From Platonic solids to quivers

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Abstract

This course will be a whirlwind tour through representation theory, a major branch of modern algebra. We being by considering the symmetry groups of the Platonic solids, which leads naturally to the notion of a reflection group and its associated root system. The classification of these reflection groups gives us our first examples of quivers (= direct graphs). Though easy to define, we'll see that the representation theory associated to quivers is very rich. We will use quivers to illustrate the key concepts, ideas and problems that appear throughout representation theory. Coming full circle, the course will culminate with the beautiful theorem by Gabriel, classifying the quivers of finite type in terms of the root systems of reflection groups. The ultimate goal of the course is to give students a glimpse of the beauty and unity of this field of research, which is today very active in the U.K.

Exercises: The Platonic solids

1. If the Schläfi symbol of the Platonic solid P is $\{p, q\}$, use Euler's formula $V - E + F = 2$ to show that

$$V = \frac{4p}{4-(p-2)(q-2)}, \quad E = \frac{2pq}{4-(p-2)(q-2)}, \quad F = \frac{4q}{4-(p-2)(q-2)},$$

   where V, E and F are the number of vertices, edges and faces respectively of P.

2. Recall from the first lecture that a *reflection* on $\mathbb{R}^n$ is an orthogonal transformation $s \in O(\mathbb{R}, n)$ such that $\dim \mathrm{Fix}_{\mathbb{R}^n}(s) = n - 1$ and $s^2 = \mathrm{id}$.

   (a) Show that $\mathrm{Fix}_{\mathbb{R}^n}(s) = \mathrm{Ker}(\mathrm{id} - s)$.

   (b) Prove that s is diagonalizable. What are the eigenvalues of s?

   (c) Deduce that $\det(s) = -1$.

   (d) Choose α such that $\mathrm{Fix}_{\mathbb{R}^n}(s) = H_\alpha$. Derive the formula

$$s_\alpha(x) = x - \frac{2(x, \alpha)}{(\alpha, \alpha)}\alpha$$

   for a reflection.

   Hint: For part (b), if $H := \mathrm{Ker}(\mathrm{id} - s)$, consider the space $H^\perp$. Show that s acts on $H^\perp$.

3. There is a purely topological proof of the fact that there are are only five Platonic solids. The key topological fact is that Euler's formula holds: $V - E + F = 2$. Using this, together with the relations $pF = 2E = qV$, show that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{E}.$$

   Deduce that there are only five Platonic solids.

4. Using the fact that $g \in W(P)$ is a reflection if and only if it has one eigenvalue equal to -1 and two eigenvalues equal to 1, count the number of reflections in $W(\mathsf{H})$ and $W(\mathsf{D})$.

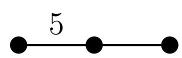
Exercises: Reflection groups and root systems

1. Let
$$E = \left\{ x = \sum_{i=1}^{n+1} x_i \epsilon_i \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i = 0 \right\},$$
where $\{\epsilon_1, \ldots, \epsilon_{n+1}\}$ is the standard basis of $\mathbb{R}^{n+1}$ with $(\epsilon_i, \epsilon_j) = \delta_{i,j}$. Let $R = \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n+1\}$.

   (i) Show that R is a *crystallographic* root system.

   (ii) Construct two different sets of simple roots for R.

   (iii) By considering the action of the reflections $s_{\epsilon_i - \epsilon_j}$ on the basis $\{\epsilon_1, \ldots, \epsilon_{n+1}\}$ of $\mathbb{R}^{n+1}$, show that the Weyl group of R is isomorphic to $\mathfrak{S}_{n+1}$.

2. Show that the symmetric matrix
$$A = \begin{pmatrix} 1 & -\cos\frac{\pi}{5} & 0 \\ -\cos\frac{\pi}{5} & 1 & -\cos\frac{\pi}{3} \\ 0 & -\cos\frac{\pi}{3} & 1 \end{pmatrix}$$
corresponding to the Coxeter graph 

of type H_3 is positive definite. What is the determinant of A? Hint: recall that $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{2}$.

3. The angle between roots in a crystallographic reflection groups. Recall the following table in section 2.5 of the lecture notes. The only possible values of $\langle \alpha, \beta \rangle$ are:

$\langle \beta, \alpha \rangle$	$\langle \alpha, \beta \rangle$	θ
0	0	$\frac{\pi}{2}$
1	1	$(\star)$
-1	-1	$\frac{2\pi}{3}$
1	2	$(\star\star)$
-1	-2	$\frac{3\pi}{4}$
1	3	$\frac{\pi}{6}$
-1	-3	$(\star\star\star)$

   (a) What are the angles θ in $(\star)$, $(\star\star)$ and $(\star\star\star)$?

