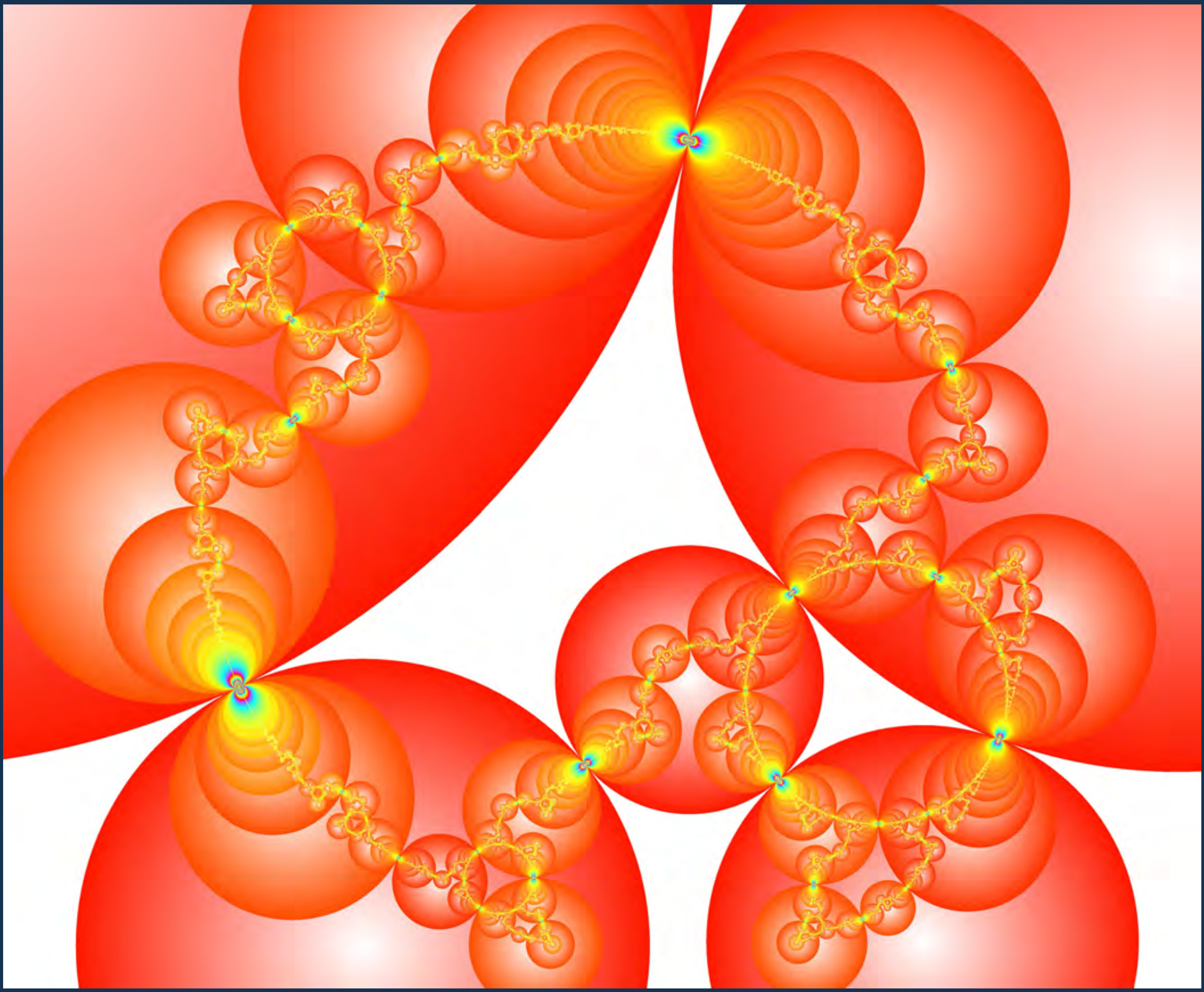
LONDON MATHEMATICAL SOCIETY
EST. 1865

NEWSLETTER

Issue: 500 - May 2022



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COVER IMAGE

A modern computer rendering of a limit set of a Kleinian group, based on an original hand drawing by Fricke and Klein, see Figure 6 in *Vorlesungen über die Theorie der automorphen Functionen,* Vol. 1, Leipzig, 1897. Created by David Wright, Oklahoma State University. Reproduced with permission.

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The Newsletter is freely available electronically at lms.ac.uk/publications/lms-newsletter.

MEMBERSHIP

Joining the LMS is a straightforward process. For membership details see lms.ac.uk/membership.

SUBMISSIONS

The Newsletter welcomes submissions of feature content, including mathematical articles, career related articles, and microtheses from members and non-members. Submission guidelines and LaTeX templates can be found at lms.ac.uk/publications/submit-to-the-lms-newsletter.

Feature content should be submitted to the editor-in-chief at newsletter.editor@lms.ac.uk.

News items should be sent to newsletter@lms.ac.uk.

Notices of events should be prepared using the template at lms.ac.uk/publications/lms-newsletter and sent to calendar@lms.ac.uk.

For advertising rates and guidelines see lms.ac.uk/publications/advertise-in-the-lms-newsletter.

CONTENTS

LMS NEWSLETTER 500: EDITORIAL

The *LMS Newsletter* celebrates its 500th issue. This gives occasion to reflect on the past and wonder what might lie ahead. What will the 1000th issue look like?

As so often, to have a glimpse of the future we must look for clues in the past, not just of the *Newsletter* but of the Society itself.

Founded in 1865, the Society is one of the oldest mathematical societies in the world (with its roots going back even further to 1717, the foundation of the Spitalfields Society). Despite its name, already in its early years, the membership of the Society represented a much broader geographic area than the capital and it quickly grew into a national society. Indeed, it became known internationally and served as a stimulus and model for many other mathematical societies around the world. I would pick two remarkable things from those early days that I feel we can be especially proud of. One is that the Society stood out among other societies, including the Royal Society, in the rigour that it applied to the refereeing process. Articles to be published in the *Proceedings of the LMS* received two independent written reports. The other is that it was open to women. Within its first twenty years the Society not only accepted several female members but also had papers presented by them.

There is of course much more to celebrate, in particular the names closely connected with the Society — De Morgan, Hardy, Pólya, Whitehead to name just a few remembered through our prizes and lecture series — and the mathematical research published by the Society. In March I was fortunate to have been able to attend the highly successful Society meeting in honour of Alan Turing. The event highlighted his paper 'On Computable Numbers, with an Application to the Entscheidungsproblem', published in the *Proceedings* in 1936 as one of the foundational papers in computer science and introducing the concept now known as a Turing machine.

The activities of the Society are not limited to research, and it is in the *Newsletter* where members can read more about these. Recent issues have reported on our *Levelling Up* programme, the Protect Pure Maths campaign, and now our reaction to and initiatives arising from the invasion of Ukraine by Russia. The Society recognises a community of mathematicians that needs to be nourished and supported.

Turning to the *Newsletter* itself, it started with a few black and white pages filled with news items, adverts and announcements of Society meetings before the electronic age. Much of this material is now most efficiently dispersed by our regular eUpdates, and the *Newsletter* was remodelled five years ago into a bi-monthly magazine packed with news items from around the wider and international community, policy round-ups, obituaries, reviews and short mathematical contributions.

Our Vice-President Cathy Hobbs mentioned in the last issue that Council is considering making the *Newsletter* an online-only publication to save costs and to support the Society's efforts to become greener. This, however, ought not change the nature of the *Newsletter* as a forum for members and as a source of information on Society activities, policy, and news.

Can we see 500 issues ahead from now, which at the present rate of publication would take us to the year 2105? Let me not predict the uncertain fate of the *Newsletter* in a world that seems to change its preferred mode of communication at faster and faster rates from email through blogs to Twitter and TikTok. Even the future nature of mathematical research lends itself to speculation. Will computer generated theorems and computer aided proofs be the norm? Can AI replace theory building mathematics in the style of Grothendieck? Whatever the answer, I am sure that mathematics will play a vital role in the next century, so I see no diminished need for a mathematical society like ours. I have every expectation that scientific rigor and inclusivity, the hallmarks of our early Society, will remain central for the advancement of mathematics. My belief in humanity also makes me think that the Society will see it fit to care for its members and the wider community. And of course, one way or another, some version of the *Newsletter* will be part of that future Society, to "provide a sense of identity, community, and connection for members".

Professor Ulrike Tillmann
LMS President

I am delighted to welcome you to the 500th issue of the LMS Newsletter. The Newsletter is a treasure of the British mathematical community and I am honoured to be the Editor-in-Chief for the jubilee issue. I am sure you will enjoy reading it as much as I did.

Dr Alina Vdovina
Editor-in-Chief, LMS Newsletter

The *LMS Newsletter:* A Look Back From Former Editors

The *LMS Newsletter* through the years, from the perspective of some former Editors-in-Chief.

On the occasion of the 500th issue of the *LMS Newsletter*, we thought it would be interesting to reflect on where we have come from through the eyes of the *Newsletter* Editors over the years. Younger readers may be surprised to know that the *Newsletter* started as a single sheet of A5 paper, sent out monthly. Through the years this evolved into a many-paged A5 booklet format, initially black and white but moving to colour in the early 2000s, and then the thorough re-launch in September 2017 to arrive at the format you read today. Nowadays the *Newsletter* is published online as well as in paper format, and a larger number of readers than ever before can access it freely via the LMS webpages. The content has evolved too, from being predominantly news about the Society and events it organised to having a much wider scope including book reviews, relevant events hosted by others, European news and now, mathematical content.

We asked a number of former editors of the *Newsletter* to reflect on their time in the role and what made it memorable. Affection for the *Newsletter,* in whatever format it has taken, comes over in their brief reflections. We hope you find this delve into history interesting.

Susan Oakes, Cathy Hobbs

David Brannan: My Role in The *LMS Newsletter* (1973–76)

When I became LMS Secretary in 1971, LMS had been conducting a review (in which Samuel James ("S.J.") Taylor of Westfield College was a key player) about 'modernising' the LMS — extending its activities, launching the LMS *Bulletin*, attracting a wider membership, increasing attendance at meetings, and so on. The review included the establishment of an *LMS Newsletter* (a mini-version of the AMS Notices) in place of the former monthly LMS postcard that was posted on 1st of the month to advertise that month's LMS meeting.

We started in September 1973, so that the LMS could advertise the Royal Society's offer of funding to support people who wished to attend the 1974 ICM in Vancouver. I negotiated with Messrs C.F. Hodgson (who had been LMS printers since 1865, but sadly closed in the mid-1980s) to print it. Pat Hodgson wanted to know what the format should be. We had rather little content, so it would not take much to fill a page of whatever size! I had recently converted (at Simon Fairthorne's suggestion) from imperial paper to metric, so I had sheets of A4 loose leaf paper in my drawer. I pulled out one sheet, folded it over, and mailed it to Pat saying "something like this size, please". So the first few issues were a single A5 sheet of paper, printed two-sided, published every month apart from August as that was a 'holiday' month. (Later it expanded to become a folded A4 sheet, then multi-sheeted and stapled.) I also made an agreement with AMS that we'd get advance proofs by air mail of the AMS Notices so we could reproduce freely anything from it that we wanted so long as we assigned the proper attribution. I edited issues 1–3.

Then in November 1973 John Britton (who had recently been appointed to Queen Elizabeth College where I worked) was elected as LMS Meetings and Membership Secretary and I became LMS Council and General Secretary (I had organised the division in two of my LMS post, with Council's approval, due to the heavy workload — an idea I had digging my allotment in Ealing in summer 1972!). Technically we were joint Editors, but John did the practical side and I simply supplied some content, dealt with queries, and assisted John.

Finally in 1976 John stood down as Secretary and was succeeded by Barry Johnson, who became the new *Newsletter* Editor in our place.

Norman Biggs: *Newsletter* Recollections (1980–83)

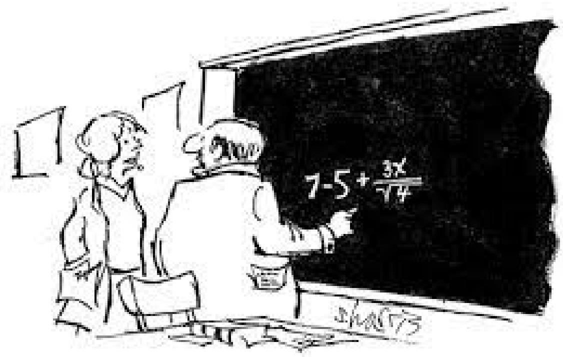
I joined the LMS Council in 1979 and was Editor of the *Newsletter* from October 1980 to December 1983 (Numbers 72–104). At that time it was an austere and serious publication, but I recall two events that sparked some liveliness.

The International Congress of Mathematicians was due to take place in Poland in 1982, but it had to be postponed because of the abnormal political situation in that country. Eventually the IMU decided that the Congress should go ahead in 1983, but there was much discussion in the community about the ethics of visiting Poland in the unusual circumstances. Frank Adams volunteered to give his view, and an open letter from him was circulated with *Newsletter* 94. It was clear that opinions differed, so I asked a few people who had close contacts with Polish mathematicians to comment. The views of Keith Devlin, Wilfrid Hodges, and Garth Dales were circulated with Newsletter 95.

The other event was the wedding of Charles and Diana in 1981. This would not normally have featured in the sober pages of the *Newsletter*, but I could not resist the opportunity for mischief of the April Fool kind. An item appeared in *Newsletter* 78 stating (wrongly) that the LMS was considering presenting a suitable gift to the happy couple, and inviting members to write to the President with suggestions by 1 April. Fortunately the President (Barry Johnson) appreciated the joke, although he had to write a few soothing letters to outraged members who had not.

David Chillingworth: *LMS Newsletter* Editor 2002–09

During my time as Editor the Newsletter transformed itself from black and white to multicolour format, involving several technical meetings with designers on layout and pantone shades: not really my comfort zone. In keeping with the smarter, brighter appearance I tried to introduce some levity with a monthly Sidney Harris cartoon (example below), although unfortunately their ironical mathematical humour sometimes misfired. Although the *Newsletter* is the official organ of the Society, I felt that it ought to be more of a forum for members to express their views, especially during the debate on the major issue of a possible merger with the IMA. Although there were of course official progress reports on this from time to time, and invitations for members to express their views, it did not seem to me that enough efforts were made to alert the readership to what was really happening: did they really want Council and the LMS staff to spend such a huge amount of time and effort on a project that might come to nothing, and did they realise if they didn't pay attention they could wake up one morning and find the LMS had disappeared? Divergence from the official track was, however, not permitted.



"YOU CAN'T IMAGINE HOW TIGHT OUR BUDGET IS. WE CAN ONLY WORK WITH SINGLE-DIGIT NUMBERS."

Tony Mann: *LMS Newsletter* Editor 2010–17

I had the privilege of being tenth editor of the *LMS Newsletter* from the June 2010 issue, when I took over from David Chillingworth, until the July 2017 issue, the final issue in the old format before the *Newsletter* evolved into the more extended magazine we now have. I was fortunate to benefit from David's guidance and to have excellent support from Reports Editors Stephen Huggett, Robert Wilson and Iain Stewart, and Reviews Editors Colva Roney-Dougal and David Singerman. But my grand title of 'General Editor' was of course a complete misnomer — all the work was done by the 'Administrative Editor', Susan Oakes, who produced an immaculate Newsletter eleven months of every year (there being no issue in August).

My memories of my time as editor include our annual team meeting, at which arguably the most important decision to be taken was the colour to be used as the basis of the Newsletter for the following year, with inspiration often being taken from the fruit bowl in front of us.

