, A^0 in the conservation law, is the interesting quantity.

Symmetry	A^0
translation in time	Energy
translation in space	Linear Momenta
rotation in space	Angular Momenta
gauge symmetry	Charge
spatial diffeomorphisms	Potential Vorticity

The motivation for Noether's Theorem was the conservation of energy in general relativity.

History and Philosophy:

Series of papers by Katherine Brading, Duke University.

Yvette Kosman-Schwarzbach, *The Noether Theorems - Invariance and Conservation*, Springer, 2011.

The complete formulae:

P.J. Olver: Applications of Lie groups to differential equations, Springer Verlag, 1st edition 1886, 2nd edition, 1993.

Implementation:

Maple's `DifferentialGeometry` package, development led by I.A. Anderson.

How to calculate Euler Lagrange equations:
partial differentiation and integration by parts!!

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \mathcal{L}[u+\epsilon v] \\
&= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \int_a^b L(x, u+\epsilon v, u_x+\epsilon v_x, u_{xx}+\epsilon v_{xx}, \ldots)\,\mathrm{d}x \\
&= \int_a^b \left(\frac{\partial L}{\partial u} v + \frac{\partial L}{\partial u_x} v_x + \frac{\partial L}{\partial u_{xx}} v_{xx} + \ldots\right)\mathrm{d}x \\
&= \int_a^b \left[\left(\frac{\partial L}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u_x} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial L}{\partial u_{xx}} + \ldots\right) v \right. \\
&\qquad \left. + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial u_x} v + \frac{\partial L}{\partial u_{xx}} v_{xx} - \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u_{xx}}\right) v + \ldots\right)\right]\mathrm{d}x \\
&= \int E(L) v\,\mathrm{d}x + \left[\frac{\partial L}{\partial u_x} v + \ldots\right]_a^b
\end{aligned}
$$

The 1-d problem: we have a functional, from the set of smooth curves defined on $[a,b] \subset \mathbb{R}$,

$$(x, u(x)) \mapsto \int_a^b L(x, u(x), u_x, u_{xx}, \dots, u_{(n\,x)})\,\mathrm{d}x \in \mathbb{R}$$

The curves which extremize this functional solve the **Euler Lagrange** equation

$$E^u(L) = 0.$$

If the integrand is invariant under a Lie group action, then Noether's Theorem

$$Q \cdot E^u(L) + \frac{\mathrm{d}}{\mathrm{d}x} A = 0,$$

guarantees first integrals of this differential equation.

Running Example: projective $SL(2)$ action

$$g \cdot x = x, \qquad g \cdot u = \frac{au+b}{cu+d}$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1$$

Via the chain rule, induce an action on u_x etc:

$$g \cdot u_x = \frac{\partial(g \cdot u)}{\partial(g \cdot x)} = \frac{u_x}{(cu+d)^2}$$

Lowest order invariant is the so-called Schwarzian derivative,

$$V = \frac{u_{xxx}}{u_x} - \frac{3}{2}\frac{u_{xx}^2}{u_x^2} := \{u; x\}.$$

Suppose our Lagrangian is

$$L(x,u,u_x,\ldots,u_{xxxxxx})\,\mathsf{d}x=\left(\left(\frac{\mathsf{d}^2}{\mathsf{d}x^2}\{u;x\}\right)^2+\frac{1}{2}\{u;x\}^2\right)\,\mathsf{d}x.$$

Then this is invariant under the induced action of $SL(2)$ and there are three first integrals, one for each dimension of $SL(2)$.

The Euler–Lagrange equation has order 10, and one of the first integrals is:

