Emmy Noether, Symmetry, and Women in Mathematics

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Noether Celebration

Emmy Noether's amazing contributions to algebra

Nathan Jacobson [edited collected papers of Emmy Noether]

Development of modern abstract algebra is largely due to Emmy Noether

- "One of the most distinctive innovations of 20th century mathematics"
- Through published papers, lectures, personal influence
- Contributions so thoroughly absorbed into math culture
- That they are rarely specifically attributed to her

Ideal Theory in Rings

Idealtheorie in Ringbereichen Math. Annalen 83 (1921), 24-66

- Of fundamental importance in the development of modern algebra
- New concepts – new framework for studying “rings”
- Decomposition/factorisation theorems – general (appropriate) context

- So fundamental, some of her ideas rediscovered by others

- Name given to various algebraic structures: notably
    - *Noetherian rings*

Rings

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Examples: Integers **Z** [positive and negative]

Characteristics (axioms) of a ring R

- Addition and multiplication possible
- R contains zero 0 and each element a has a negative –a so a + (-a) = 0
- Parentheses often not needed:
  - (a + b) + c = a + (b + c) and (a.b) . c = a . (b.c)
- Distributive rule: (a + b) . c = a.c + b.c

- Emmy Noether's rings were commutative: a + b = b + a and a . b = b . a

Not every ring contains a "1"
e.g. $2Z = \{ 2\,x \mid x \in Z\}$

As well as Z: many rings had been studied

- **Ring of Gaussian integers a+bi with a, b in Z**
- **Polynomial rings R[x] with coefficients in a ring R**

Emmy Noether's aim in her 1921 paper

To convert known "decomposition theorems"

- for Z or algebraic integers like Z[i], or polynomial rings Z[x]

into theorems for "ideals" in arbitrary "integral domains" (or rings in general)

- Analogue of prime-power decomposition of integers
  - $a = p_1^{r_1} p_2^{r_2} \dots p_s^{r_s} = q_1 q_2 \dots q_s$
  - Think of q_i as "components" of the decomposition
  - We know: for integers decomposition "essentially unique" – number s, the primes p_i, the exponents r_i
  - Only issue is "-1" – discuss later
  - Similar unique decomposition results for some rings of algebraic integers and polynomial rings.

Example: $60 = 2^2\ 3^1\ 5^1 = 4 \cdot 3 \cdot 5$

Emmy Noether: introduced general properties – which could be interpreted in all these contexts and others - to get a general decomposition theorem whenever these properties held

Some "characteristic properties"

Capture the idea of prime-power decomposition for integers

In a decomposition $a = q_1 q_2 \ldots q_s$

- Components q_i should be *pairwise coprime* and q_i *not decomposable* as product of pairwise coprime numbers (irreducible)

E.g. 60 = 6 x 10
Or 60 = 2 x 2 x 3 x 5
Not allowed

- Components q_i should be *primary*
  - If q_i divides bc but not b then q_i divides some power of c

E.g. 60 = 4 x 3 x 5
Component 4 is primary
Since if 4 divides $b.c$ but not b,
Then c even, so 4 divides c^2

- And two more properties …
- Properties formulated to apply to any ring:
  - "q divides b" means $b = qc$ for some c

Ideals: to deal with the “uncertainty” caused by “-1”

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Ideal N in a ring R

Definition: N subset of R such that

- If N contains b then N also contains $a.b$ for arbitrary a in N
- If N contains b and c then N also contains $b - c$

E.g. $\mathrm{N} = 60Z = \{\, 60\, x \mid x \in Z\}$
Is an ideal of Z

Transfers language

- If N contains b then say b divisible by N
- If M is an ideal contained in N, then say M divisible by N

E.g. 60Z is contained in 12Z
So 60Z is divisible by 12Z

Definition: N is finitely generated by subset $B = \{b_1, \dots, b_s\}$

- if each a in N can be written as $a = a_1 b_1 + \cdots + a_s b_s + n_1 b_1 + \cdots + n_s b_s$

- [this is modern language: Emmy Noether calls B an “ideal basis” for N]

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Finite Chain Theorem: If every ideal is finitely generated, then each infinite ascending chain of ideals terminates after a finite number of steps.

Ideals: to deal with the "uncertainty" caused by "-1"

Finite Chain Theorem: If every ideal is finitely generated, then for a countably infinite sequence of ideals such that

$$N_1 \subseteq N_2 \subseteq \cdots \subseteq N_r \subseteq \cdots$$

there exists an integer n such that $N_n = N_{n+1} = \cdots$.

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Why does it apply to the integers Z ?