There were controversies. If I remember correctly, the biggest postbag received during my time as editor came when the *Newsletter* changed to a glossy paper (for reasons of economy). No fewer than two members wrote to us about this change! If my memory is to be trusted, the correspondents were equally divided for and against the new paper, with the ability to read the *Newsletter* in one's bath being cited as a positive benefit of the change.

I thoroughly enjoyed my time as General Editor, and I am very grateful to everyone named above for their support, and especially to Susan and the production team, for making my job so easy. And I'd also like to congratulate the subsequent editors and their teams, who have taken the *Newsletter* to a new level.

Iain Moffatt: *LMS Newsletter* Editor 2017-19

As Editor, I took the *Newsletter* through its most recent redesign, changing it from its previous A5 format to its current layout, and introducing mathematical feature articles. I interviewed for the editorship during December 2016. The idea for a redesign had already been circulating in the LMS, and in advance of my appointment the LMS had agreed that the redesigned Newsletter should include feature articles, have an Editorial Board, be produced in LaTeX, and have a 'clean' design and particular physical dimensions. Otherwise, they allowed me a blank page to make things work. But they also set the target of September 2017 for the first issue of the new format, which, because of printing times, gave a very ambitious six months or so to bring everything together!

First up was creating the initial Editorial Board, and deciding on the broad content and visual design. There is a proper order to do this in, but the tight timeline meant ignoring it. So that December and January was a blur of phone calls to potential Editorial Board members, conversations with designers and LMS staff, and cutting up old issues of the *Newsletter* with scissors to glue together new ones.

Once the layout design was decided, and armed with some knowledge of how much would fit on a page, I could start reaching out to mathematicians for feature articles. I am extremely grateful to everyone who contributed articles, but especially to those who took the leap in the dark to write for the first few issues. Then with a small Editorial Board we could start refining the content, and, with LMS staff, the production processes. As the printer's deadline for the first issue closed in, it was the switch to producing the *Newsletter* in LaTeX that nearly derailed things. The new class file arrived close to the deadline, and there was a last-minute scramble of hacks and workarounds to get the layout we wanted. But we made it thanks to the efforts of everyone involved.

With the first issue of the redesigned *Newsletter* out, our focus changed to regular production and to incremental improvements. Since stepping down as editor in December 2019 I've enjoyed seeing the redesigned *Newsletter* growing and improving under its subsequent editors and editorial boards.

LMS Newsletter Editors 1973–present

1973	D.A. Brannan
1973–76:	D.A. Brannan and J.L. Britton
1976–80:	B.E. Johnson
1980–83:	N.L. Biggs
1984–87:	C. Kosniowski
1988–02:	S.M. Oakes (joint with A.R. Pears and D.J.H. Garling)
1993–98:	A.R. Pears
1998–02:	D.J.H. Garling
2002–09:	D.R.J. Chillingworth
2010–17:	A.J.S. Mann
2017–19:	I. Moffatt
2020–21:	E. Lingham
2021–present:	A. Vdovina

Ukraine

From the LMS President

At the time of writing in mid-March 2022, a catastrophe continues to unfold following the unprovoked invasion of Ukraine by Russian government forces. The consequences for the people of Ukraine, their homes, their livelihoods and their institutions are devastating. It is in equal parts heartbreaking and infuriating to witness this destruction and the repression and victimisation of the brave Russians who dare to oppose the actions of their government. The Society stands in solidarity with everyone affected by this horrible violence.

In these circumstances, holding the 2022 International Congress of Mathematicians (ICM) in St Petersburg in July was clearly going against the ICM's aim of promoting international cooperation in mathematics. The Society announced in February that, as the Adhering Organisation to the International Mathematical Union (IMU) for the UK, it would not send delegates to an ICM in St Petersburg. The IMU decided shortly thereafter that the ICM 2022 would take place as a fully virtual event, hosted outside Russia.

As I write, the Society is reviewing all its activities in light of the invasion and UK government sanctions on Russia. The Society champions the sharing of mathematical knowledge and the development of an international community of mathematicians. But the unprovoked aggression of the Russian government renders continued joint working with Russian institutions impossible.

In relation to its academic publishing activities, the Society must consider the three Russian mathematical journals, the English translation of which the Society co-publishes. These journals are: *Russian Mathematical Surveys; Izvestiya: Mathematics;* and *Sbornik: Mathematics*. The Society must also consider the future of the journal exchanges it operates with Russian institutions, and the approach that it will take to papers submitted to all its journals for publication authored or co-authored by individuals affiliated with or funded by Russian institutions.

The Society is also seeking to work with other UK and international mathematical societies to assist mathematicians who seek refuge in the UK. We want to help them reconnect with their mathematical community and restart their careers, and to signpost them to sources of practical and emotional support. It is vital that our mathematical colleagues know that they are not alone in facing the disastrous events that have overtaken them. We know that they would do the same for us.

Professor Ulrike Tillmann

The Only Constant Is Change...

A farewell from Caroline Wallace, outgoing Executive Secretary

As Members will know, I have made the difficult decision to depart from the Society at the end of May. This follows a change in my personal circumstances that has opened up some new and unexpected opportunities for me. I feel very lucky indeed to have had the privilege of leading the Society's staff team, and to have contributed to the Society's vital work supporting research mathematicians, advancing mathematical knowledge and promoting mathematics. It has been a genuine delight to work with such dedicated Trustees, Committee members and staff. And mathematics is, and always will be, close to my heart in terms of its codification of reasoning, logical thinking and scholarly imagination - some of the most inspiring characteristics of our species.

My time at the Society has been marked (but not marred) by some truly world-changing events, not least the covid-19 pandemic. These challenging times have helped me find new reservoirs of resilience and pragmatism that I didn't know I had. As others have articulated much more clearly than me, we can't choose the times we live in; "all we have to decide is what to do with the time that is given to us". And I couldn't have asked for a better group of people with whom to weather the storm — thank you.

I wish the Society and my successor all the very best for the future as you continue to support the mathematics community and share the profound advances and insights of mathematics with the world.

Caroline Wallace
LMS Executive Secretary

Environmental Sustainability at the LMS

The LMS has recently begun to work through a plan that addresses the environmental sustainability of the Society, in terms of the activities it undertakes, the activities it sponsors and the physical premises occupied by the Society. The climate emergency will affect everyone in all parts of the world and Council has agreed that it is critical that the Society joins in the efforts to take action now, however difficult or overwhelming this may seem.

The LMS is organising its efforts under three general headings.

(1) It will address its own environmental sustainability as an organisation.

(2) It will examine the steps it can take to reduce the harmful effects to the environment that its activities can cause, while maintaining their high quality and overall effectiveness.

(3) It will try to build a community of action within the mathematical sciences community, raising awareness of environmental issues in our working lives and indicating ideas and good practice about how to address these, as well as encouraging and celebrating mathematical research that contributes to our understanding of the climate and the environment.

We will be reporting regularly on the specific steps that we are taking as each is rolled out. To begin, Andrew Dorward, the Head of Buildings and Conferences, explains below some of the steps we have taken as an organisation.

Iain Gordon
LMS Vice-President and
Council Environment Champion

An initial focus has been on De Morgan House, and a programme of energy saving initiatives is currently underway. The De Morgan House Buildings Team has completed the upgrade to energy-efficient LED lighting in most of the communal areas; the staircase alone has 96 candle lightbulbs, which have all been changed to LED. Room by room, old fluorescent lighting tubes are being removed and replaced with energy efficient LED panels.

In 2020 new energy-efficient 'A' rated boilers were installed at De Morgan House which now provide all the heating needs of the building. A policy is also in place that, prior to purchase, all new equipment is reviewed to assess its energy efficiency, which is considered when deciding what to purchase.

The use of electricity and gas at De Morgan House is also actively managed by the Head of Buildings and Conferences, with heating timers installed, and radiators manually turned off when rooms in the building are not in use. This has become ever more important owing to new working patterns and ensures empty rooms are not heated unnecessarily.

De Morgan House electricity is provided from fully renewable sources, via its energy provider. De Morgan House achieved a rating of 'C' in a recent energy performance survey, noting that a less energy-efficient 'D' rating is typical for similar buildings. The buildings team will continue to try to improve on this rating with further energy efficient improvements planned for 2022 and 2023. Energy efficiency is now a key priority when setting the annual budget for the running of De Morgan House, as we strive to reduce our carbon footprint as much as possible.

In addition, to reduce the known environmental impact from meat consumption, the De Morgan House Conference Business is also looking to move to more vegetarian and vegan catering options. The aim is to provide non-meat catering as standard for all events.

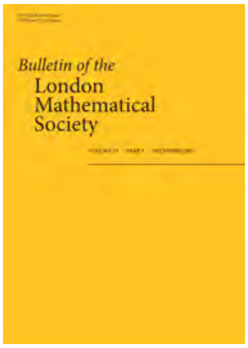
The move towards sustainability is also guiding the Society's choice of investments. The majority of the assets are held in funds administered by Cazenove Capital, part of the Schroders Group, and their performance is assessed annually. At the last meeting with our fund managers, there was a special focus on sustainable investment, and we were satisfied that Schroders is well placed to help us in this respect. Following subsequent discussions at Council and the Investment Sub-Committee, it was agreed to transfer initially 20% of the Society's investment portfolio from Cazenove's Charity Multi-Asset Fund to its Responsible Multi-Asset Fund (RMAF). The RMAF has enhanced Environment, Social and Governance (ESG) indices, reducing risk in all these areas. It excludes (for example) fossil fuels and armaments, and its investors will want to know the extent to which its activity is supporting human rights. The Society is continuing the move to sustainable investment at a steady rate, yet acknowledges that the performance (and the definition) of ESG funds will come under scrutiny in the changing world.

Andrew Dorward
Head of Buildings and Conferences, De Morgan House

Legacy for the Society

The London Mathematical Society acknowledges a generous bequest from the late Professor Stephen J. Pride FRSE, formerly of the University of Glasgow towards its work of advancing the development of mathematical knowledge. Professor Pride was an LMS member, and served as a member of the LMS Editorial Board for 10 years. The Society is extremely grateful to all Members who wish to remember the Society in their wills.

Recently Published Survey Articles in the *Bulletin*



The *Bulletin of the London Mathematical Society* continues to publish definitive survey articles, something it has done since its very first issue in 1969. Three survey articles have been published so far in 2022, all with open access.

- 'Some recent advances in topological Hochschild homology'; Akhil Mathew; doi.org/10.1112/blms.12558
- 'A user's guide to the local arithmetic of hyperelliptic curves'; Alex Best, Alex Betts, Matthew Bisatt et al.; doi.org/10.1112/blms.12604
- 'Non-commutative amoebas; Grigory Mikhalkin and Mikhail Shkolnikov; doi.org/10.1112/blms.12622

The Managing Editors of the Bulletin, Andrey Lazarev and Sibylle Schroll, welcome the submission of survey articles through the journal's EditFlow submission site: bit.ly/35qBkMY.

LMS Publications Committee Strategic Retreat

The LMS Publications Committee will be holding a strategic retreat in June to discuss long-term goals and how to achieve them. Broadly, there are two questions: "What should the LMS be publishing in 10 or 15 years' time?" and "How do we get there from here?" The first question is to be answered principally on mathematical grounds, and will consider our journals' identity, content and coverage independent of considerations of finance, contracts and the transition to open access, while the second question considers the latter issues and journal development more broadly. I would welcome thoughts from LMS members on these or related issues.

Niall MacKay
LMS Publications Secretary
Email: publications.secretary@lms.ac.uk

Cecil King Travel Scholarships

Prachi Sahjwani and Valentin Kunz

The Cecil King Travel Scholarships are funded by the Cecil King Memorial Foundation. They are designed to advance the educational and vocational training of young people who display outstanding potential in the sphere of mathematics.

Each year the LMS administers two £6,000 travel awards to support early career mathematicians for a period of study or research abroad, typically three months. One scholarship is usually awarded to a mathematician in any area of mathematics and one to a mathematician whose research is applied in a discipline other than mathematics.

The awards are competitive and are based on written proposals describing the intended programme of study or research and the benefits to be gained from such a visit. The Early Career Research Committee of the LMS has a panel of specialists representing a broad spectrum of mathematics who are tasked with analysing the quality and standard of the applications, along with the scientific merit of the research project.

Applicants should be mathematicians in the United Kingdom or the Republic of Ireland who are registered for a doctoral degree or have completed one within 12 months of the closing date for applications. The LMS encourages applications from women, disabled, Black, Asian and Minority Ethnic candidates, as these groups are under-represented in the field of mathematics in both the United Kingdom and the Republic of Ireland.

The application deadline for the 2023 programme is 15 November 2022. The application form will be available on the LMS website later in 2022. Shortlisted applicants will be invited to an interview during which they will be expected to make a short presentation on their proposal. Interviews will take place in January 2023.

Following this year's round of applications, we are delighted to announce that the 2022 Cecil King Travel Scholarships were awarded to Prachi Sahjwani, PhD student at the University of Cardiff, and to Valentin Kunz, PhD student at the University of Manchester.

Prachi Sahjwani proposed a study programme at the University of Freiburg, Germany. Her main research interests are in Geometric Analysis, specifically applying analysis and geometric flow techniques to solve problems in geometry.

Valentin Kunz is due to visit Harvard University, USA and his research will be on invariant Cauchy-Riemann mappings between spheres, which have many surprising properties.

It is hoped that the Cecil King Travel Scholarship will be of great benefit to both recipients and will support the development of their academic careers. More information about the Scholarships is available at tinyurl.com/2p92jzpz.

Valeriya Kolesnykova
Accounts, Fellowships & Membership Assistant

Forthcoming LMS Events

The following events will take place in the next few months:

LMS Meeting and Hirst Lecture:
6 May, De Morgan House, London

Northern Regional Meeting:
24 May, Leeds

LMS/Gresham Lecture: The Maths of Gyroscopes and Boomerangs:
25 May, Museum of London and online

LMS Meeting at the BMC:
7 June, King's College, London

LMS Meeting and Aitken Lecture:
1 July, BMA House, London

LMS–INI–Bath Symposium: K-Theory and Representation Theory:
18-22 July, University of Bath

LMS–Bath Symposium: New Directions in Water Waves: 18-29 July

LMS-Bath Symposium: Combinatorial Algebraic Geometry: 1-5 August 2022

A full listing of forthcoming LMS events can be found on page 70.

OTHER NEWS

Isaac Newton Institute Solidarity Initiatives

To help our fellow mathematicians who have become refugees, the INI has initiated a Solidarity list (tinyurl.com/2p8fdhh6) to connect those who need help and those who are able to offer it. Refugee researchers in the mathematical sciences can also be offered six months of accommodation and subsistence through our special INI Programme Solidarity for mathematicians (newton.ac.uk/event/slm). An accompanying satellite programme is being set up which will allow researchers to take their six months stipends also to other universities in the UK. For more information write to Ulrike Tillmann (oms@newton.ac.uk).

Professor Sir John Ball Elected President of RSE

Former LMS President Professor Sir John Ball has been elected as President of the Royal Society of Edinburgh. Sir John succeeds Professor Dame Jocelyn Bell Burnell, who commented:

"I am delighted to hand over to Professor Sir John Ball as the new President of the Royal Society of Edinburgh. During my Presidency, I set out RSE's desire to play an even greater role in the wellbeing of Scotland and all its people. Returning to RSE as interim President in May, I have been very pleased to see the progress that has been made over recent years through initiatives such as the Women in Science exhibition and, most recently, the Post-covid-19 Futures Commission. I look forward to seeing RSE's continued development and evolution under Sir John's Presidency."