   (b) What about $\langle \beta, \alpha \rangle = \langle \alpha, \beta \rangle = \pm 2$?

(c) Let $\alpha, \beta \in \mathbb{R}^n$. Show that $s_\beta s_\alpha$ is a rotation of $\mathbb{R}^n$. Hint: decompose $\mathbb{R}^n = \mathbb{R}\{\alpha, \beta\} \oplus H_\alpha \cap H_\beta$ and consider $s_\beta s_\alpha$ acting on $\mathbb{R}\{\alpha, \beta\}$. If e_1, e_2 is an orthonormal basis of $\mathbb{R}^2$, write out s_α and s_β explicit.

4. The *hypercube* H_n is the n-dimensional analogue of the square ($n = 2$), or cube ($n = 3$). Concretely, we can realize H_n in $\mathbb{R}^n$ as the set of points

$$\mathsf{H}_n = \{v \in \mathbb{R}^n \mid -1 \leq v_i \leq 1 \ i = 1, \dots, n\}.$$

The group of symmetries of H_n is denoted BC_n. It is called the *hyperoctahedral group*.

(i) How many vertices does the H_n have? How about edges, or faces?

(ii) The $(n-1)$-dimensional faces of H_n are the copies $F_i^\pm$ of H_{n-1} given by $\{v \in \mathsf{H}_n \mid v_i = \pm 1\}$. Since BC_n permutes these $(n-1)$-dimensional faces, it will permute their midpoints $\{e_i^\pm \mid i = 1, \dots, n\}$, where

$$e_i^\pm = (0, \dots, 0, \pm 1, 0, \dots, 0).$$

Deduce that w is a sign permutation matrix i.e. a matrix where each row has only one non-zero entry which is either a 1 or -1, and similarly for the columns.

(iii) What is the order of the group BC_n?

(iv) The *hyperoctahedron* is dual to the hypercube. It is defined to be

$$\mathsf{O}_n = \{x \in \mathbb{R}^n \mid (x, v) \leq 1 \text{ for all vertices } v \text{ of } \mathsf{H}_n\}.$$

Check for $n = 2$ and $n = 3$ that one gets the (rotated by $\frac{\pi}{4}$) square and octahedron respectively.

(v) Show directly from the definition that the symmetries of H_n are also symmetries of O_n. This shows that $W(\mathsf{H}_n) \subset W(\mathsf{O}_n)$.

(vi) Notice that the $e_i^\pm$ are the vertices of O_n. Deduce that $W(\mathsf{O}_n) = BC_n$.

Exercises: Quivers

1. A *homomorphism* between representations. Let $M = \{(\mathbb{C}^{v_i}, \varphi_\alpha)\}$ and $N = \{(\mathbb{C}^{w_i}, \psi_\alpha)\}$ be representations of a quiver Q. Then a homomorphism $\mathbf{f} : M \to N$ is a collection of linear maps $f_i \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{v_i}, \mathbb{C}^{w_i})$ for each $i \in Q_0$ such that the diagrams

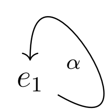
$$\begin{array}{ccc} \mathbb{C}^{v_{t(\alpha)}} & \xrightarrow{\varphi_\alpha} & \mathbb{C}^{v_{h(\alpha)}} \\ \big\downarrow f_{t(\alpha)} & & \big\downarrow f_{h(\alpha)} \\ \mathbb{C}^{w_{t(\alpha)}} & \xrightarrow{\psi_\alpha} & \mathbb{C}^{w_{h(\alpha)}} \end{array}$$

commute for all $\alpha \in Q_1$. The space of all homomorphisms from M to N is denoted $\mathrm{Hom}_Q(M, N)$.

   (a) Consider the representations

$$M: \quad \mathbb{C}^2 \underset{(c,d)}{\overset{(a,b)}{\rightrightarrows}} \mathbb{C} \qquad\qquad N: \quad \mathbb{C} \underset{y}{\overset{x}{\rightrightarrows}} \mathbb{C}$$

   where $a, b, c, d, x, y \in \mathbb{C}$. If $(a, b) = (2, 1)$, $(c, d) = (6, 3)$, $x = 1$ and $y = 3$, construct a non-zero homomorphism $\mathbf{f} : M \to N$. Are there any homomorphisms $\mathbf{f} : M \to N$ when $(a, b) = (2, 2)$, $(c, d) = (6, 4)$, $x = 2$ and $y = 2$? In general, what conditions do a, b, c, d, x and y need to satisfy for $\mathrm{Hom}_Q(M, N)$ to be non-zero? What is the dimension of $\mathrm{Hom}_Q(M, N)$ in this case?