Using Maple's `DifferentialGeometry` package

> $Noether\left(\frac{1}{2} \cdot u[\,]^2 \cdot D\_u[\,],\ lambda2\right);$

$$
\begin{aligned}
&-\frac{1}{4}\,\frac{1}{u_1^9}\Big(2\,u_{[\,]}^2\,u_1^7\,u_{1,1,1,1,1}+4\,u_{[\,]}^2\,u_1^7\,u_{1,1,1,1,1,1,1,1,1}-8\,u_{[\,]}^2\,u_1^6\,u_{1,1}\,u_{1,1,1,1}-32\,u_{[\,]}^2\,u_1^6\,u_{1,1}\,u_{1,1,1,1,1,1,1,1}-6\\
&\quad u_{[\,]}^2\,u_1^6\,u_{1,1,1}^2-132\,u_{[\,]}^2\,u_1^6\,u_{1,1,1}\,u_{1,1,1,1,1,1,1}-284\,u_{[\,]}^2\,u_1^6\,u_{1,1,1,1}\,u_{1,1,1,1,1,1}-180\,u_{[\,]}^2\,u_1^6\,u_{1,1,1,1,1}^2+21\,u_{[\,]}^2\\
&\quad u_1^5\,u_{1,1}^2\,u_{1,1,1}+240\,u_{[\,]}^2\,u_1^5\,u_{1,1}^2\,u_{1,1,1,1,1,1,1}+1620\,u_{[\,]}^2\,u_1^5\,u_{1,1}\,u_{1,1,1}\,u_{1,1,1,1,1,1}+2820\,u_{[\,]}^2\\
&\quad u_1^5\,u_{1,1}\,u_{1,1,1,1}\,u_{1,1,1,1,1}+2204\,u_{[\,]}^2\,u_1^5\,u_{1,1,1}^2\,u_{1,1,1,1,1}+3028\,u_{[\,]}^2\,u_1^5\,u_{1,1,1}\,u_{1,1,1,1}^2-9\,u_{[\,]}^2\,u_1^4\,u_{1,1}^4-1536\,u_{[\,]}^2\\
&\quad u_1^4\,u_{1,1}^3\,u_{1,1,1,1,1,1}-12508\,u_{[\,]}^2\,u_1^4\,u_{1,1}^2\,u_{1,1,1}\,u_{1,1,1,1,1}-8548\,u_{[\,]}^2\,u_1^4\,u_{1,1}^2\,u_{1,1,1,1}^2-26624\,u_{[\,]}^2\,u_1^4\,u_{1,1}\\
&\quad u_{1,1,1}^2\,u_{1,1,1,1}-3432\,u_{[\,]}^2\,u_1^4\,u_{1,1,1}^4+8064\,u_{[\,]}^2\,u_1^3\,u_{1,1}^4\,u_{1,1,1,1,1}+68392\,u_{[\,]}^2\,u_1^3\,u_{1,1}^3\,u_{1,1,1}\,u_{1,1,1,1}+53004\,u_{[\,]}^2\\
&\quad u_1^3\,u_{1,1}^2\,u_{1,1,1}^3-33408\,u_{[\,]}^2\,u_1^2\,u_{1,1}^5\,u_{1,1,1,1}-129300\,u_{[\,]}^2\,u_1^2\,u_{1,1}^4\,u_{1,1,1}^2+101808\,u_{[\,]}^2\,u_1\,u_{1,1}^6\,u_{1,1,1}-25200\,u_{[\,]}^2\,u_{1,1}^8\\
&\quad -4\,u_{[\,]}\,u_1^8\,u_{1,1,1,1}-8\,u_{[\,]}\,u_1^8\,u_{1,1,1,1,1,1,1,1}+12\,u_{[\,]}\,u_1^7\,u_{1,1}\,u_{1,1,1}+56\,u_{[\,]}\,u_1^7\,u_{1,1}\,u_{1,1,1,1,1,1,1}+208\,u_{[\,]}\\
&\quad u_1^7\,u_{1,1,1}\,u_{1,1,1,1,1,1}+360\,u_{[\,]}\,u_1^7\,u_{1,1,1,1}\,u_{1,1,1,1,1}-6\,u_{[\,]}\,u_1^6\,u_{1,1}^3-368\,u_{[\,]}\,u_1^6\,u_{1,1}^2\,u_{1,1,1,1,1,1}-2088\,u_{[\,]}\\
&\quad u_1^6\,u_{1,1}\,u_{1,1,1}\,u_{1,1,1,1,1}-1416\,u_{[\,]}\,u_1^6\,u_{1,1}\,u_{1,1,1,1}^2-2320\,u_{[\,]}\,u_1^6\,u_{1,1,1}^2\,u_{1,1,1,1}+1968\,u_{[\,]}\,u_1^5\,u_{1,1}^3\,u_{1,1,1,1,1}\\
&\quad +12848\,u_{[\,]}\,u_1^5\,u_{1,1}^2\,u_{1,1,1}\,u_{1,1,1,1}+6864\,u_{[\,]}\,u_1^5\,u_{1,1}\,u_{1,1,1}^3-8256\,u_{[\,]}\,u_1^4\,u_{1,1}^4\,u_{1,1,1,1}-26184\,u_{[\,]}\,u_1^4\,u_{1,1}^3\,u_{1,1,1}^2\\
&\quad +25536\,u_{[\,]}\,u_1^3\,u_{1,1}^5\,u_{1,1,1}-7200\,u_{[\,]}\,u_1^2\,u_{1,1}^7+4\,u_1^9\,u_{1,1,1}+8\,u_1^9\,u_{1,1,1,1,1,1,1}-6\,u_1^8\,u_{1,1}^2-56\\
&\quad u_1^8\,u_{1,1}\,u_{1,1,1,1,1,1}-152\,u_1^8\,u_{1,1,1}\,u_{1,1,1,1,1}-104\,u_1^8\,u_{1,1,1,1}^2+312\,u_1^7\,u_{1,1}^2\,u_{1,1,1,1,1}+1312\\
&\quad u_1^7\,u_{1,1}\,u_{1,1,1}\,u_{1,1,1,1}+336\,u_1^7\,u_{1,1,1}^3-1344\,u_1^6\,u_{1,1}^3\,u_{1,1,1,1}-3096\,u_1^6\,u_{1,1}^2\,u_{1,1,1}^2+4224\,u_1^5\,u_{1,1}^4\,u_{1,1,1}-1440\\
&\quad u_1^4\,u_{1,1}^6\Big)
\end{aligned}
$$