LMS Atiyah Fellow 2021–22

Professor Maciej Dunajski, the LMS Atiyah Fellow 2021–22, has taken up his Fellowship to make two short visits to the Center for Advanced Mathematical Sciences in the American University of Beirut. In April he gave a mini-course on Calculus of Variations and Geometry aimed jointly at students in the AUB and at the Lebanese University. He also gave a public Colloquium on geometry, based on his popular book *Geometry, A Very Short Introduction*, published by OUP. Professor Dunajski hopes to make his second visit to CAMS in September this year.

Abel Prize Laureate 2022

Photo credit: John Griffin/Stony Brook University/Abel Prize

Professor Dennis P. Sullivan, an Honorary Member of the London Mathematical Society, has been awarded the 2022 Abel Prize. Dennis Sullivan is best known for his groundbreaking work on topology and dynamical systems, two fields in which ideas about geometric structure play a central role.

Since it was first awarded in 2003, the Abel Prize has come to represent a lifetime achievement award. The past 24 Abel laureates carried out most of their renowned work in the mid-to-late twentieth century. "It's nice to be included in such an illustrious list", said Sullivan, who has appointments at both Stony Brook University in Long Island, New York and at the City University of New York. Read Sullivan's full biography at https://tinyurl.com/28bzy5z8.

The Abel Prize Week will take place in Oslo from 23–25 May, and the award ceremony for the 2022 laureate will take place on 24 May at the University Aula, Oslo (16:00 – 16.40 local time). It will be a physical event, for which registration is required (details at tinyurl.com/5dkjw46a). The event will also be streamed live on The Abel Prize YouTube Channel.

OPPORTUNITIES

LMS Grant Schemes

The next closing date for research grant applications (Schemes 1,2,4,5,6 and AMMSI) is 15 May 2022. Applications are invited for the following grants to be considered by the Research Grants Committee at its June 2022 meeting. Applicants for LMS Grants should be mathematicians based in the UK, the Isle of Man or the Channel Islands. For grants to support conferences/workshops, the event must be held in the UK, the Isle of Man or the Channel Islands:

Conferences (Scheme 1)

Grants of up to £5,500 are available to provide partial support for conferences. This includes travel, accommodation and subsistence expenses for principal speakers, UK-based research students, participants from Scheme 5 countries and Caring Costs for attendees who have dependents.

Visits to the UK (Scheme 2)

Grants of up to £1,500 are available to provide partial support for a visitor who will give lectures in at least three separate institutions. Awards are made to the host towards the travel, accommodation and subsistence costs of the visitor. Potential applicants should note that the host institutions are expected to contribute to the costs of the visitor. In addition, the Society allows a further amount (of up to £200) to cover Caring Costs for those who have dependents.

Research in Pairs (Scheme 4)

For those mathematicians inviting a collaborator, grants of up to £1,200 are available to support a visit for collaborative research either by the grant holder to another institution abroad, or by a named mathematician from abroad to the home base of the grant holder. For those mathematicians collaborating with another UK-based mathematician, grants of up to £600 are available to support a visit for collaborative research either by the grant holder to another institution or by a named mathematician to the home base of the grant holder. In addition, the Society allows a further amount (of up to £200) to cover Caring Costs for those who have dependents.

Collaborations with Developing Countries (Scheme 5)

For those mathematicians inviting a collaborator to the UK, grants of up to £3,000 are available to support a visit for collaborative research, by a named mathematician from a country in which mathematics could be considered to be in a disadvantaged position, to the home base of the grant holder. For those mathematicians going to their collaborator's institution, grants of up to £2,000 are available to support a visit for collaborative research by the grant holder to a country in which mathematics could be considered to be in a disadvantaged position. Applicants will be expected to explain in their application why the proposed country fits the circumstances considered eligible for Scheme 5 funding. In addition, the Society allows a further amount (of up to £200) to cover Caring Costs for those who have dependents. Contact the Grants team if you are unsure whether the proposed country is eligible, or check the IMU's Commission for Developing Countries definition of developing countries (tinyurl.com/y9dw364o).

Research Workshop Grants (Scheme 6)

Grants of up to £10,000 are available to provide support for Research Workshops. Research Workshops should be an opportunity for a small group of active researchers to work together for a concentrated period on a specialised topic. Applications for Research Workshop Grants can be made at any time but should normally be submitted at least six months before the proposed workshop.

Computer Science Small Grants (Scheme 7)

Grants of up to £750 are available to support a visit for collaborative research at the interface of Mathematics and Computer Science either by the grant holder to another institution within the UK or abroad, or by a named mathematician from within the UK or abroad to the home base of the grant holder. Priority will be given to applications from early career researchers.

African Mathematics Millennium Science Initiative (AMMSI)

Grants of up to £2,000 are available to support the attendance of postgraduate students at conferences in Africa organised or supported by AMMSI. Application forms for LMS-AMMSI grants are available at ammsi.or.ke.

ECR Travel Grants

The next closing date for applications for Early Career Research Travel Grants is 13 May 2022. Applications will be considered by the Early Career Research Committee at its June 2022 meeting. Grants of up £500 are available to provide partial travel and/or accommodation support for UK-based early career researchers to attend conferences or undertake research visits either in the UK or overseas.

More Information

For full details of these grant schemes, and to find information on how to submit application forms, visit the LMS website: lms.ac.uk/content/research-grants. Queries regarding applications can be addressed to the Grants Administrator Lucy Covington (020 7927 0807, grants@lms.ac.uk), who will be pleased to discuss proposals informally with potential applicants and give advice on the submission of an application.

LMS Research Schools 2022

The following Research Schools will take place in 2022:

Methods of Random Matrix Theory & Applications
Reading, 16–20 May 2022
Main Lecturers: Estelle Basor (American Institute of Mathematics); Tamara Grava (University of Bristol and SISSA); Alexander Its (Indiana University-Purdue University Indianapolis).

Point Configurations: Deformations and Rigidity
UCL, 27 June – 1 July 2022
Main Lecturers: Keith Ball (Warwick University); Werner Krauth (École Normale Supérieure); Franz Merkl (Ludwig Maximillian University of Munich); Sylvia Serfaty (New York University).

Bicategories, Categorification and Quantum Theory
Leeds, 11–15 July 2022
Main Lecturers: Richard Garner (Macquarie University, Sydney); Marco Mackaaij (University of Algarve); Chris Huenen (Radboud University Nijmegen); Sonia Natale (Universidad Nacional de Còdoba).

Unimod 2022
Leeds, 18–22 July 2022
Main Lecturers: Amador Martin-Pizarro (Albert Ludwig Universität in Freiburg); Assaf Hasson (Ben Gurion University of the Negev); Zoé Chatzidakis (École Normale Supérieure)

Elliptic Curves
Baskerville Hall, Hay-on-Wye, 8–12 August 2022
Main Lecturers: Alina Cojocaru (University of Illinois Chicago); Jennifer Park (Ohio State University); Chao Li (Colombia University); Joseph Silverman (Brown University); Jan Vonk (University of Leiden); Christian Wuthrich (University of Nottingham).

For more information and links to the events visit: https://www.lms.ac.uk/events/lms-research-schools.

EUROPEAN MATHEMATICAL SOCIETY NEWS

Ukraine

Following the Russian invasion of Ukraine, the EMS Executive Committee issued a statement expressing solidarity with the people of Ukraine, and our colleagues there. The committee also called for academic cooperation with state institutions and business enterprises in Russia to be frozen.

Russian Mathematicians

The following is an English translation of an open letter to the President of the Russian Federation from Russian mathematicians against the war in Ukraine. The original is hosted on the webpage of the Russian scientific publication *Troitsky Variant*.

Mathematicians Against War

President of the Russian Federation
V. V. Putin
Mr President!

We, mathematicians working in the Russian Federation, strongly protest against the military invasion of the territory of Ukraine launched by the Russian army on February 24, 2022.

The standard of living in a country and its position in the world are largely determined by the level of its science. Scientists around the world are working on problems that have no national and territorial restrictions, for the well-being of all mankind; international cooperation, lack of borders for the dissemination of knowledge and humanistic values are the foundation on which science is built. Our long-standing efforts to strengthen the reputation of Russia as a leading mathematical centre have been completely devalued in consequence of unprovoked military aggression initiated by our country.

Mathematics has always been one of the few areas of fundamental science in which Russia has maintained a leading world position. As confirmation of this, Russia was supposed to host the most prestigious mathematical conference in the world, the International Congress of Mathematicians, in the summer of 2022. The International Mathematical Union cancelled this decision in connection with the Russian attack on Ukraine. In a situation where our country has become a military aggressor and, as a result, a rogue state, Russia's leading positions in world mathematics will be irretrievably lost.

In the orders of the President of 4 December 2020, mathematics was named a priority area for the development of the Russian Federation; goals were identified both in the field of fundamental science and in education. These goals, of course, cannot be achieved in the current conditions, when the lives of our closest colleagues – scientists in Ukraine, with whom we have been connected by many years of successful joint work, are daily exposed to danger, the source of which is the Russian army, and when Russia has found itself in international isolation, without the possibility of intensive scientific exchange and cooperation with scientists from other countries.

We are convinced that no geopolitical interests can justify casualties and bloodshed. War will only make the country lose its future, for which we work.

We demand an immediate cessation of military actions and the withdrawal of Russian troops from the territory of Ukraine.

ICM 2022 online

The International Mathematical Union has announced that the International Congress of Mathematicians July 2022 will be held fully (and freely) online, instead of in Russia. The EMS is an affiliate member of the IMU, and the EMS Executive Committee welcomes this decision.

EMS 30th Anniversary Celebration

The European Mathematical Society 30th Anniversary Meeting, originally scheduled for 2020 but postponed because of the pandemic, was held as a special event on 31 March at ICMS in Edinburgh.

Institut des Hautes Études Scientifiques

Events organized at IHES (France) during the coming months include:

- Conference: 100 years of the Ising model: 30 May – 3 June
- Hadamard Lectures by Carinna Ulcigrai: 1, 2, 3 June at the Institut de Mathématiques d'Orsay; 7, 8, 9 June at IHES
- Summer school on the Langlands programme: 11–29 July

For further details visit ihes.fr/en.

Institut Henri Poincaré

The IHP has been promoting and hosting worldwide scientific interdisciplinarity and interaction at the heart of Paris since 1928. From 5 September to 9 December 2022 the institute will host a three-month program entitled Geometry and Statistics in Data Sciences. Registration is free but mandatory.

European Society for Mathematical and Theoretical Biology

The European Conference on Mathematical and Theoretical Biology (ECMTB) 2022 will take place 19–23 September in Heidelberg, Germany. It is a joint event organised by the ESMTB and the SMB (Society for Mathematical Biology).

EMS Magazine

The latest version of the EMS Magazine is available to read. Highlights include a message regarding predatory publishing practices (Thierry Bouche, Stefan Jackowski, Betül Tanbay), T. Tao and the Syracuse conjecture (J.-P. Allouche) and a report on the EWM Panel Discussion on Gender Balance in Mathematics at the European Congress of Mathematics (Eugénie Hunsicker), with much more besides!

French Mathematical Society (SMF)

In 2022 the SMF will be 150 years old. Celebration days will take place on 16, 17 and 18 March 2022. The programme can be found online at https://smf.emath.fr/150-ans-smf.

Heidelberg Laureate Forum (HLF)

Calling Outstanding Young Researchers — apply now for the 9th Heidelberg Laureate Forum, September 18–23, 2022! Young researchers in computer science and mathematics from all over the world are encouraged to apply for one of the 200 coveted spots. The HLF offers all accepted young researchers the opportunity to personally interact with the laureates of the most prestigious prizes in the fields of mathematics and computer science: the recipients of the Abel Prize, the ACM A.M. Turing Award, the ACM Prize in Computing, the Fields Medal, and the Nevanlinna Prize engage in a cross-generational scientific dialogue with young researchers in Heidelberg, Germany. More information at heidelberg-laureate-forum.org.

European Women in Mathematics (EWM)

The EWM's General Meeting 2022 (EWM GM 2022) will be held 22–26 August 2022, in Aalto University, Espoo, Finland. The European Mathematical Society lecturer will be Claire Voisin (CNRS, Institut de mathématiques de Jussieu-Paris, France). The plenary speakers will be: Kathrin Bringmann (Cologne), Maria Bruna (Cambridge), Nina Holden (ETH Zürich and Courant Institute of Mathematical Sciences), Kaisa Miettinen (Jyväskylä), Ilaria Perugia (Vienna). The Gender speaker will be Jessica Wade (Imperial College London).

Deadlines:

- February 28, 2022: submission of minisymposium proposals
- March 31, 2022: application for travel/accommodation grants
- May 31, 2022: submission of abstracts

The conference is supported by Aalto Science Institute, Federation of Finnish Learned Societies and Foundation Compositio. EWM is supportedby G-Research and Smith Institute.

The IDM Governing Board hopes to increase significantly the number of schools celebrating the IDM, either in the classroom, or through a larger event. One way to interpret the 2022 IDM theme ”Mathematics Unites” is the fact the whole planet shares the same mathematical language. To illustrate this through a school activity, some proofs without words have been added to the material proposed for classroom activities. The IDM Governing Board needs your help for reaching the schools and school networks of your country: invite them to celebrate and to joint the IDM community be registering to the IDM newsletter.

EMS News prepared by David Chillingworth
LMS/EMS Correspondent

Note: items included in the European Mathematical Society News represent news from the EMS and are not necessarily endorsed by the Editorial Board or the LMS.

MATHEMATICS POLICY DIGEST

LMS Welcomes R&D Tax Relief Reform

The Society is delighted that in the Spring Statement of 23 March, the Chancellor Rishi Sunak announced that the definition of research and development (R&D) for tax reliefs will be expanded by clarifying that pure mathematics is a qualifying cost. This will support the growing volume of R&D underpinned by mathematical advances.

In May 2021, the CMS (supported by the LMS) wrote to Her Majesty's Treasury regarding the fact that work in pure mathematics was specifically excluded from R&D tax relief, which supports companies that work on innovative projects in science and technology. This followed an earlier proposal in the Bond Review that mathematical sciences be encompassed in the HMRC definition of science and technology and included in the tax credit scheme. The Society was pleased to see the impact of this work in the changes announced by the Chancellor.

National Academy: Next Steps

The task and finish group which was convened to analyse responses to the recent consultations on the National Academy for Mathematics and Connected Centres Network has now outlined a proposal for next steps. At the time of writing a document summarising that proposal has been sent to CMS member societies with a request for feedback.

Multiply Programme

Multiply is a new £559 million government programme to help transform the lives of adults across the UK, by improving their functional numeracy skills through free personal traininng, digital training and flexible courses. Through this scheme, local areas are encouraged to invest in meaningful participation that boosts people's ability to use mathematics in their daily life and work. For local allocations in England, Multiply has invited the Greater London Authority, Mayoral Combined Authorities and upper tier authorities outside of these areas to develop investment plans against a national menu of interventions. For local allocations in Scotland, Wales and Northern Ireland, Multiply will be delivered alongside wider programmes of UK Shared Prosperity Fund (UKSPF) activity; further details can be found in the UKSPF prospectus at bit.ly/38OscTy. These programmes aim to start in the 2022/23 academic year. For more information about the Multiply programme, see bit.ly/36n0uwi.

Digest prepared by Katherine Wright
Society Business, Research &
Communications Officer

Note: items included in the Mathematics Policy Digest are not necessarily endorsed by the Editorial Board or the LMS.

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SOCIETY INFORMATION

London Mathematical Society

New layout and style for the LMS journals and Mathematika

Ola Törnkvist (LMS Editorial Manager)

London Mathematical Society, London, UK

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Funding information

Abstract
Since the start of 2022, the *Bulletin, Journal, Proceedings, Transactions* of the London Mathematical Society, the *Journal of Topology* and *Mathematika* are published in a new journal design, which is demonstrated on this page.