- Each ideal M is equal to $\mathrm{M} = mZ = \{\, m\,x \mid x \in Z\}$ for some m
- Each ideal N containing M is $\mathrm{N} = nZ = \{\, n\,x \mid x \in Z\}$ for some n dividing m
- So M contained in only finitely many larger ideals

E.g. $60Z \subset\ 12Z \subset\ 4Z \subset\ 2Z$

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Theorem was known previously for several special cases:

- Context of modules – Dedekind, [Supplement XI to "Zahlentheorie"]
- Polynomial rings – Lasker, 1905

Emmy Noether gives much wider-ranging applications of this theorem than previously for the special cases

Noetherian rings

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Ascending chain condition (ACC)

- Each countably infinite ascending chain of ideals terminates after a finite number of steps.

ACC defined also on chains of left ideals and chains of right ideals in non-commutative rings

Definition: Ring is Noetherian if it has the ACC on left and right ideals

Next Noether's applications of Finite Chain Theorem: depend on Axim of Choice

Decompositions of ideals I

Least common multiple $\ell.c.m.\{N_1, N_2, \dots, N_k\}$ of ideals is the intersection $N_1 \cap N_2 \cap \cdots \cap N_k$

Example $60Z = 12Z \cap 10Z$ the ideals $12Z, 10Z$ the components of decomposition

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Irreducible ideal: if N not the intersection (l.c.m.) of two larger ideals

So $4Z$ is irreducible (only larger ideal is $2Z$)

But none of $60Z = 12Z \cap 10Z$, $12Z = 4Z \cap 3Z$, or $10Z = 2Z \cap 5Z$ irreducible

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Emmy's next theorem: [Theorem II]

Every ideal is an intersection of finitely many irreducible ideals

What about uniqueness?

Always assuming:
all ideals finitely generated

Analogue of: every integer is a product of finitely many prime-powers

Decompositions of ideals II

Suppose ideal $\mathrm{N} = N_1 \cap N_2 \cap \cdots \cap N_k$ with the N_i irreducible

Shortest decomposition: cannot omit any component, that is,

No N_i contains the intersection of the remaining ideals

Example: $60Z = 4Z \cap 3Z \cap 2Z \cap 5Z$ not shortest since $60Z = 4Z \cap 3Z \cap 5Z \subset 2Z$

NICE decomposition: $\mathrm{N} = N_1 \cap N_2 \cap \cdots \cap N_k$ if “shortest” and all N_i irreducible

Emmy’s next theorem: [Theorem IV]

Any two NICE decompositions of an ideal have same number of components

Decompositions of ideals III

Primary and prime ideals: two definitions (products of elements and products of ideals) – shows equivalent

- Primary ideals contained in unique prime ideal [Theorem V]
- Irreducible ideals are primary [Theorem VI]

- Hence in NICE decomposition $\mathrm{N} = N_1 \cap N_2 \cap \cdots \cap N_k$ each N_i contained in unique prime ideal P_i

Emmy's next theorem: [Theorem VII]

Any two NICE decompositions involve exactly the same set of prime ideals

- Some information "exponents" but not uniqueness [Theorem VII]
- 11 pages [and five Theorems] refining the structure

Study of modules

Section 9: Precursor of Emmy Noether's work on Group Rings

Modules are like vector spaces, but the scalars come from a ring (not necessarily a field)

Involves

- Extension of theory to non-commutative rings
- Introduces concept of sub-module
- Assuming that each submodule is finitely generated, obtains structural results which "lay the foundations for uniqueness theorems"

Later work

- 1927 Noether (her letter to Brauer)
  - *"I am very glad that you have now realized the connection between representation theory and the theory of noncommutative rings …"*
- 1929 Noether [Math.Z.]
  - *Roquet: "Her great paper on representation theory and group rings"*
- 1932 [J Reine und Angew. Math.] Brauer--Hasse--Noether Theorem on division algebras over number fields
  - *Weyl: "a high watermark in the history of algebra"*

Final applications

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Section 10: Connection of her theorems with known work on polynomial rings

Section 11: Examples from Number Theory

Section 12: Example from “elementary divisor theory” – matrix algebras

It is an extraordinary paper

Hermann Weyl

Her influence is not just in her papers

Hermann Weyl:

“she had great stimulating power and many of her suggestions took shape only in the works of her pupils and co-workers.”

Albert Einstein May 1, 1935 - New York Times

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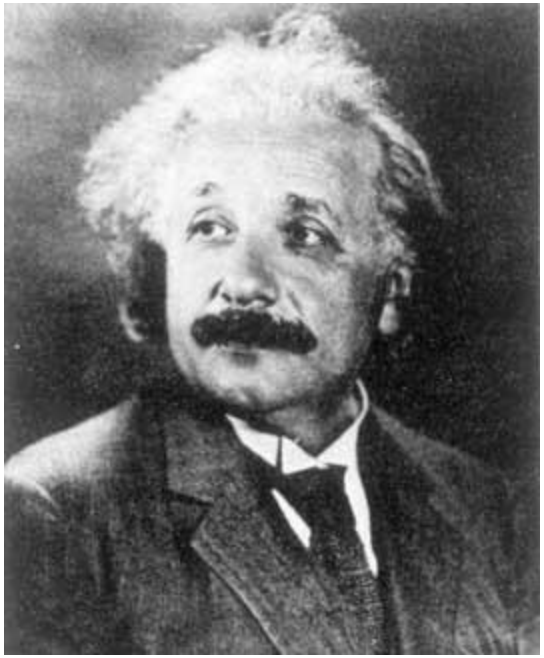
Obituary: describes Emmy Noether as:

“the most significant creative mathematical genius thus far produced since the higher education of women began.”