(1

We can use the Lie group action to cut down the expression swell. We can use the power of the Lie group based moving frames to derive

• Results using a trivariational complex and exterior calculus:
I. Kogan and P.J. Olver, Acta Appl. Math **76** (2003)

• Results, directly using the invariant calculus, with links to Lie group integrators:
ELM, A practical guide to the invariant calculus, Cambridge Univ., Press., 2010.
T.M.N. Gonçalves and ELM, Moving frames and Noether's conservation laws – the general case. Forum of Math., Sigma, (2016).

From mathematical wallpaper to structure

Set $V = \{u; x\}$, the Schwarzian derivative. Then the Euler–Lagrange equation $E^u(L) = 0$ for

$$u \mapsto \int L(V, V_x, \dots, V_{(nx)})\, \mathrm{d}x$$

in terms of V is

$$\mathcal{H}^*(E^V(L)) = -(\frac{\partial^3}{\partial x^3} + 2V\frac{\partial}{\partial x} + V_x)(E^V(L)) = 0$$

The operator in front of $E^V(L)$ is easily calculated, it comes from the identity, or syzygy, connecting the two invariants,

$$V = \{u; x\}, \quad W = \frac{u_x}{u_t}, \quad \frac{\partial}{\partial t}V = \mathcal{H}(W)$$

where t is a dummy variable chosen to effect the variational calculation.

For Lagrangians of the form $\mathcal{L}[u] = \int L(V, V_x, \ldots)\,\mathrm{d}x$ where $V = \{u; x\}$,

$$\mathbf{c} = \underbrace{\begin{pmatrix} a^2 & -ac & -c^2 \\ -2ab & ad+bc & 2dc \\ -b^2 & bd & d^2 \end{pmatrix}}_{R(g)^{-1}} \Bigg|_{g=\rho} \begin{pmatrix} \frac{\partial^2}{\partial x^2} E^V(L) + V E^V(L) \\ -2\frac{\partial}{\partial x} E^V(L) \\ -2E^V(L) \end{pmatrix}$$

where

$$\rho: \quad a = \frac{1}{\sqrt{u_x}}, \qquad b = -\frac{u}{\sqrt{u_x}}, \qquad c = \frac{u_{xx}}{2(u_x)^{3/2}}, \qquad ad - bc = 1.$$

- $R(gh) = R(g)R(h)$, and $R(\rho(u, u_x, u_{xx}))$ is equivariant

Which representation yields $R(g)$? How to find ρ? And how to calculate the vector of invariants directly?

Answers and observations:

$$\begin{pmatrix} c^1 \\ c^2 \\ c^3 \end{pmatrix} = \underbrace{\begin{pmatrix} a^2 & -ac & -c^2 \\ -2ab & ad+bc & 2dc \\ -b^2 & bd & d^2 \end{pmatrix}}\Bigg|_{g=\rho} \begin{pmatrix} \frac{\partial^2}{\partial x^2}E^V(L) + VE^V(L) \\ -2\frac{\partial}{\partial x}E^V(L) \\ -2E^V(L) \end{pmatrix}$$

$$\rho: \quad a = \frac{1}{\sqrt{u_x}}, \qquad b = -\frac{u}{\sqrt{u_x}}, \qquad c = \frac{u_{xx}}{2(u_x)^{3/2}}, \qquad ad-bc = 1.$$