KEYWORDS
00-XX (primary)

1 | NEW JOURNAL DESIGN

From 2022, the LMS journals and *Mathematika* are published in *this* layout and style. Also, bibliographies are typeset according to the AMS referencing style [1], with examples given below. (There are no single quotes around article titles; the title of the item cited will always be typeset in italics, and the remainder of the reference will be typeset in an upright roman typeface.)

The LMS welcomes PDF manuscripts prepared in any style of TeX or LaTeX. There is no template or LaTeX class file for the new journal design; rather, the typesetters will convert any submission format into the new design.

To submit, please go to www.lms.ac.uk/publications/journals.

JOURNAL INFORMATION

The LMS journals are wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

REFERENCES

1. M. Letourneau and J. Wright Sharp, *AMS Style Guide: Journals*, American Mathematical Society, Providence, RI, 2017, Chapter 10, pp. 63–66.
2. A. N. Author, *Example of a title of a journal article*, Bull. Lond. Math. Soc. **53** (2021), 14–28.

LMS Council Diary — A Personal View

Council met via video conference on Friday 4 February. After a welcome from the President, with a particular welcome to those members new to Council, there was an update on President's business. This included a report on the Council for Mathematical Sciences (CMS) Town Hall meeting on 11 January, where green papers had been considered on a National Academy for Mathematical Sciences and a Connected Centres Network, which is being followed up by a working group chaired by the CMS Chair Alison Etheridge, and an update on the Levelling Up Scheme, for which further funding is providing a firm foundation for a second cohort of universities to join the scheme. The President also outlined recent activities of the Protect Pure Maths Campaign, such as collection of data on the demand for mathematics skills in the UK labour market and a submission to the House of Commons Science and Technology Select Committee on Diversity in STEM, and congratulated the Executive Secretary on having a letter on the Protect Pure Maths campaign published as the Editor's Choice in the *New Scientist* magazine.

There was discussion on several matters arising from the Minutes of the previous Council meeting. A report of Vice-President Hobbs on intake to Mathematical Science degrees was considered, which highlights that the number of mathematics undergraduates is growing in some institutions but shrinking in others. Vice-President Hobbs will speak at the April HoDoMs Conference about her report as well as at a Joint Mathematical Council committee workshop in February. An update was also given on the recruitment of a new Executive secretary, for which interviews will be held in April. Vice-President Gordon also reported on discussions with the chairs of the Early Career Research, Research Grants, and Society Lectures and Meetings Committees about how a pilot scheme for taking into account sustainability in the Society's events and grant giving activities might be implemented, which will be discussed at a later Council meeting.

After the Treasurer presented the First Quarter Financial Review and revised budgets for 2021-22, Council discussed an item on a proposed Code of Conduct for Trustees, and it was agreed that such a Code of Conduct should be introduced and would be made public. Other business included a report from the Elections Scrutineer, a paper on Committee Membership and Representatives, and a discussion on the Membership Census 2022, which showed that the Society had lost a relatively high number of Ordinary members over the last two years but there had been a large increase in the number of members in their twenties. The importance of emphasising the benefits of membership was agreed and it was noted that a membership benefit is highlighted in each issue of the *Newsletter*.

The meeting concluded with the President thanking members of Council for their contributions.

Elaine Crooks
Member-At-Large

Levelling Up: Maths — The First Year

"A very promising start", says founder Tony Hill.

Levelling Up: STEM nurtures A-level students from underrepresented groups in Maths, Physics and Chemistry who may be planning to study a STEM subject at university. It combines high quality academic material with an integrated pastoral programme and is delivered by STEM undergraduates from a local university. It begins midway through Year 12 and continues until the A-level examinations in Year 13. The first cohort, of 225 students, started in March 2021.

The programme is currently in the (highly resource-intensive) start-up phase with the first cohort of students completing the programme in April 2022. The broad aim of the second (roll-out) phase is to increase the number of participating universities and refine the programme, as necessary, over a three year period. In *Levelling Up: Maths*, this expansion is already well underway. Working together, the LMS and the IMA are now supporting eight English universities for the second cohort, starting in Spring 2022. In addition to first cohort universities Durham and Leicester, the newly-joined universities comprise Birmingham, Greenwich, Manchester Metropolitan, Middlesex, Southampton and UEA.

Students from underrepresented groups can be hard to reach. Accordingly, our central principle is that participating universities should intervene close to home and make their own determination of the precise underrepresentation focus. For example, amongst other things, the focus could be on socio-economic, gender or ethnic underrepresentation, or a mixture of these. The programme is open to all UK universities. I'd like to see *Levelling Up* working with established regional groupings, such as the five universities in the North East (NERAP). My ambition, by Spring 2025, is to have an annual national cohort of around 3000 students across the whole *Levelling Up: STEM* programme. Due to its central role in STEM, it is expected that a significant proportion of these students will be receiving tutoring in Maths.

A key principle of this scheme is that students, undergraduates and universities will all benefit from engagement with *Levelling Up: STEM*. Our goal is to:

(1) Increase aspiration and attainment for students from underrepresented groups
(2) Enhance the professional skills of undergraduates from such groups
(3) Engender a deeper understanding of access challenges within universities

A detailed evaluation of the impact of the programme on both the students and undergraduate tutors/mentors is being conducted. The School of Education at Durham University is co-ordinating the analysis across the six university departments involved in the first cohort. Interim findings are now being used to refine our approach, in preparation for our second cohort joining this Spring. The final report will be available in Summer 2022. Future student evaluation will centre on measuring the improvement in attainment.

Dr Tony Hill

Report: International Day of Mathematics

The International Day of Mathematics (IDM) 2022 celebrations took place under the theme Mathematics Unites with at least 1,650 celebrations in approximately 100 countries. The theme Mathematics Unites proposed by a student of Canada, Yulija Nesterova, can be explored through the following citations at tinyurl.com/bdfx94ec.

The Mathematics Unites Photo Challenge generated more than 3,200 entries and the most beautiful and inspiring photos are shown in galleries.

There were five global virtual live celebrations in each of the five international languages native to the speakers and chair. The celebrations have been recorded and can be accessed from at tinyurl.com/2p8pathx.

- Arabic: (Chair: Djamel Eddine Cheriet; Speakers: Tarig Abdelgadir, Samia Achour, Taous Meriem Laleg-Kirati)
- Portuguese: (Chair: Maria de Natividade; Speakers: Ines Guimarães, Marcos Cherinda, Humberto Bortolossi)
- English: (Chair: Sujatha Ramdorai; Speakers: Katie Steckles, Wilfred Ndifon, Steven Strogatz, Laura Wynter)
- French: (Chair: Raïssa Malu; Speakers: Wendelin Werner, Marie-Françoise Ouedraogo, Moreno Andreatta, Christian Genest)
- Spanish: (Chair: Jeanette Shakalli; Speakers: Alicia Dickenstein, Bernardo Recamán, Natalia Jonard, Eduardo Sáenz de Cabezón)

The celebrations were integrated into a 48 hour Live Coverage on 14 March 2022, starting 00:00 New Zealand time and ending 24:00 Pacific time and for which the IDM website got more than 30,000 hits from different people.

UNESCO launched the Open Access tool kit 'Mathematics for Action: Supporting Science-Based Decision Making' as an IDM event. This tool kit consists of a collection of lively two-page briefs promoting mathematically-grounded solutions to global challenges and highlighting the role of mathematics in addressing the SDGs of the UN 2030 Agenda. It was produced by a consortium of expert organisations.

The United Nations proclaimed the International Year of Basic Science for Sustainable Development 2022 (IYBSSD 2022), of which IMU is an organiser. The 2022 theme 'Mathematics Unites' for IDM 2022 expresses in particular that 'Mathematics unites all sciences, giving them a common language'. IDM 2022 is naturally part of IYBSSD 2022.

Press releases in five languages can be found at idm314.org/press_kit.html and the message of the Director General of UNESCO can be found at message of UNESCO DG.

Christiane Rousseau
University of Montreal

Report: LMS Invited Lectures on the Mathematics of Deep Learning

The LMS Invited Lectures on the Mathematics of Deep Learning were held at Isaac Newton Institute in Cambridge from 28 February to 4 March 2022 and were organised by Professor Carola-Bibiane Schönlieb, Professor Colm-Cille Caufield, Dr Marcello Carioni and Dr Subhadip Mukherjee from the University of Cambridge with the financial support of London Mathematical Society, University of Cambridge, Isaac Newton Institute, Newton Gateway to Mathematics and Cantab Capital Institute for the Mathematics of Information.

This workshop aimed at giving an insight to theoretical questions arising in the deep learning community and [amazing] speakers tried to answer some of them throughout the week. The main lectures were given by Professor Gitta Kutyniok (Ludwig-Maximilians-Universität München) and four accompanying lectures were given by Professor Rebecca Willett (University of Chicago), Professor Klaus-Robert Müller (Technische Universität Berlin), Professor Peter Bartlett (University of California, Berkeley) and Professor Weinan E (Princeton University).

Mathematics for deep learning or deep learning for mathematics? This is the question that best summarises the whole week. The lectures investigated how to open up the black box of deep learning methods mathematically but also how deep learning can be used to solve mathematical problems. These two aspects are intertwined since understanding the functioning of networks, their training, stability and reliability, is crucial for the resolution of inverse problems or partial differential equations using them.

Deep learning methods are nowadays everywhere and in aspects of everyday life. Have you ever wondered how the facial unlocking feature of your smartphone works? Or how can Google Maps predict with very good accuracy the time needed to go from one place to another, taking into account the real-time traffic conditions? We trust our smartphones blindly. But would you trust a Neural Network to decide whether a medical scan shows any cancer cells or maybe a qualified doctor would have been your first choice?

When using neural networks, the first choice one has to make is about the structure of the network. The architecture of the neural network inevitably has an impact on its performance and must yield good approximation results taking into account expressivity and complexity. Another crucial aspect is the training, usually performed with stochastic gradient descent. Convergence of the algorithm to a minimiser of the energy functional and the shape of energy's landscape (convexity, global and local minima, saddle points) are objects of study. Alongside training algorithms, the training dataset is equally, if not more, important. A biased training dataset might lead to wrong decisions. The creation of unbiased dataset is a current topic of sociological discussion. As far as robustness and stability are concerned, it has been proved that deep learning based techniques can fail under small perturbations, i.e. adversarial examples in image classification can be easily constructed. How robust neural networks can be is now under investigation.

Going back to inverse problems in imaging, we know that they can be solved with the classical variational approach (based on the knowledge of the model and of the noise/data distribution) and in a pure deep learning approach (training a network from a dataset and ignoring completely extra information). A third way is possible: hybrid methods combine known information about the problem on the noise or on the model and use a variational approach with a learned fidelity or learned penalty, respectively.

Scientific discussions were happening during the breakfast/coffee/lunch breaks the whole week. However, on Tuesday, between the lectures of Professor Gitta Kutyniok, all the attendees participated in a very interesting coordinated discussion about the Challenges of Machine Learning. Different topics were addressed, mainly about the ethics of machine/deep Learning, while also about privacy and associated legal aspects. In addition, the speakers, organisers and senior researchers present in the room gave valuable advice to the attendees about future career plans in academia, about interesting conferences, books, as well as reading groups to help them navigate through the extremely wide literature of deep learning.

Another highlight of the week was the Wednesday afternoon poster session, which sparked great discussions among the online and in-presence participants. The attendees were able to discuss their projects, the challenges they are facing in their own research related to deep Learning, as well as to establish possible new collaborations. Finally, from this great week we couldn't forget Wednesday night's formal dinner at (the outstanding) Selwyn College. It gave all the attendees, speakers and organisers another chance to meet and discuss, as well as a glimpse of college life.

Videos and presentations of the talks are available at gateway.newton.ac.uk/event/tgm109/programme.

Marta Lazzaretti (I3S Laboratory, second year PhD student at University of Genoa and University of Cote d'Azur)
Vasiliki Stergiopoulou (I3S Laboratory, last year PhD student at University of Cote d'Azur)

LMS Fellowship Report

In the autumn of 2019, my PhD studies turned a corner. After three years of battling with stochastic modelling and trying to understand T cell biology, my supervisors and I stumbled upon a new concept that would shape the following two years. We hypothesized that the magnitude of the immune response to a virus depended on the rate at which information (in bits per second) is transferred from a virus-infected cell to a T cell. But with less than a year of PhD funding remaining, I felt that we would need more time to do this idea justice.

After returning from the Christmas break in January 2020 I noticed that the LMS Early Career Fellowship (ECF) applications had a deadline of February. All my attention turned to putting together a convincing research proposal, and getting my PhD supervisors and proposed host at Oxford on board. Everyone seemed keen, and I felt reinvigorated about my research. But then came March 2020, and the world changed…

Suddenly, my son's bedroom became a day-time office. One day, I was working on my PhD, the next day I was home-schooling. Fortunately, some light then appeared at the end of that dark tunnel. The LMS approved my ECF application, and I knew that I had six months of funding lined up following the submission of my thesis. It's hard to overstate the impact this had on reducing my anxiety levels. In a world that appeared to have stopped hiring new employees, I reminded myself daily that I was one of the lucky ones.

In the end, it took a full year from being offered the ECF to starting the proposed research in March 2021. During this time, I was in regular contact with the LMS, who were extremely accommodating in delaying my start date due to pandemic related setbacks. Shortly after starting, we received extensive feedback from two peer-reviewers of a manuscript that we had submitted 3 weeks earlier. I then received similar feedback from my *viva voce* examiners. In both cases, more convincing experimental evidence had been requested to support our theoretical model.

Fortunately, my host in Oxford, Dr Omer Dushek, and his team were not only able to advise on potential historical datasets but were also able to offer their own novel datasets in support of the theory. This led to a significantly improved manuscript, jointly authored with my PhD supervisors and my host team in Oxford. A year on from starting the ECF, this paper has now been published in the Journal of the Royal Society Interface at tinyurl.com/mr3rhwz6.

I'm currently in the process of preparing a follow-up paper based on the new mathematical research that was performed during the ECF. I also successfully applied for a 3-year post-doc position midway through the ECF which I'm now undertaking at University College London. None of this would have been possible without the LMS so I am extremely grateful for all their support.

Joseph R. Egan
University College London

Report: Making Waves in Swansea

The Coastal Engineering Research Group in the Faculty of Science and Engineering, Swansea University, were very pleased to welcome Dr Ikha Magdalena from the Mathematics Department of Institut Teknologi Bandung, Indonesia for a 10 day visit. The visit was funded through a LMS Scheme 5 Travel Grant. Dr Magdalena met members of the Coastal Engineering Group as well as members of the Department of Mathematics and Computer Science, and gave a talk entitled An Integrated Study of Wave Attenuation by Vegetation, delivered in person and via Zoom. During her visit Dr Magdalena developed some analytical solutions mimicking the Swansea experiments, in which waves were generated by a sudden vertical movement of a panel in the base of a large wave tank. She also tested and improved nonlinear, nonhydrostatic computational solutions.

Note: The Welsh covid-19 regulations governing indoor meetings required that masks were worn.

l to r: Dr Jose Horrillo-Caraballo, Mr Xin Wang, Professor Dominic Reeve, Dr Ikha Magdalena, Dr Ditra Matin. Backdrop is a photograph of the laboratory experiment undertaken at Swansea University that provided test measurements for Dr Magdalena's mathematical models of tsunami generation and propagation

Dominic Reeve
Swansea University

The LMS Women in Mathematics Committee A History, 1995–2021

CAROLINE SERIES

Abstract. This is an account of the history and activities of the London Mathematical Society's Women in Mathematics Committee (LMS WiMC) and the British Women in Maths Days (BWM Days) which preceded it.