“In the realm of algebra, …she discovered methods which have proved of enormous importance in the development of the present-day younger generation of mathematicians”



Emmy Noether's legacy

For Women in Mathematics.

International Congresses of Mathematicians (ICM)

Emmy was first in many things.

- 1897 First ICM - Zurich
  - 4 women out of 208 participants
  - No women speakers
- 1932 ICM – again in Zurich
  - Emmy Noether first woman to give a plenary lecture at an ICM
- 1990 ICM in Kyoto
  - Karen Uhlenbeck second woman ICM plenary speaker
  - 58 years later!

Women in Mathematics Associations

Before the 1990 ICM

Associations for women in mathematics formed:

1971 Association for Women in Mathematics AWM

- (in the US) – with men and women as members

1980 AWM initiated: Emmy Noether Lecture

- *"to honour women who have made fundamental and sustained contributions to the mathematical sciences"*
- Annually at Joint Mathematics meeting in January
- 2013: AWM-AMS Noether lectures [co-sponsored by American Math Soc]

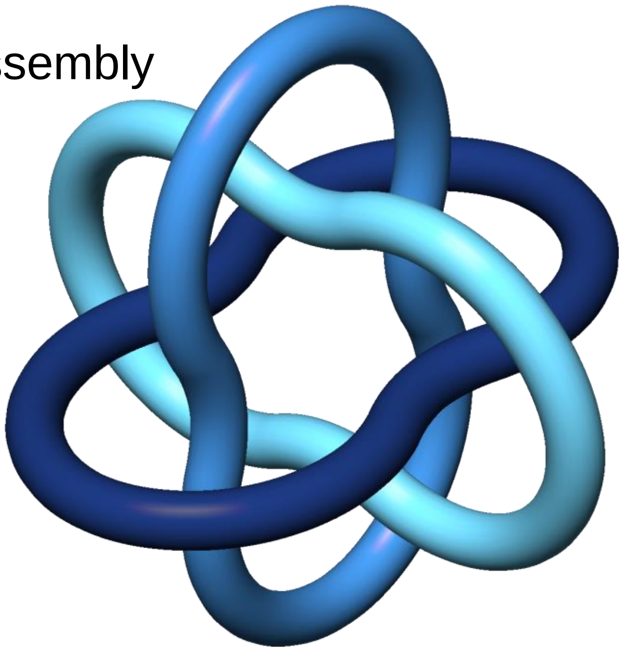
AWM

ASSOCIATION FOR WOMEN IN MATHEMATICS

ICM Emmy Noether Lecture

Additional AWM Emmy Noether lectures

- 1994 Organised by AWM – presented at ICM [each four years]
- 2002 International Mathematical Union: General Assembly
  - agreed to formalize the ICM Emmy Noether Lecture
  - for following two ICMs
- 2010 International Mathematical Union: General Assembly
  - established ICM Emmy Noether Lecture on-going



Women and the International Mathematical Union (IMU)

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1951 IMU established

IMU Executive Committee

- (now) 11 persons – President, 2 VPs, 6 Members-at-Large and immediate past President – 4 year terms
- **2003-2006 First time a woman on the IMU EC:**

- Ragni Piene (Norway)
- Ragni chaired Emmy Noether Committee for 2006

- Members at large
- May serve maximum of two terms
- Terms are four years

Women and the International Mathematical Union (IMU)

2007-2010 First time two women on IMU EC

- Ragni and Cheryl Praeger
- Cheryl chaired second ICM Emmy Noether Committee - for 2010 ICM
- ICM Emmy Noether Lecture established by IMU GA 2010

Women and the International Mathematical Union (IMU)

20011-2014 First woman President of IMU

– Ingrid Daubechies

2014 Established IMU Committee

– Committee for Women in Mathematics CWM

And exceptionally …..

Women and the International Mathematical Union (IMU)

2014 International Congress of Mathematicians in Seoul

Only Woman to receive a Fields Medal - in more than 80 years!

- Awarded by the IMU at the ICM
- Up to four – every four years – for outstanding achievements by mathematicians under 40
- First FM awarded in **1936**

- There's a long way to go

Thank you Emmy