• $R(g)$ is the Adjoint representation of $SL(2)$ on its Lie algebra

• We have three equations for u, u_x and u_{xx}. Writing the vector of invariants as $(v^1, v^2, v^3)^T$ and simplifying yields

$$\begin{aligned} 4c^1c^3 + (c^2)^2 &= 4v^1v^3 + (v^2)^2 \\ v^3u_x &= -c^1u^2 + c^2u + c^3 \end{aligned}$$

In general, to solve an invariant ODE of the form $\Delta(x, V, V_x, V_{xx}, \ldots, V_{nx}) = 0$, with $V = \{u; x\}$, once you have solved for V, you are faced with solving

$$\frac{u_{xxx}}{u_x} - \frac{3}{2}\frac{u_{xx}^2}{u_x^2} = V(x)$$

for u, However, a moving frame satisfies a linear differential equation, in this case

$$\rho_x = \underbrace{\begin{pmatrix} 0 & -1 \\ \frac{1}{2}\{u; x\} & 0 \end{pmatrix}}_{\in \mathfrak{sl}(2)} \rho$$

so Lie group integrators can be brought to bear. Further, it can be seen that $u = \rho^{-1} \cdot 0$, so that once you have the frame, you have u without further integration.

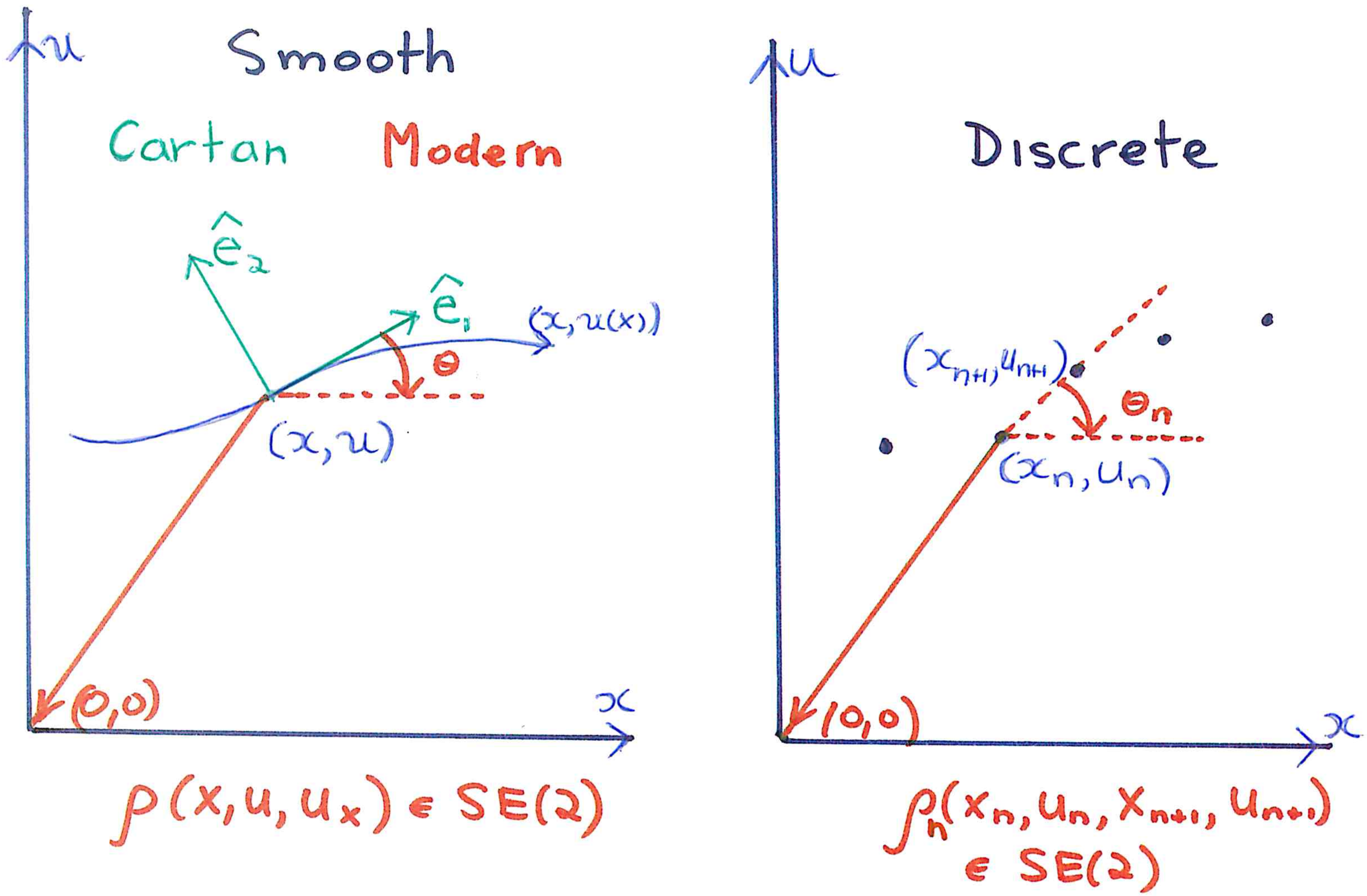
Zadra and ELM: The numerical Lie group integrators do indeed commute when used in different directions, and can therefore be used to integrate conservation laws in more than one dimension.