Introduction

The LMS WiMC has been very active and has led the way to what has been a complete change of climate for female mathematicians in the UK. Thus it seemed worthwhile to compile an account of the history before memories fade away. Similar accounts of trans-European activity for women mathematicians, in particular European Women in Mathematics, and the European Mathematical Society's Women in Maths Committee, can be found in [6], while a history of the more recent international activity around the IMU's Committee for Women in Mathematic appears in [7].

Prehistory

In 1986, I took part in a panel discussion organised by the US based Association for Women in Mathematics (AWM) during the International Congress of Mathematicians in Berkeley, 1986. In my report in the AWM Newsletter of that year [1] I wrote: *"This spring I circulated a letter to women members of the London Mathematical Society asking for their ideas on the subject of the panel. I sent out about 70 letters, roughly half to institutional addresses. Of the replies I received, the general impression was of little change, with many problems stemming from the primary school level. Lady Jeffreys, the distinguished applied mathematician and former Mistress of Girton College, Cambridge, who is now 83, writes: "It is 65 years since I began my studies, and it is disappointing that it is still considered rather odd for a woman to be mathematical. Something has to be done in the home (Your mother couldn't do it either, dear) and in the primary school, giving the girls confidence, which the little boys have. At all stages confidence is important."*

I added: *"One of the changes which I do see over the last eight years is that there are now enough women involved in serious mathematical research that collaboration between women has become not only possible but quite natural, without compromising standards or field of research. I find this very exciting, and consider myself truly fortunate to be part of what is probably the first generation in history where this has been possible."*

As a result of the panel, European Women in Mathematics (EWM) was set up in 1986 as a sister organisation to the largely US based Association for Women in Mathematics. Some British women took part, and indeed the third meeting of EWM took place in Warwick in 1988, see [6].

British Women in Maths Days

Despite this European activity, activities for women mathematicians in the UK did not really take off nationally until a suggestion from Dusa McDuff[1] who wrote about her idea at some length in the 2nd EWM Newsletter, January 1995 [2], on one of her extended visits to the UK. I quote: *"I have been visiting the Newton Institute at Cambridge recently, and the University too, and was saddened to find that the situation for women in mathematics at Cambridge seems little better than when I was a student there in the late sixties. I have also travelled about a little… and it seems that in many places in Britain there are so few women mathematicians that they get very isolated. Even if that doesn't lead to discouragement, it can make it very hard to interact fruitfully with colleagues. (I am very aware of that problem since I suffered from it for many years.)*

I think that the AWM (Association for Women in Mathematics) has done a great deal to help improve the lot of women mathematicians in the States and so I am proposing that we start a similar kind of organisation in the UK.

[1]Distinguished mathematician Dusa McDuff FRS was born and educated in the UK and moved to Stony Brook in 1978, but has always maintained her close ties with the UK. An article about her experiences appears in this issue; see page 29.

Things this organisation might do: (i) Organise meetings . . . the speakers would be women at a variety of stages . . . and we'd try to get as many women to come as possible so that they could meet each other. I have been told that the LMS would almost surely supply some money to support such meetings . . . provided that the lectures are open to everyone. . ."

The idea took root and the first British Women in Maths Day took place in Imperial College, London in September, 1995, see photographs 1, 2. Nearly 50 women attended from all over the British Isles. The main organisers were Ruth Williams (Cambridge) and Lynda White (London). There were short talks followed by a lively discussion about 'what next' with numerous suggestions for further activities. It was agreed that a similar meeting should be held the following year [3].

Not only the following year but, in some form or another ever since, BWM days have been an annual event. From early on, the days were supported by the LMS and for a number of years were organised by Helen Robinson (Coventry University). Once the LMS committee was established, it took over the organisation with both financial and administrative support from the LMS, which made an enormous difference to the volunteer organisers. The workshops are mathematical in content and open to all, but all the speakers are female. In 1999, a BWM workshop was held in Edinburgh, and in 2001 for the first time the event was held at the LMS headquarters, De Morgan House, in London.

Dinner at the 1995 workshop at Imperial College

Dusa is centre back

The LMS Women in Maths Committee: Early Days

Between its foundation in 1865 and 1998, the LMS awarded in total only four prizes to women mathematicians, and only one woman (Mary Cartwright, see below) became its President. Whilst in the early days of the Society this no doubt reflected the make-up of the profession, by 1998 this was certainly not the case. By then, around 38% of graduates and 18% of lecturers were female. However, at the upper end only 2% of professors of mathematics were female. Few women were invited to showcase their work at the prestigious Society meetings (3 speakers out of 21 were female in that year) and there were very few women on the LMS Council.

Concerned at the under-representation of women in the discipline, especially at the highest levels, and also inspired by international comparisons particularly of the kind documented by European Women in Mathematics, the LMS Council discussed what it could do to address the challenges the profession had in recruiting and retaining talented women mathematicians. The upshot was the creation in 1999 of the Women in Mathematics Committee of the Society. Cathy Hobbs (University of the West of England), who was on the LMS Council, chaired the committee from 1999-2001 and the work to develop initiatives continued under the leadership of Helen Robinson and then Alice Rogers (King's College London). The committee was tasked with undertaking activities with four aims:

- Raising the profile of women in mathematics,
- Supporting women in the mathematics profession,
- Improving practice in the mathematics community as regards gender diversity,
- Collecting and disseminating data about women in mathematics.

One of the committee's first actions was to suggest that each year the LMS should have an invited lecture given by a prominent woman mathematician as part of an LMS Society Meeting. The lecture, organised by the committee, is called the Mary Cartwright Lecture after the distinguished mathematician Dame Mary Cartwright (1900-1998), student of G.H. Hardy and the first female mathematician to be elected to the Royal Society, as well as the first female President of the LMS. These lectures have been given every year, usually but not always in London, since 2000.

Gwyneth Stallard took over as the committee chair in 2006 and continued with many new initiatives. In terms of removing barriers, an issue for many parents was that the additional costs incurred to cover childcare while on short visits to collaborators or attending a conference were not covered by any existing mechanism. Grant-giving bodies did not recognise these costs as being valid claims alongside travel and subsistence, yet they are very real financial barriers which tend to affect women more than men — an example of indirect discrimination. The LMS WiM Committee decided to take direct action on this by using some of its budget to make small grants of up to £200 to parents who needed money to fund childcare during short visits. Applicants have to make a case for the importance to their career of attending the conference or making the visit and give a breakdown of costs, but they can choose to use the money in the way most effective for their own circumstances, such as the extra cost incurred of taking someone with them to look after the children, or paying for extra childcare at home.

Another barrier to women is that they often find themselves the 'trailing partner' to another academic. Typically their partners are a few years older and higher up the academic ladder, so when the partner is offered a promotion elsewhere the couple make the pragmatic decision to move. This can leave a highly qualified and talented woman moving to a new area with no commensurate job. The WiM Committee bid for funds from the LMS to provide fellowships for those who find themselves in such a position (female or male). The Grace Chisholm Young Fellowships recognise the holder's academic ability and gives them an official position within the host university, providing a small amount of funding for travel and other expenses and a contribution to the host university. They are named for the mathematician Grace Chisholm Young (1868 – 1944) who looked after the family home and children while her husband, another mathematician, travelled for work and (apparently by mutual agreement) published many of their joint papers in his name alone.

The LMS Women in Maths Committee: Expansion

Although the WiM Committee was very active right from its beginning in 1999, the work of trying to support women in mathematics was for many years carried out by a relatively small number of people. This changed dramatically following the International Review of Mathematics in the UK in 2010 which included as one of its main findings that "action about gender diversity is not a sufficiently high priority for the UK mathematical sciences research community" and recommended that urgent action was needed. The following year, Research Councils UK began to talk about making research funding conditional on action being taken on equality and diversity. These two events caused a sea change in the mathematical community with Heads of Departments keen to know what they should do.

The WiM Committee developed a 'Good Practice Scheme' [4] to support departments and, in 2012, ran the first ever UK wide Benchmarking Survey of practice in mathematical departments. This provided data on the number of women at various career stages and information on practices currently adopted by university departments, with lots of examples of what could be done. The report was launched at the House of Commons – the first ever LMS event to be held there, and the work of the Committee was now mainstream. This was particularly important in view of the national Athena SWAN award scheme, a national initiative to encourage women into science, see tinyurl.com/yckmkaax. The award, which has to be applied for by individual university departments, requires the collection of many statistics along with the development of numerous actions and action plans, and was found by many to be extremely burdensome. The Benchmarking Survey provided the kick start that many departments needed and, between 2013 and 2016, the number of mathematics departments with Athena SWAN awards increased from 3 to 39. Good Practice Workshops continue to be run at regular intervals.

The LMS also reviewed its own practices to ensure that work to support women in mathematics is embedded across its work, issuing a Council Statement that has been used as a model by many other similar societies, see www.lms.ac.uk/womeninmaths. The LMS Programme Committee (now renamed SLAM) which awards grants for conferences has added new questions to its application forms to ensure that conference organisers include a good proportion of female speakers and support those with childcare needs, providing role models and ensuring that women are given opportunities to speak about their work. After an initial year or two when many conferences were refused funding or asked to reapply because of the low numbers of proposed women speakers, the community is now taking this issue seriously.

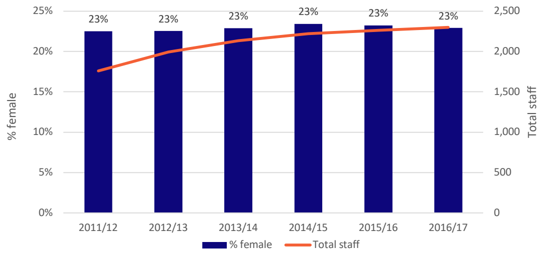
In 2013, Anne Bennett, a senior and much valued member of staff at the LMS, died very suddenly. She had always had an interest in women in mathematics, and on the recommendation of the WiM Committee, the LMS established the Anne Bennett prizes in her memory. Awarded to both junior and senior mathematicians in different years, the prizes recognise both work in and influence on mathematics, and also having acted as an inspiration for women mathematicians. In 2020, the senior AB prize was awarded to Peter Clarkson who took over as chair of the Good Practice Scheme Steering Group from Cathy Hobbs. There have also been significant efforts to encourage the nomination of women for all the LMS prizes. Indeed in almost every year since 2000, at least one of the winners of the Whitehead prizes (for mathematicians within 15 years of their PhD) has been a woman, as have a significant number of winners of the more senior prizes.

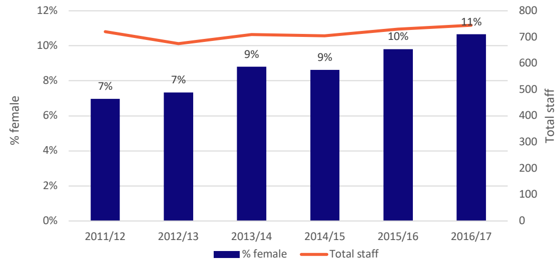
The Present

From 2015-2021 the chair of the WiMC was Eugenie Hunsicker (Loughborough). The committee now includes representatives from the other mathematical societies in the UK (IMA, EdMS, ORS) and also from EWM. It is working to increase the number of women and girls in mathematical events supported by the Society as well as to attract a broader range institutions and greater geographical diversity within the UK. As well as the Women in Maths days, it is now also possible for schools to bid for LMS funding to help run Girls in Maths Days. The committee is developing online and print resources such as posters for use in schools, showcasing a broad range of individuals in mathematical careers and a broad range of careers that involve mathematics. This project, entitled *Success Stories in Mathematics* was launched in the British Library in 2018 and can be found at www.lms.ac.uk/success-stories. The benchmarking survey has been recently updated [5]. In 2020, thanks to a generous donation from the Liber Stiftung, in 2020 the committee awarded four special Emmy Noether Fellowships to support career development of mid-career women mathematicians with substantial caring responsibilities. The Foundation was delighted with the use made of its grant and repeated this donation in 2021.

The committee is also broadening the range of equality and diversity issues it addresses and has recently changed its name to the LMS Women and Diversity Committee, to recognise that now gender diversity has become more mainstream it is time to focus on wider diversity issues as well. It is looking at ways to collaborate with other STEM groups nationally and mathematical groups internationally on issues related to ensuring equal access and opportunity regardless of ethnicity, class, religion, LGBT status or disability. In 2020 and again in 2021 the Society organised a hugely successful online meeting *Celebrating Black Heroes of Mathematics*, in partnership with the Institute of Mathematics and its Applications, the British Society for the History of Mathematics, and the International Centre of Mathematical Sciences in Edinburgh.

In 2020, the Society amended its bye-laws to include a named 'Women and Diversity' member-at-large on Council. In 2021, Sara Lombardo (Loughborough) was elected to this post and now chairs the WiMC.





Proportion of UK female lecturers (top) and professors (bottom) in mathematics 2011–2017, from [5].

Conclusion

One indication of the dramatic increase in engagement of the community with the work of the WiMC over the last few years is the number of participants at the annual Women in Maths Days which have been expanded to invite applications from several institutions to host such an event each year. This has grown from about 30 participants in the early days to around 100 at triennial two day events introduced in 2010. There were several hundred participants (including undergraduates and school girls for the first time) at a four day event in Oxford in 2015, one of the highlights of the 150th anniversary of the LMS. Overall well over 1000 women have attended a WiM Day. The number of female members of the Society has increased from 12% in 2010 to almost 20% in 2019. Cathy Hobbs is now one of the two LMS Vice-Presidents, while the author was the President 2017-19 and Ulrike Tillman is now in office 2021-23.

Moreover 8 out of 20 Council members have been female for most years since 2013. In recent years the various grant giving committees have always been chaired by men who have been hugely supportive of initiatives to insist on more female speakers. For example, in 2015, 42 out of 106 speakers at LMS events were women, and the Society's Lectures and Meetings Committee (SLAM) is for the first time chaired by a woman.

All this work won a remarkable tribute in 2016 when the committee was awarded the Royal Society's inaugural Athena Prize "in recognition of their work in introducing a broad range of initiatives in the field of mathematics resulting in a change of culture that has happened nationwide, leading the way in increasing the number of women in mathematics." The prize is awarded biennially to teams working in UK academic and research communities, who have contributed most to the advancement of diversity in science, technology, engineering and mathematics (STEM) within their communities. The recipients of the prize receive a medal and a gift of £5,000. Gwyneth Stallard was also recognised individually by the award of an OBE in 2015 for her work in supporting women in mathematics.

Among the many different models of how best to address the issues faced by female mathematicians, the UK experience shows the great advantages to be gained by working through an established body with a budget and good administrative support and structures. From the start, the committee greatly benefitted from very strong support from senior male mathematicians, which was very important in making progress. These included Sir Martin Taylor (LMS President at the time the committee started), Peter Clarkson mentioned above, Charles Goldie and Malcolm McCallum. It was also crucial that it has always worked alongside Heads of Mathematics Departments (within the UK this is an organised group called HoDoMS, of which both Cathy and Peter have been members). Non-mathematicians working to promote diversity have commented how impressive the achievements have been with a relatively small budget and operation. By 2017, approaching 11% of professors of mathematics in the UK were female – still small, but nearly triple the percentage in 1998. This and much further data can be found in the updated Benchmarking Survey [5]. The indirect effects of the work of the committee have made a tremendous difference to mathematical life in the UK.