   (b) Recall that the representations of the quiver  are simply pairs $(\mathbb{C}^n, A)$,

   where $A : \mathbb{C}^n \to \mathbb{C}^n$ is an $n \times n$ matrix. If $M = (\mathbb{C}^n, A)$, show that $\mathrm{Hom}_Q(M, M) = \{B : \mathbb{C}^n \to \mathbb{C}^n \,|[A, B] = 0\}$, where $[A, B] := AB - BA$ is the *commutator* of A and B.

2. Let Q be a quiver. Recall that, for each $i \in Q_0$, we have defined the representation $E(i)$ of Q.

   (a) Show that the representation $E(i)$ is simple.

   (b) If Q has no oriented cycles, show that every simple representation equals $E(i)$ for some $i \in Q_0$.

(c) Consider the quiver $e_1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} e_2$. Show that the representation $\mathbb{C} \underset{2}{\overset{3}{\rightleftarrows}} \mathbb{C}$ is simple.

3. Let Q be the quiver

$$\begin{array}{ccccc} & & e_2 & & \\ & & \Big\downarrow{\scriptstyle \alpha} & & \\ e_1 & \xrightarrow{\ \beta\ } & e_5 & \xrightarrow{\ \gamma\ } & e_3 \\ & & \Big\downarrow{\scriptstyle \delta} & & \\ & & e_4 & & \end{array}$$

Write down the basis of paths for the path algebra $\mathbb{C}Q$. What is $\dim \mathbb{C}Q$?

4. Let Q be the quiver $e_1 \xrightarrow{\ \alpha\ } e_2 \xrightarrow{\ \beta\ } e_3$ and let

$$A = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \;\middle|\; a, b, c, d, e, f \in \mathbb{C} \right\}$$

be the algebra of upper triangular 3×3 matrices, where multiplication is just the usual matrix multiplication. Construct an explicit isomorphism of algebras $\mathbb{C}Q \xrightarrow{\sim} A$.

Exercises: Gabriel's Theorem

1. You'll notice that the positive definite Euler graphs are precisely the positive definite Coxeter graphs that are *simply laced* i.e. have at most one edge between any two vertices. Let $(-,-)_C$, resp. $(-,-)_E$, be the Coxeter form, resp. the Euler form, associated to a graph Γ.

   (a) Show that if Γ is simply laced then $(-,-)_E = 2(-,-)_C$.

   (b) If Γ is not simply laced, show that there is no $\lambda \in \mathbb{R}$ such that $(-,-)_E = \lambda(-,-)_C$.

   (c) Show that the symmetric matrix

   $$\begin{pmatrix} 2 & -m \\ -m & 2 \end{pmatrix}$$

   corresponding to the Euler graph $\bullet \overset{m}{\longrightarrow\!\!\!\!-} \bullet$ is positive definite if and only if $m = 1$. When is it positive semi-definite?

   (d) By considering the subgraphs $\bullet \overset{m}{\longrightarrow\!\!\!\!-} \bullet$ with $m > 1$ of Γ, show that a non-simply laced Euler graph is not positive definite.

   (e) Deduce Theorem 4.8 from Theorem 2.18.

2. Let $i \in Q_0$ be a sink. Show that $S_i^+(E(i)) = 0$.

3. Consider the representation M given by

$$\begin{array}{ccccc} & & \mathbb{C} & & \\ & & {\scriptstyle (1,0)}\uparrow & & \\ \mathbb{C} & \xleftarrow[(1,2)]{} & \mathbb{C}^2 & \xrightarrow{(0,1)} & \mathbb{C} \\ & & \downarrow{\scriptstyle (1,1)} & & \\ & & \mathbb{C} & & \end{array}$$

   If we label the central vertex by i, what is $S_i^-(M)$?

4. Let Q be the quiver $e_1 \xrightarrow{\alpha} e_2 \xleftarrow[\beta]{} e_3$ of type A_3. The corresponding root system, with reflection group $\mathfrak{S}_4$ was considered in the first exercise on reflection groups and root systems. Thus, the positive roots are

$$R^+ = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_3 - \epsilon_4, \epsilon_1 - \epsilon_3, \epsilon_2 - \epsilon_4, \epsilon_1 - \epsilon_4\},$$

which, under the identification $e_i \mapsto \epsilon_i - \epsilon_{i+1}$, corresponds to

$$R^+ = \{e_1, e_2, e_3, e_1 + e_2, e_2 + e_3, e_1 + e_2 + e_3\}.$$

For each of the above dimension vectors construct an explicit indecomposable representation of Q.