Many authors have worked on Noether's laws for finite difference Lagrangians, including Hereman, Peng, Hydon, ELM, . . .

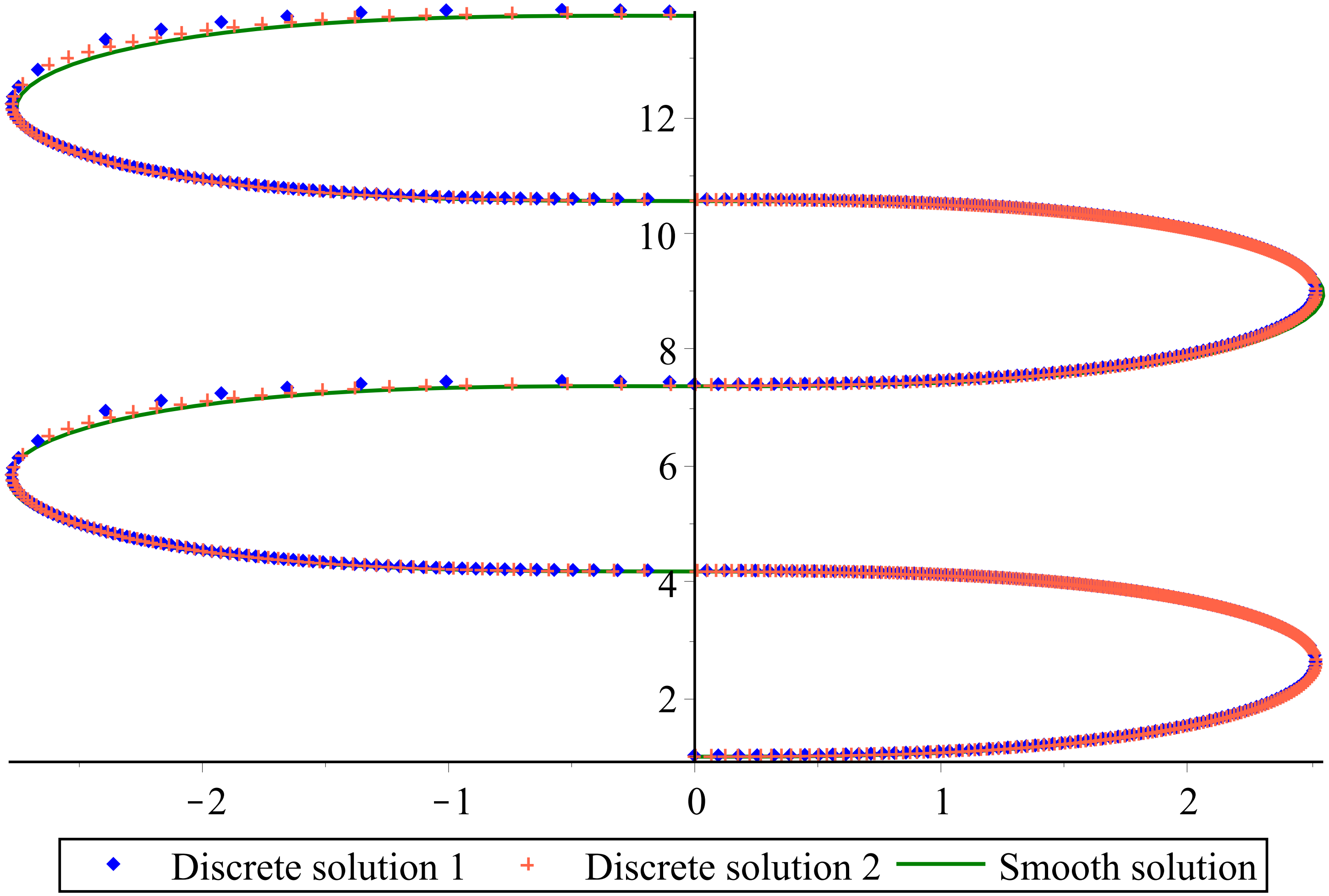
ELM, Rojo, Hydon and Peng: 80pp on moving frames and finite difference Noether's Theorem, Parts 1 and 2, now available on the ArXiv.