Acknowledgements

Apart from personal memories and material from Newsletters of European Women in Mathematics, much of this article has been extracted from accounts by Cathy Hobbs, Eugenie Hunsicker and Gwyneth Stallard, all former chairs of the LMS WiMC. I would like to thank them all for their help

FURTHER READING

[1] AWM Newsletter, September-October 1986, tinyurl.com/ycbuvwkd
[2] EWM Newsletter Number 2, January 1995. https://tinyurl.com/y2frxm9k
[3] EWM Newsletter Number 3, January 1996, ibid
[4] Advancing Women in mathematics: Good Practice in UK University Departments, 2013, tinyurl.com/2p857k7n, Accessed 25 May April 2020.
[5] National Benchmarking Study, 2017, tinyurl.com/4v5phpu9, Accessed 28 May April 2020.
[6] Marie-Françoise Roy and Caroline Series. European Women in Mathematics and the European Mathematical Society's Women in Mathematics Committee, in *Association for Women in Mathematics: The First Fifty Years*, eds. J.L.Beery, S. J. Greenswald, C. Kessel, Springer, 2022.
[7] Marie-Françoise Roy and Caroline Series. International initiatives for women mathematicians, ibid.

Caroline Series

Caroline Series is Emeritus Professor of Mathematics at the University of Warwick, where she spent most of her career.

She is well known for her work on the coding of dynamical systems associated to hyperbolic surfaces and, more recently, for her work on hyperbolic three manifolds whose associated fractal limit sets are beautifully explained and illustrated in her book Indra's Pearls, coauthored by David Mumford and David Wright.

In 1986, Caroline was a founder member of European Women in Mathematics and has since served as Chair of the European Maths Society's Committee for Women in Mathematics and also as the first Vice-Chair of the IMU's Women in Maths Committee CWM.

Caroline was elected a Fellow of the Royal Society in 2016 and is also a Fellow of the IMA. As readers will know, she was President of the LMS 2017-19 and in 2021 she was awarded the LMS-IMA Crighton Medal in recognition both of her mathematical work and her outstanding service to the mathematical community.

On Being a Woman in Mathematics, Then and Now

DUSA MCDUFF

Abstract. This article describes how the author made her way as an aspiring mathematician some sixty years ago, with few guides and no senior women mathematicians to advise her. Indeed her election to the Royal Society came almost fifty years after Dame Mary Cartwright's, with no female mathematician in between.

Dusa McDuff FRS

Dusa McDuff FRS is a British mathematician who has spent the greater part of her career in the USA. Born in 1945 in London, she did her first degree in Edinburgh and then went to Cambridge for her Ph. D. Subsequently she spent seven months in Moscow with her first husband, during which time she came under the influence of the brilliant Russian mathematician Israel Gel'fand. After another brief period as a postdoc in Cambridge, followed by a few years as a lecturer at York and then Warwick Universities, she moved to the U.S. to take up a position at SUNY, Stony Brook. She is now Helen Lyttle Kimmel '42 Professor of Mathematics at Barnard College, Columbia University.

Dusa is especially known for her pioneering contributions to the new field of symplectic geometry and topology. Her work has been recognised by many honours. In 1994, she became only the second female mathematician (following Dame Mary Cartwright) to be elected to the Royal Society. She was elected an honorary member of the LMS in 2007 and in 2018 became only the second woman (also following Mary Cartwright) to be awarded the Sylvester medal of the Royal Society.

It is hard to remember now that people thought this way, but when I was young I was once told that it was unnatural for a woman to be good at mathematics, the life sciences maybe, but not something so abstract and austere as mathematics. I imagine that no-one would say that out loud today — though they might think it. At the time this reaction was something I was used to; since my name is Dusa (after Medusa) my schoolmates would often tease me that my glance would turn them to stone.

As a young girl, I was very diligent, doing what everyone expected of me. I always loved mathematics because of its beauty and precision, even when that just meant doing sums. I did well in exams, and was very much encouraged, especially by my mother, who was an architect with (unusually for the time) a full-time job in the civil service. But in my midteens I rebelled; I was devoted to an unsuitable boyfriend (now a distinguished literary translator), and outgrew my school. It was a girls' school, the best my parents could find in Edinburgh, but (despite its wonderful maths teacher) inferior in what they offered in maths to the corresponding boys' school and inferior in English teaching to the school my boyfriend went to. It was the early sixties. My father, an eminent geneticist, brought home brilliantly patterned paper dresses and wonderful jazz records from New York. I essentially had two lives, one as a poetry groupie and proto-hippy, and one that I kept to myself as a student of mathematics; playing music served as a shaky bridge.

It took me a long time to reconcile these two strands in my life. I refused my scholarship to Cambridge, remaining in Edinburgh, but made no contact with other maths students there. When I did get to Cambridge as a graduate student, I married my boyfriend fairly soon after, and did not fit into any accepted social framework. I never went to Girton College (which I nominally belonged to) because it was so far away, and there was neither anything mathematical for me there nor any provisions for married students; I was excluded from the main colleges (where people had lunch and dinner) because I was female. I did talk to a few maths students in my functional analysis group as well as to my supervisor, so I wasn't as isolated as before. I also managed to write a thesis (on a problem suggested by my supervisor) that was published in the *Annals of Mathematics* — that success is probably the main reason I managed to survive as a mathematician. However, it was really only in Moscow (where I went in 1969-70 so that my husband could work on his dissertation) that the contradictions in my life started to resolve. I had the unplanned good fortune to work with Gel'fand; his world included music and poetry as well as a vast realm of beautiful mathematics that he opened up to me.

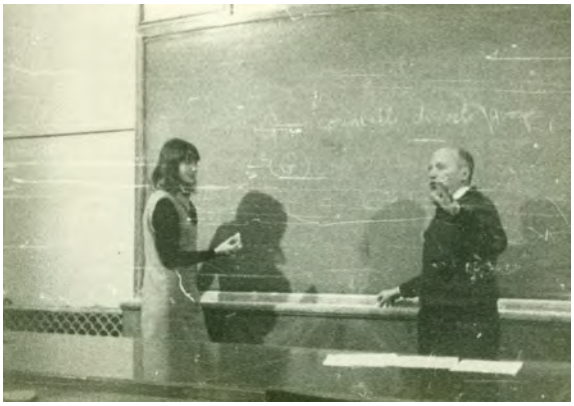


Figure 1. Dusa McDuff with Israel Gel'fand, 1971, Moscow

On my return from Moscow, I completely switched fields, so that during my two years as a postdoc in Cambridge I was again very isolated, with only tenuous mathematical contacts. However, in spring 1972 a chance encounter with Cassels (then Chair of the Cambridge mathematics department) provided the opportunity to attend a conference in Seattle, where I met Graeme Segal. I wrote what was essentially a second Ph.D. under Graeme's guidance, and slowly began building a useful foundation of mathematical knowledge in topology that led to my later work in symplectic geometry. I am very glad that I had time to develop my mathematics without too many demands being put on me.

Although I had a more limited undergraduate education at Edinburgh than I would have obtained in Cambridge, it is not clear that I would have been any better off if I had taken the expected path. I am sure there were other mathematically talented girls who just didn't make it. When I got to Cambridge, there were three other female graduate students, but after graduation they were planning either to go back to their home country, or to get married rather than pursue a career. I never knew them really well. (Of course, at the time I was married — but I didn't see the contradiction since I expected to be the one to support the family: mathematics paid better than poetry.) I never met Dame Mary Cartwright as a mathematician: my only meeting with her was in my entrance interview, when as Mistress of Girton she handed out delicate tea cups to the group of potential students seated around her. I also never met Sheila Edmonds, another considerable mathematician, but who never had a University lectureship and was immersed in college life. I did talk to Helen Alderson, a somewhat older woman, once a mathematical child prodigy in Moscow and now trying to get back to research, after emigration, marriage and having a family. People helped her find fellowships, but she was very marginalized and had a difficult life. There were no other women in the maths department then. That was the situation for women in Cambridge for a long time because of structural problems. The positions easily available to women were in colleges rather than in the university, and thus were essentially teaching positions which did not 'count'.

I was definitely happier, and began to fit in more, when I got a job as a lecturer, first in York and then Warwick, both 'new universities' with fewer traditions and prejudices. In both places I was the first female lecturer, but felt accepted by my colleagues. I didn't socialize much or have many friends — I didn't have time, what with teaching, doing mathematics, and looking after my young daughter. I still knew hardly anyone with the same kind of life and with whom I felt free to talk. I heard of the women's movement from Graeme, who was surprised to find how little I knew about it. I thought I was beyond all that: separated from my husband by the time I got to Warwick, I was already earning my own living, trying to bring up a child and set on making my mark as a mathematician. And anyway I knew of no like-minded women with whom to form a consciousness-raising group.

I learnt later that the faculty wives in Warwick felt sorry for me, I am not sure why; I loved my job, though I was very busy, and my mathematics was finally reviving. I would have been happy to stay there — but I fell in love with John Milnor and moved to the U.S. to be closer to him, giving up a tenured lectureship at Warwick for an untenured Assistant Professorship at Stony Brook. When I left Warwick, Caroline Series had just completed her Ph.D. at Harvard and spent a year as a Research Fellow at Newnham College, Cambridge. She was the first female British mathematician I met; I advised her to apply for my soon-to-be vacated position in Warwick — which she did. She enjoyed it, but for many years was the only female faculty member. Things moved very slowly in the U.K.

The situation in the U.S. was a little different: the women's movement was more visible, and had been making demands that universities reform themselves. In the early 70s, the Ivy League was opening its campuses to female students and getting rid of nepotism rules, perhaps even thinking of offering proper jobs to female faculty, though that was controversial and in many cases came considerably later. I first went to the U.S. as a Visiting Professor at M.I.T. in 1974, invited out of the blue because they were looking for a suitable woman to add to their faculty. (My name must have come up via Isadore Singer's close connection with Gel'fand.) This was a wonderful opportunity. I had a real mathematical idea again, writing a paper with Graeme on the group-completion theorem that is still cited today, and I started to be more proactive in my career, applying to visit the Institute for Advanced Study. During my year at M.I.T., I met Ragni Piene from Oslo, who was a graduate student there. She was the first woman mathematician I had the chance to know. She has been a life-long friend, as has Caroline.

I learned recently that at that time Harvard admitted one or two female math graduate students every year or so. Nancy Hingston, Linda Ness, and Caroline Series are the mathematicians that I know from that group. A few other universities and individual mathematicians were also making similar efforts to seek out and train promising young women, laying the foundations for a slowly growing cohort of strong female mathematicians. The fact that some universities (such as M.I.T. and, a little later, Stony Brook) were actively seeking women to hire, was a help. Also, a few women came from Europe and flourished, feeling liberated from the expectations of their home societies and being blissfully unaware of the corresponding societal strictures in the U.S. Over the years I have enjoyed working with a variety of female students, encouraging them to pursue their interest in mathematics in whatever way suits them best. But the progress in building a visible presence of women in mathematics in the U.S. has been slow. In many leading departments there are still too few women faculty and graduate students.

I have participated in many efforts to bring female mathematicians together, mostly in the States, but also via wonderful European Women in Mathematics meetings, and the British Women in Mathematics Days now sponsored by the LMS. Even today, these programs have a purpose. Although there are more female mathematicians, they are not evenly distributed among the different universities and research groups, and still too many women feel intimidated into silence, as I was for many years. I help organize the Women and Mathematics program at IAS, Princeton, started by Karen Uhlenbeck, a one week program for (in non-covid days) about 60 participants. The students almost uniformly say how refreshing and liberating it is to be in an all-female environment for once. Some of them are surprised by the difference it makes, not having realized the extent to which they had felt intimidated.

Recently, the Association for Women in Mathematics (AWM) has become much more active in promoting research. Under the leadership of Kristin Lauter (who worked at Microsoft for many years), it has recently been organizing networks of women in different fields, that every so often run workshops in which groups, with participants ranging from graduate students to senior faculty, come together to work on specific problems. At first glance, this format seems artificial and industry-inspired — but often it works beautifully, leading to the development of new ideas and unexpected collaborations, with new talent coming to the fore.

I can't of course talk about what it is like to be a young female mathematician today. Women are more visible, and the possibility of their having real mathematical talent that is worth nurturing is much more widely acknowledged. It is still very hard to navigate the issues of two-career families, or how to look after children (even more impossible in covid times). However, there is much more awareness and discussion of these questions and there also has been some effective action. Both mathematical societies AMS and LMS have recently had female presidents, and in their different ways are working hard to bring about real change in the common practises around the issues of diversity, equity and inclusion.

Extremal Kähler metrics and convex analysis

SIMON DONALDSON

The aim of this article is to give an impression of some contemporary developments in complex differential geometry through the particular case of toric manifolds where the constructions can be expressed in elementary terms.

A variational problem

Our starting point is a partial differential equation for a function u of n real variables $x_1, \ldots, x_n$. We require the function to be strictly convex, by which we mean that the matrix of second derivatives

$$u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$$

is positive definite at each point. Let (u^{ij}) be the inverse matrix. The partial differential equation is

$$\sum_{ij} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j} = -A, \tag{1}$$

where A is a given function of $x_1, \ldots x_n$. (We are mainly interested in the cases when A is a constant, or a linear function.) This equation was first written down by Miguel Abreu in [1]. It is a nonlinear fourth order PDE, the nonlinearity coming from the nonlinear map which takes the matrix (u_{ij}) to its inverse. (u^{ij}). The equation is closely related to Monge-Ampère equations which arise in many parts of pure and applied mathematics. These are second order PDE which have the form

$$\det(u_{ij}) = F,$$

where F is a given function of x, u and the first derivatives of u. For example, it is an exercise to show that a solution of the Monge-Ampère equation with $F = 1$ is a solution of (1) with $A = 0$.

We want to consider a function u on a convex polytope $P \subset \mathbf{R}^n$. So P is a bounded set defined by a finite number of inequalities $\lambda_j(\mathbf{x}) < 0$, for affine-linear functions λ_j. (By affine-linear we mean a function of the form $C + \sum c_i x_i$.) We also fix a measure $d\sigma$ on the boundary of P. This is to be just a multiple of the standard $(n-1)$-dimensional volume measure on each face of the boundary. It is elementary that there is a unique affine-linear function A such that for any affine-linear function f

$$\int_P f A d\mu = \int_{\partial P} f d\sigma, \tag{2}$$

where $d\mu$ is the Lebesgue measure on $\mathbf{R}^n$. Now, with this function A, we want to solve the PDE (1) in P for a function u satisfying certain boundary conditions. These are, roughly speaking, that as we approach a point p on a face on which the measure $d\sigma$ is m_α times the volume measure the function u should behave like $m_\alpha D \log D + u(p)$ where D is the distance to the boundary. The boundary conditions can be built into a variational formulation of the problem. For a function f on the closure $\overline{P}$ define

$$L_{\overline{P}}(f) = \int_{\partial P} f d\sigma - \int_P f A d\mu.$$

Now define a functional on convex functions u on $\overline{P}$, smooth in the interior, by

$$\mathcal{M}(u) = -\int_P \log \det(u_{ij}) + L_{\overline{P}}(u). \tag{3}$$

The function $-\log \det H$ on positive symmetric matrices H is convex, so the same is true of the functional $\mathcal{M}$ and any critical point is a minimum. A variational analysis shows that a minimiser is the same as a solution of equation (1) satisfying the boundary conditions. The relevance of the condition (2) on A is clear from this variational point of view, because if it did not hold the functional is obviously not bounded below, since adding an affine-linear function to u does not change $\log \det(u_{ij})$.