The key to getting the finite difference approximation to match the smooth laws, seems to be, to match the smooth with the discrete moving frames.

Example: Euler's Elastica: match famous frame for $SE(2)$ with the obvious matching discrete frame, use this to read off the discrete invariant which matches Euclidean curvature.



Smooth
Cartan
Modern
u
x
$\hat{e}_2$
$\hat{e}_1$
$(x, u(x))$
θ
(x, u)
$(0,0)$
$\rho(x, u, u_x) \in SE(2)$
Discrete
u
x
(x_{n+1}, u_{n+1})
θ_n
(x_n, u_n)
$(0,0)$
$\rho_n(x_n, u_n, x_{n+1}, u_{n+1})$
$\in SE(2)$



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Discrete solution 1
Discrete solution 2
Smooth solution

To incorporate the physics into the numerical model, need to get round the Ge and Marsden "no go" theorem, so instead:

♢ make the discrete Lagrangian to be

  ♡ invariant under the induced action on the approximation data, and

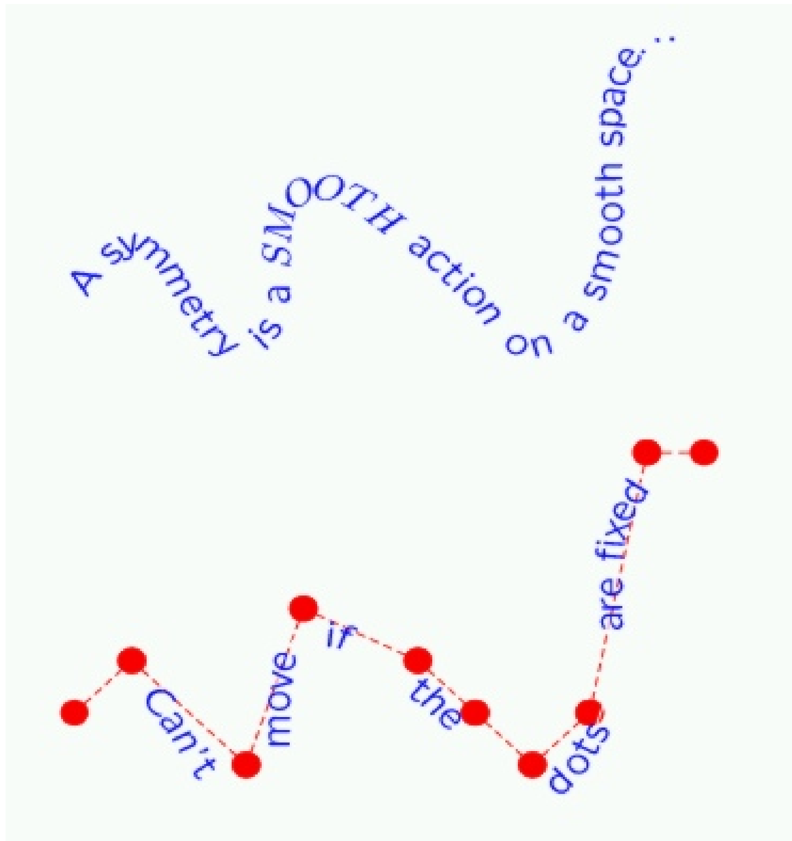
  ♡ have the correct continuum limit

♢ write down the exactly conserved (in approximation space), discrete Noether law

♢ prove the discrete Euler–Lagrange equation and the discrete conservation laws, converge to the desired smooth equations and laws in some useful sense.

ELM and Pryer: Noether-type Discrete Conserved Quantities arising from a Finite Element approximation of a variational problem, FoCM, **17** (3) 2017.

Challenge: find where the group action has gone to!



For Finite Difference methods, where the approximation data is the value at a point, you have to have the coordinates of the independent variables as new dependent variables, whose values are referred to a fixed (dummy) grid.