Toric geometry

To explain where the PDE (1) comes from, we begin with the case of surfaces of revolution. Away from the fixed points we can choose “equiareal” co-ordinates

(x,θ) in which the metric has the form $hdx^2 + h^{-1}d\theta^2$ where h is a function of x and the circle action rotates the θ co-ordinate. (Equiareal means that the area form of the metric is the standard form $dxd\theta$ in these co-ordinates.) The Gauss curvature is given by the formula

$$K = -\frac{1}{2}\frac{d^2\ h^{-1}}{dx^2}. \tag{4}$$

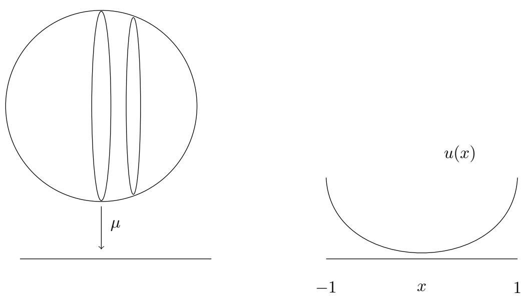
If we integrate twice to write $h = \frac{d^2u}{dx^2}$ for a convex function $u(x)$ this gives the expression on the left hand side of (1) up to a factor $-1/2$, so the equation (1) is prescribing the Gauss curvature as a given function $A(x)$. Take, for example, the case of the standard round 2-sphere in $\mathbf{R}^3$ rotating about an axis. Then, by a result of Archimedes, the equiareal co-ordinate x is the projection onto this axis and the metric is

$$(1-x^2)^{-1}dx^2 + (1-x^2)d\theta^2$$

so $h(x) = (1-x^2)^{-1}$ and

$$u(x) = \frac{1}{2}\left((1-x)\log(1-x) + (1+x)\log(1+x)\right)$$

on the interval $(-1,1)$, which is our polytope P in this case.



The round 2-sphere and its symplectic potential function

The introduction of u may seem artificial in this 1-dimensional case but becomes essential in higher dimensions. The general setting is a *Kähler metric* on a manifold of dimension $2n$, with an isometric action of a n-dimensional torus T^n. Thus, on the subset where the action is free, we have n angular co-ordinates $\theta_1,\dots,\theta_n$ and it can be shown that that there are additional co-ordinates $x_1,\dots,x_n$ and a "symplectic potential" function $u(x_1,\dots,x_n)$ such that the metric has the form

$$\sum u_{ij}dx_i dx_j + u^{ij}d\theta_i d\theta_j, \tag{5}$$

where u_{ij} and u^{ij} are defined as before. The expression on the left hand side of (1) gives minus the *scalar curvature* of this metric. Solutions of the equation (1) with a constant A give constant scalar curvature Kähler (CSCK) metrics. When A is an affine-linear function they give *extremal Kähler metrics*, a notion introduced by Calabi. For the purposes of this article the reader does need to know this differential geometric background: the point is that CSCK and extremal metrics are natural higher dimensional generalisations of constant Gauss curvature surfaces. On the 2-sphere there are two points where the θ co-ordinate is not defined, the fixed points of the rotation action. We have a map $\mu : S^2 \to [-1,1]$ mapping these two points to the endpoints of the interval and the description above is valid over the interior $(-1,1)$. The general story for a compact $2n$-dimensional Kähler manifold X with T^n action is that there is a map $\mu : X \to \mathbf{R}^n$ with image a closed convex polytope $\overline{P}$. Over the interior P of $\overline{P}$ the fibres of μ are free T^n-orbits but over boundary points the fibres are lower dimensional tori. The polytopes that arise in this way form a special class called *Delzant polytopes*. The definition involves an integrality condition: there must be n faces meeting at each vertex and these must be equivalent, under the action of $GL(n,\mathbf{Z})$ and translations, to the standard co-ordinate hyperplanes. The integral structure defines a measure on each face of the boundary of $\overline{P}$, for the face is contained in a hyperplane $H + p$ and we have a lattice $H \cap \mathbf{Z}^n$ in H which fixes a measure.

Kähler geometry is the intersection of symplectic geometry and complex geometry and the discussion above is the symplectic picture. We could go on to write down a complex structure on X in which the action of the torus T^n extends to a holomorphic action of $T^n_{\mathbf{C}} = (\mathbf{C}^*)^n$, with an open dense orbit—just as for $\mathbf{C}^* \subset S^2$. But to keep things short let us move on to an algebro-geometric point of view.

Any convex set $\Pi \subset \mathbf{R}^n$ defines a graded algebra $R = R_\Pi$. First take the cone on Π, the set

$$C(\Pi) = \{(x,h) \in \mathbf{R}^n \times \mathbf{R} : h \geq 0, x \in h\Pi\},$$

and let Σ_Π be the intersection of $C(\Pi)$ with the integer lattice $\mathbf{Z}^n \times \mathbf{Z}$. The algebra R_Π has an additive basis s_ν corresponding to points $\nu \in \Sigma_\Pi$ and multiplication defined by $s_\lambda s_\nu = s_{\lambda+\nu}$. The grading is provided by the $\mathbf{Z}$ component of ν. Similarly, there is an obvious action of $T^n_{\mathbf{C}}$ on R. For a general convex set Π this algebra will not be finitely generated but in the case of a closed polytope $\overline{P}$ which is the convex hull of a finite number of points in the integer lattice $\mathbf{Z}^n \subset \mathbf{R}^n$ it will be. In that case, by general foundational results in algebraic geometry (the "Proj" construction), $R_{\overline{P}}$ is the coordinate ring of a "toric variety" $X \subset \mathbf{CP}^N$ with a $T^n_{\mathbf{C}}$ action on X induced by that on $R_{\overline{P}}$. If the polytope $\overline{P}$ also satisfies the Delzant condition, X will be a complex projective manifold. We will call such polytopes *integral Delzant polytopes*.

For an example, let $\overline{P}$ be the interval $[-1,1]$ in $\mathbf{R}$. Then the ring $R_{\overline{P}}$ is generated by the three elements U,V,W corresponding to the lattice points $(-1,1),(0,1),(1,1)$ with a single relation $V^2 = UW$. This is the co-ordinate ring of the conic curve in $\mathbf{CP}^2$ defined by the same equation in homogeneous co-ordinates.

The differential geometric and algebro-geometric discussions are compatible, so for an integral Delzant polytope $\overline{P}$ the T^n-invariant Kähler metrics on the complex projective manifold X we defined algebraically above correspond to convex functions u on P satisfying our boundary conditions. (More precisely, the correspondence is with Kähler metrics in the cohomology class determined by the projective embedding.)

With this background we have reached the main point. An important question in complex differential geometry is: when does a projective manifold admit an extremal metric? This includes (for the special class of manifolds with vanishing first Chern class) the question of the existence of Calabi-Yau metrics with zero Ricci curvature, which was famously answered by Yau in 1978. But in general the extremal condition is the right one to consider. By what we have said, in the case of toric manifolds X this question comes down to the solubility of our PDE (1). (The Calabi-Yau condition, in the toric case, becomes a Monge-Ampère equation.)

The existence theorem

Fix a base point p_0 in the interior of our polytope $\overline{P}$ and call a convex function u *normalised* if $u \geq 0$ and $u(p_0) = 0$. By adding affine-linear functions we can restrict attention to normalised functions u. Contemplating the formula (3) one sees that the minimisation problem involves two competing effects. To make the integral of $-\log\det(u_{ij})$ small we should make $\det(u_{ij})$ large, so we should make the second derivatives of u large in at least some directions, but that will make the function u large on the boundary so the term in (3) involving the integral of u over the boundary will be large. The question is whether a balance between these two effects can be achieved. An answer to this question is known, at least for Delzant polytopes.

Theorem 1. *Let $\overline{P} \subset \mathbf{R}^n$ be a Delzant polytope and $L_{\overline{P}}, \mathcal{M}_{\overline{P}}$ be the corresponding functionals. There is a minimiser of the functional $\mathcal{M}_{\overline{P}}$ on normalised functions u if and only if $L_{\overline{P}}(f) > 0$ for all non-zero normalised convex functions f on $\overline{P}$. This minimiser is unique.*

This statement combines work of many people, the final step being achieved in the recent preprint [6]. Earlier work of the author [4], [5] and B. Chen, Li, Sheng [3] dealt with the case $n = 2$. For higher dimensions, the breakthrough comes from work of X. Chen and Cheng [2], in the larger setting we discuss in the next section. The entire proof involves a mountain of analysis and we only attempt to make the statement plausible.

The convexity of the functional $\mathcal{M}$ gives the uniqueness part of Theorem 1. Another simple fact is that if there is a smooth convex function f with $L_{\overline{P}}(f) < 0$ then $\mathcal{M}$ is not bounded below. For if we take any convex function u satisfying the boundary conditions and set $u^{(s)} = u + sf$ then for $s \geq 0$ the function $u^{(s)}$ is convex and also satisfies the boundary conditions. We have $\mathcal{M}(u^{(s)}) \leq \mathcal{M}(u_0) + sL_{\overline{P}}(f)$ since $\det(u^{(s)})_{ij}) \geq \det(u_{ij})$, hence $\mathcal{M}(u^{(s)}) \to -\infty$ as $s \to \infty$. Turning to the existence question, consider a finite-dimensional analogue of our infinite dimensional situation, with a function F on a Euclidean space $\mathbf{R}^N$. The lack of compactness of $\mathbf{R}^N$ means that, even if F is bounded below, there may be no minimum: for example the function $F(x) = e^x$ on $\mathbf{R}$. But a "coercive inequality" of the form $F(x) \geq \epsilon\|x\| - C$ for some $\epsilon > 0$ implies that a minimiser must exist. In our problem, suppose we know that there is a bound, for some $\lambda > 0$ and all normalised convex functions f on $\overline{P}$:

$$L_{\overline{P}}(f) \geq \lambda \|f\|_{L^1(\overline{P})}. \tag{6}$$

Then it is not hard to show, using the slow growth of the logarithm function, that this implies that the nonlinear functional $\mathcal{F}$ satisfies a coercive inequality, for normalised u:

$$\mathcal{M}(u) \geq \epsilon \|u\|_{L^1(\overline{P})} - C, \tag{7}$$

for some $\epsilon > 0$. One of the main results of Chen and Cheng is that such an inequality implies the existence of a minimiser—the infinite-dimensional problem behaves like the finite-dimensional analogue. The recent work of Li, Zian and Shen [6] establishes a convex analysis result, that the positivity hypothesis in the statement of Theorem 1 is equivalent to a "uniform" inequality (6), which is an *a priori* stronger condition.

Degeneration of a conic

Stability of complex projective manifolds

The theorem of the previous section gives, in a sense, a complete answer to the question of the existence of extremal Kähler metrics on toric manifolds. It fits into a larger picture, for general projective manifolds, where the final answer is not yet known.

Recall that any convex set in $\mathbf{R}^n$ defines a graded ring. Let f be a convex function on our polytope $\overline{P}$ and define a convex subset Q of $\mathbf{R}^{n+1} = \mathbf{R}^n \times \mathbf{R}$ by

$$Q = \{(x,h) \in \overline{P} \times \mathbf{R} : h \geq f(x)\},$$

so we get a ring R_Q. The translation $(x,h) \mapsto (x,h+1)$ induces the structure on R_Q of an algebra over the polynomial ring $\mathbf{C}[t]$. We can also obtain R_Q as the *Rees algebra* of a filtration of the graded $R_{\overline{P}}$. In general, let R be an algebra over $\mathbf{C}$ with a filtration by vector subspaces

$$0 = \mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots,$$

such that

$$\mathcal{F}_a.\mathcal{F}_b \subset \mathcal{F}_{a+b}. \tag{8}$$

The Rees algebra is the algebra over $\mathbf{C}[t]$

$$\mathrm{Rees}(R,\mathcal{F}_*) = \bigoplus_a \mathcal{F}_a t^a \subset R[t].$$

In the case at hand, let Σ_a be the set of lattice points

$$\Sigma_a = \{(x,k) \in \mathbf{Z}^n \times \mathbf{Z} : k \geq 0, x \in k\overline{P}, f(x/k) \leq a/k\}.$$

This is a subset of the set of lattice points defining $R_{\overline{P}}$ and we define $\mathcal{F}_a$ to be the subspace spanned by these basis elements. The convexity of f implies that $\Sigma_a + \Sigma_b \subset \Sigma_{a+b}$ and this gives the multiplicative property (8). From the definitions, the Rees algebra of this filtration of $R_{\overline{P}}$ is canonically identified with R_Q.

Suppose that $\overline{P}$ is an integral Delzant polytope and that f is a piecewise linear convex function of the form

$$f(x) = \max(\mu_1(x), \ldots, \mu_r(x)) \tag{9}$$

where μ_i are affine-linear functions with integral coefficients. Then the Rees algebra R_Q has an algebro-geometric interpretation. It is finitely generated over $\mathbf{C}[t]$ and the Proj construction over $\mathbf{C}[t]$ defines a variety $\mathcal{X} \subset \mathbf{CP}^N \times \mathbf{C}$. Projection to the second factor gives a map $\pi : \mathcal{X} \to \mathbf{C}$ with the property that for $t \neq 0$ the fibre $\pi^{-1}(t)$ is a copy of our complex manifold X but the central fibre $\pi^{-1}(0)$ is a different variety: a degeneration of X. For example, if $\overline{P}$ is the interval $[-1,1]$ in $\mathbf{R}$—so X is the Riemann sphere embedded as a conic curve in $\mathbf{CP}^2$—and f is the function $f(x) = \max(x,-x)$ the degeneration has central fibre a pair of lines in the plane; a singular conic. In general a function of the form (9) defines a decomposition of $\overline{P}$ into a union of convex pieces on each of which f is affine-linear, and the central fibre is a reducible variety with components corresponding to these pieces. From the more algebraic point of view, for any filtered algebra R one considers $\mathrm{Rees}(R,\mathcal{F}_*) \otimes_{\mathbf{C}[t]} \mathbf{C}$, where $\mathbf{C}[t]$ acts on $\mathbf{C}$ by evaluating t at some $\tau \in \mathbf{C}$. If $\tau \neq 0$ this tensor product is isomorphic to R but for $\tau = 0$ it is the associated graded ring

$$\bigoplus_a \mathcal{F}_a/\mathcal{F}_{a-1}.$$

The differential-geometric and algebro-geometric constructions we have encountered all extend beyond the toric case. For any complex projective manifold X there is a *Mabuchi functional* on the space of Kähler metrics and the problem of finding an extremal metric is the problem of minimising this functional. The work of Chen and Cheng shows that the existence of a minimiser is equivalent to a coercive inequality like (7) but a complete algebro-geometric criterion for this is not yet known.

Whatever the final answer may be it must be bound up with algebro-geometric notions of "stability" which stretch back to Mumford's Geometric Invariant Theory from the 1960's and, further, to Hilbert. In place of the positivity criterion on convex functions we expect to see a criterion involving filtrations of the co-ordinate ring $R(X)$. Filtrations which satisfy a finite-generation condition correspond to degenerations of X and there is a numerical invariant of these—the Futaki invariant—which reduces in the toric case to $L(f)$. The manifold X is called *K-stable* if the Futaki invariant is positive for all non-trivial degenerations. In the toric case this corresponds to the positivity of $L(f)$ for all functions f of the form (9) where the μ_i have rational co-efficients. (Multiplying $\overline{P}$ by a scale factor one can then reduce to the case of integral co-efficients.) The extension to more general filtrations was made by Székelyhidi in [7] (whose treatment we have followed above). This leads to a strengthening of the notion of K-stability to $\hat{K}$-stability, which corresponds in the toric case to the positivity criterion in our Theorem. In another direction, there is a notion of *uniform K-stability* which corresponds in the toric case to the existence of an inequality (6). In the toric case, $\hat{K}$-stability, uniform K-stability and the existence of an extremal metric are all equivalent, and perhaps the same will turn out to be true in general. For some classes of manifolds, such as Fano manifolds, the condition of K-stability is also equivalent but this is not expected to be true in general.

In any case there is much current activity in this area and much to be done, both in proving abstract existence theorems and in understanding more deeply these interactions between algebraic and differential geometry.