For Finite Elements, where the approximation data takes the form of average values over edges and faces, we can induce actions as follows,

$$\int_\sigma f(x,u)\,\mathrm{d}x \mapsto \int_\sigma f(g\cdot x, g\cdot u)\frac{\partial(g\cdot x)}{\partial x}\,\mathrm{d}x.$$

Recall the link between extremisation and Noether's laws starts with:

$$\underbrace{0 =}_{\text{at extremal}} \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \int L(x, u + \epsilon v, u_x + \epsilon v_x, \dots, u_{(nx)} + \epsilon v_{(nx)})\,\mathrm{d}x$$

versus

$$\underbrace{0 =}_{\text{invariance}} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int L(g(t)\cdot x, g(t)\cdot u, g(t)\cdot u_x, \dots, g(t)\cdot u_{(nx)}) \frac{\mathrm{d}(g(t)\cdot x)}{\mathrm{d}x}\,\mathrm{d}x$$

with $g(t) \subset G$ and $g(0) = e$, the identity element.

And we take this to be the the starting point for the discrete Noether's Theorem. If $\mathbf{p}$ is the approximation data, we have

$$\underbrace{0 =}_{\text{at extremal}} \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \int \bar{L}(p_1 + \epsilon v_1, p_2 + \epsilon v_2, \dots, p_n + \epsilon v_n)\, \overline{\mathrm{d}x}$$

and

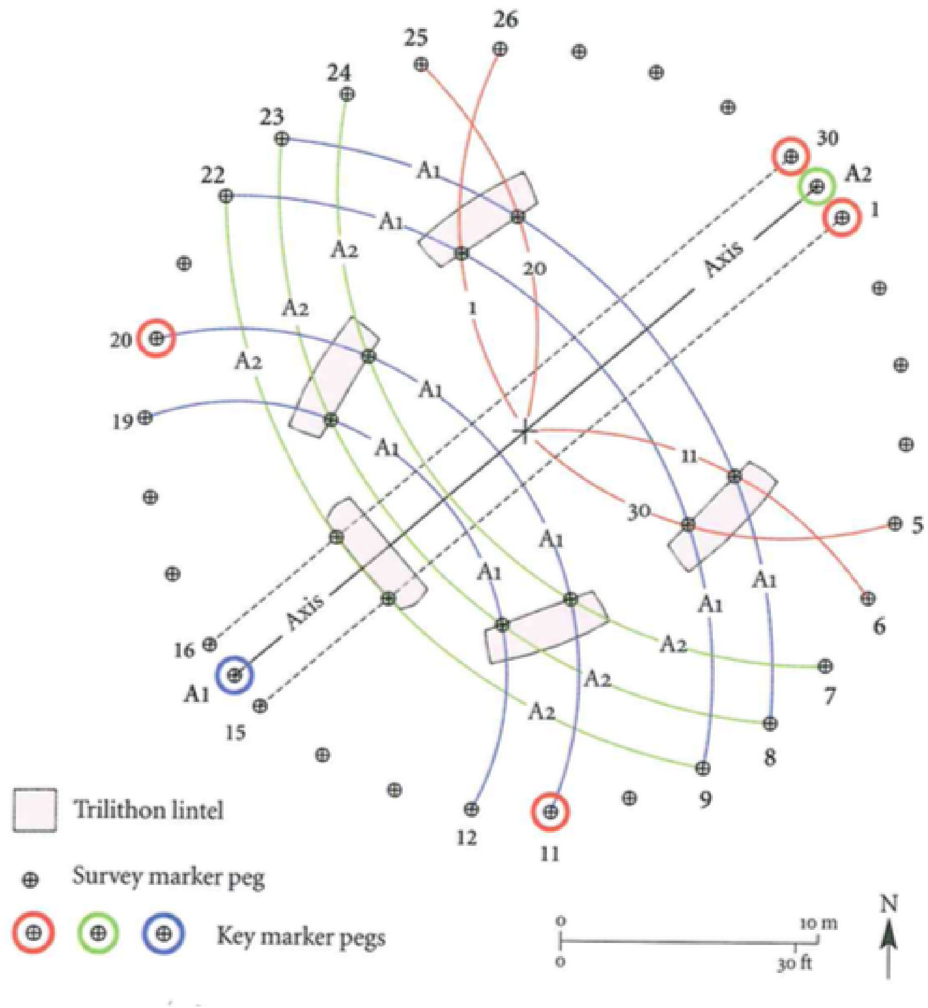
$$\underbrace{0 =}_{\text{invariance}} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int \bar{L}(g(t) \cdot p_1, g(t) \cdot p_2, g(t) \cdot p_3, \dots, g(t) \cdot p_n)\, g(t) \cdot \overline{\mathrm{d}x}$$

with $\bar{L}$ the approximate Lagrangian and $\overline{\mathrm{d}x}$ the approximate volume form.

A bit of fun: Emmy Noether at Stonehenge: in which we show evolving "sound" waves of a drum beating in the heart of Stonehenge

- shallow water-type equations and FEM
- exact energy conservation using a trick from geometric integration of ODES called "the discrete gradient method"
- weak conservation of linear and angular momentum, à la ELM and Pryer.

We use a precise survey of Stonehenge, using only pegs and ropes, discovered by Anthony Johnson, "Solving Stonehenge: the new key to an ancient enigma", Thames & Hudson, 2008.



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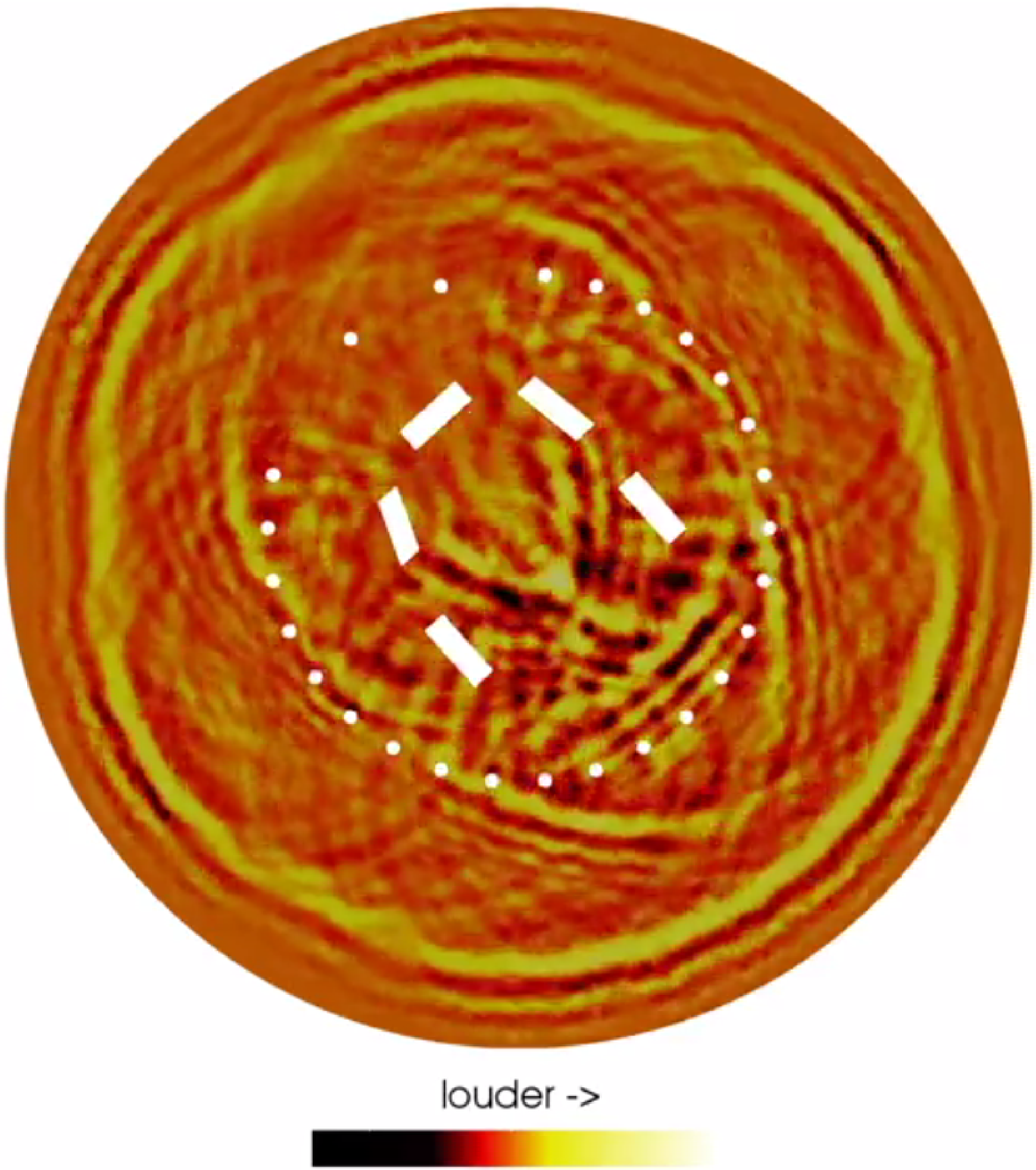
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THANK YOU!!