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The Enduring Appeal of Szemerédi's Theorem

TIMOTHY GOWERS

Szemerédi's theorem on arithmetic progressions is a centerpiece of additive combinatorics and continues to stimulate developments in the area. In this article I try to give some idea of why it has had so many ramifications and of what some of them are.

What does the theorem say?

With the exception of Ramsey's theorem itself, the best known result of Ramsey theory is the following very appealing theorem of van der Waerden from 1927.

Theorem 1. *For every pair of positive integers r and k there exists a positive integer n such that however the integers from 1 to n are coloured with r colours, there is necessarily an arithmetic progression of length k consisting of numbers of the same colour.*

The proof of the theorem is a clever double induction, which yields a bound for n that has an Ackermann dependence on k. It is of course tempting to ask whether such a huge bound was necessary — a question I shall return to later.

In 1936, Erdős and Turán conjectured that a significantly stronger statement was true. Their conjecture was the following statement, which remained open until 1975 but is now a famous theorem of Szemerédi.

Theorem 2. *For every $\delta > 0$ and every positive integer k there exists a positive integer n such that every set $A \subset \{1, 2, ..., n\}$ of size at least δn contains an arithmetic progression of length k.*

To see that this implies van der Waerden's theorem, one simply sets $\delta = 1/r$ and observes that at least one of the colour classes must be a set of size at least $1/r$. Szemerédi's theorem is often referred to as the *density version* of van der Waerden's theorem.

There seems to be no way of modifying the double-induction argument to yield the density version, which raised the hope (and this was part of the reason Erdős and Turán asked the question) that a proof of the conjecture would lead to better bounds for van der Waerden's theorem. Initially, however, this hope was not fulfilled: in fact, Szemerédi used van der Waerden's theorem as a lemma in his proof.

A further conjecture, possibly the most famous of all of Erdős's problems, was the following.

Conjecture 1. Let $A \subset \mathbb{N}$ be a set such that $\sum_{a \in A} a^{-1} = \infty$. Then for every k, A contains an arithmetic progression of length k.

This is a striking statement, and, as Erdős observed, since the sum of the reciprocals of the primes diverges, it would imply that the primes contained arbitrarily long arithmetic progressions. However, it is roughly equivalent to a more mundane statement that is a bit more transparent. For each k, Szemerédi's theorem guarantees the existence of a function $\delta_k : \mathbb{N} \to \mathbb{R}$ that tends to zero such that if $A \subset \{1, ..., n\}$ has density at least $\delta_k(n)$ (this means simply that $n^{-1}|A| \geq \delta_k(n)$), then A contains an arithmetic progression of length k. We can think of $\delta_k(n)$ as the density required to guarantee a progression of length k. Erdős's conjecture is more or less the same as the statement that for each k, δ_k tends to zero at least as fast as $1/\log n$, which, given the prime number theorem (or even just Chebyshev's theorem) explains why it has consequences for the primes. However, there is no particular reason to believe that $1/\log n$ is an important threshold for Szemerédi's theorem — this again is a question to which I shall return.

What makes Szemerédi's theorem so interesting?

If a theorem is described as a centerpiece for an entire branch of mathematics, one might expect it to have a large number of applications. However, although it does occasionally happen that one is presented with a dense set of integers and can exploit the fact that the set contains a long arithmetic progression, that is not a very common situation, and there are other results in combinatorics that are used far more often. So why is it that Szemerédi's theorem has captured the imagination of so many people? There are two main reasons.

The first is what one might call indirect applications. There is something about the contrast between the simplicity of the statement and the difficulty of proving it that has stimulated the discovery of a number of different proofs. Each of these proofs has led to the development of tools that themselves have had many applications. Thus, one can confidently say that in the unlikely event that nobody had formulated the Erdős-Turán conjecture, combinatorics would be a much less rich subject than it is today.

The second is that Szemerédi's theorem has been generalized and strengthened in several different directions. It is now just one of a large number of density theorems, some of them very surprising.

I shall talk about these two aspects of the theorem in turn.

Proofs of Szemerédi's theorem

The first proof of a non-trivial special case of Szemerédi's theorem was due to Roth in 1953. He used the circle method from analytic number theory to prove the theorem for progressions of length 3, though his argument is now normally presented as an application of discrete Fourier analysis. Very roughly, the idea is to take the characteristic function of a set $A \subset \{1, \ldots, n\}$, regard it as a function defined on the cyclic group $\mathbb{Z}/n\mathbb{Z}$, and look at the discrete Fourier transform of that function. If there are no unexpectedly large Fourier coefficients, then A behaves rather like a random set, and therefore contains many arithmetic progressions of length 3. If not, then one can show that A is "biased", meaning that there is a long arithmetic progression P such that $|A \cap P|/|P|$ is noticeably larger than $|A|/n$. One can then restrict to P and start again. This leads to an iteration that has to terminate, since the density cannot increase beyond 1, so at some point one obtains a progression of length 3.

Unfortunately, Roth's proof did not generalize to longer progressions. The difficulty was that it is possible for A to have only very small Fourier coefficients (except a trivial one at zero) and yet to behave very *unlike* a random set from the point of view of containing progressions of length 4. An example of a problematic set of this kind is the subset A of $\mathbb{Z}/p\mathbb{Z}$ (where p is a large prime) that consists of all x such that $x^2 \in [-\theta p, \theta p]$ mod p, where θ is some reasonably small constant. The identity $x^2 - 3(x+d)^2 + 3(x+2d)^2 - (x+3d)^2 = 0$ implies that if $x, x+d$, and $x+2d$ all belong to this set, then $x+3d$ has an unexpectedly good chance of belonging to the set as well — much greater than the probability 2θ one would expect if A was a random set.

In 1969, Szemerédi found a very different proof of Roth's theorem, and this was to be the starting point for the purely combinatorial argument that he went on to obtain. Interestingly, the notion of sets "behaving as though they are random" played an important role in this argument as well. A key lemma in the proof was a graph-theoretic statement that says, very roughly, that the vertex set of any dense graph can be partitioned into a bounded number of pieces in such a way that almost all the bipartite graphs one obtains from two of those pieces "look like random graphs". (One can also insist that the pieces are not too large, so as to rule out trivialities such as taking the entire vertex set as a single piece.) This lemma was later developed into Szemerédi's regularity lemma, one of the most useful tools in extremal graph theory, which has had innumerable applications.

In 1977, a couple of years after Szemerédi's theorem was proved, Hillel Furstenberg found a second proof, and in so doing created a new subdiscipline of mathematics, now often called ergodic Ramsey theory. His starting point was to show that Szemerédi's theorem is equivalent to a statement about dynamical systems. A *measure-preserving* system is a set X with a probability measure μ on it, and a map $T : X \to X$ with the property that $\mu(T^{-1}A) = \mu(A)$ for every measurable subset $A \subset X$. Furstenberg proved that for each k, Szemerédi's theorem for progressions of length k is equivalent to the following statement.

Theorem 3. *Let (X, μ, T) be a measure-preserving system and let $A \subset X$ be a set of positive measure. Then for any positive integer k there exists n such that $\mu(A \cap T^{-n}A \cap \cdots \cap T^{-(k-1)n}A) > 0$.*

The proof of this equivalence, which is known as Furstenberg's correspondence principle, turns out not to be too difficult. What is considerably less obvious is how to prove this ergodic reformulation of Szemerédi's theorem. Furstenberg did it by a two-step argument: either the system is "weak mixing", which roughly speaking means that all sets get spread about under repeated applications of

T^{-1}, or it has a non-trivial "compact factor", which roughly speaking means that it can be decomposed as a kind of product, one term of which is highly structured. Thus, even though the proof is quite different from Szemerédi's proof, and also from Roth's proof for progressions of length 3, it is still based on a contrast between random-like behaviour and structured behaviour: random-like systems behave nicely, and systems that are not random-like have some kind of correlation with structured sets, which can be exploited in an iteration.

As I mentioned earlier, Szemerédi used van der Waerden's theorem as a lemma, which resulted in a bound that was even worse than the Ackermann function — as far as I am aware nobody has worked out what it was. As for Furstenberg's argument, the correspondence principle was proved via a non-constructive limiting argument, so it did not lead to quantitative bounds even in principle. (Much later Terence Tao managed to find a quantitative argument that was strongly inspired by Furstenberg's proof, but the bound from this argument was still at least of Ackermann type.)

At this point, the only reasonable quantitative bounds were for the case $k = 3$: Roth's argument shows that $\delta_3(n)$ is at most $C(\log\log n)^{-1}$ for an absolute constant C, and Szemerédi's proof for this case also gives a sensible bound. In the late 1990s I managed to find a way of developing Roth's approach to give a proof first for progressions of length 4 and later for progressions of all lengths. The example presented earlier gives a small hint about the ideas that were involved: discrete Fourier analysis involves functions of the form $x \mapsto e^{2\pi i\alpha x}$, but proves to be inadequate for dealing with progressions of length 4 owing to the existence of "quadratic examples". To deal with these, one needs to introduce functions of the form $x \mapsto e^{2\pi i(\alpha x^2+\beta x)}$ and to develop a kind of "quadratic Fourier analysis". However, it turns out that these functions, which are known as *pure quadratic phase functions*, are not enough, and one needs to consider "generalized quadratic phase functions" as well. To give an example of such a function, let $n = m^2$ and for each $x \in \{0,1,\dots,n-1\}$ write it in base m — that is, choose $y,z \in \{0,1,\dots,m-1\}$ such that $y + mz = x$. Then define $\phi(x)$ to be $e^{2\pi i(\alpha y^2+\beta yz+\gamma z^2)}$. The exponent resembles a quadratic form in two dimensions, and gives a flavour of the kinds of functions that arise in the proof.

The broad strategy of the proof was again to show that either the dense set A was random-like or it had some structure that could be exploited. In the case of progressions of length 4, that structure was closely related to the generalized quadratic phase functions just mentioned — in fact, by a later theorem of Green and Tao, one can show that if A is not quasirandom in the relevant sense, then its characteristic function must correlate in a non-trivial way with such a function. I proved a weaker statement that was still sufficient to deduce that A has increased density inside some long progression.

This approach led to the first quantitative bounds for the general case of Szemerédi's theorem: the smallest density $\delta_k(n)$ needed to guarantee a progression of length k in a subset of $\{1,2,\dots,n\}$ is at most $(\log\log n)^{-1/C_k}$, where $C_k = 2^{2^{k+9}}$.

A few years earlier, Shelah had found a new combinatorial proof of van der Waerden's theorem that improved the bound from an Ackermann bound to the first known primitive recursive bound. Where addition is obtained by repeatedly taking the successor function, multiplication is repeated addition, and exponentiation is repeated multiplication, the tower function is obtained by repeated exponentiation: for example, one can define $T(n)$ inductively by setting $T_1 = 1$ and $T(n) = 2^{T(n-1)}$. Shelah's bound for van der Waerden's theorem was the next level up in the hierarchy: that is, it was of so-called "wowzer" type, which one gets by setting $W(1) = 1$ and $W(n) = T(W(n-1))$. My proof of Szemerédi's theorem improved that to a six-fold exponential.

It is not obvious how to obtain multidimensional structure from the assumption that the set A is not quasirandom (in an appropriate sense that I shall not discuss here, though it was very important to the argument). However, it turned out to be possible to use a remarkable result of Gregory Freiman, or more precisely some equally remarkable ideas from a proof of Freiman's theorem due to Imre Ruzsa. There is not space to explain how to use the result, but I can at least state it, and demonstrate in that way that results that yielded multidimensional structure existed already.

If A is a set of integers, the *sumset* $A + A$ is defined to be the set $\{x + y : x, y \in A\}$. It is a simple exercise to show that if $|A| = n$, then $|A + A| \geq 2n - 1$, with equality if and only if A is an arithmetic progression. In the other direction, $A + A$ clearly cannot have size greater than $n(n+1)/2$. Freiman's theorem concerns what can be said about A if $|A + A| \leq C|A|$, where one thinks of C as a fixed constant and A as being large. As just remarked, arithmetic progressions give examples of such sets, but so do slightly more general sets called multidimensional arithmetic progressions. Here is a precise definition.

Definition 1. Let k be a positive integer. A k-*dimensional arithmetic progression* is a set of the form $\{x_0 + \sum_{i=1}^{k} x_i d_i : 0 \le x_i \le m_i\}$ for some $d_1, \ldots, d_k$.

For example, the set $\{1, 2, 3, 11, 12, 13, 21, 22, 23, 31, 32, 33\}$ is a 2-dimensional arithmetic progression, as, slightly less obviously, is the set $\{3, 8, 10, 13, 15, 17, 20, 22, 27\}$. (It is not required by the definition that all the numbers $x_0 + \sum_{i=1}^{k} x_i d_i$ should be distinct: when they are, we say that the progression is *proper*.)

It can be shown that if A is a k-dimensional arithmetic progression, then $|A + A| \le 2^k|A|$ (this is easy when the progression is proper). Freiman's theorem tells us that these are essentially the only examples.

Theorem 4. *For every C there exist constants k and K such that if $A \subset \mathbb{Z}$ is any set with $|A+A| \le C|A|$, then there is a multidimensional arithmetic progression P of dimension at most k such that $A \subset P$ and $|P| \le K|A|$.*

Yet another approach to Szemerédi's theorem arose out of the following deceptively simple result of Ruzsa and Szemerédi, which is known as the triangle removal lemma.

Theorem 5. *For every $\epsilon > 0$ there exists $\delta > 0$ such that every graph G with n vertices that contains at most δn^3 triangles contains a set H of at most ϵn^2 edges such that $G \setminus H$ is triangle free.*

Like Szemerédi's theorem, this is a result that one would expect to be either false or fairly easy to prove. An indication that that is not the case is that there is a huge gap between the best known upper and lower bounds for δ: it is known that the smallest possible value of δ^{-1} is at least $\log(\epsilon^{-1})^2$ and at most $T(C\log(\epsilon^{-1}))$, where T is the tower function mentioned earlier. (The lower bound follows from a 1946 construction of Behrend and the upper bound is due to Jacob Fox.)

The connection with Szemerédi's theorem is that the triangle removal lemma implies Roth's theorem. If $A \subset \{1, 2, \ldots, n\}$ is a set with no arithmetic progression of length 3, one can construct a tripartite graph with vertex sets X, Y and Z, each of which is a copy of $\{1, 2, \ldots, 2n\}$. We join $x \in X$ to $y \in Y$ if $y - x \in A$, $y \in Y$ to $z \in Z$ if $z - y \in A$, and $x \in X$ to $z \in Z$ if $(z - x)/2 \in A$. A triangle in this graph gives us a triple (x, y, z) such that $y - x, z - y$, and $(z - x)/2$ all belong to A. But $(y - x) + (z - y) = 2(z - x)/2$, so we obtain an arithmetic progression, which is non-degenerate unless $y - x = z - y$. It is not hard to check that the number of triangles is $O(n^2)$, that there are at least $|A|n$ triangles that correspond to degenerate arithmetic progressions, and that those triangles are edge disjoint. If A is dense and n is sufficiently large, this contradicts the triangle removal lemma, since it yields a graph with very few triangles with the property that in order to destroy all triangles one must remove a large number of edges.

It was natural to try to generalize this proof to longer progressions, but that turned out to be surprisingly difficult. The relatively easy part is to come up with an appropriate analogue of the triangle removal lemma. A k-*uniform hypergraph* is a collection of subsets of size k of a set X (so a 2-uniform hypergraph is just a graph). The elements of X are called *vertices* and the subsets are called *edges*, or sometimes *hyperedges*. A *simplex* in a k-uniform hypergraph is a set of $k+1$ vertices such that any k of them form an edge. And now the *simplex removal lemma* is the obvious generalization of the triangle removal lemma.

Theorem 6. *For every $\epsilon > 0$ and every positive integer $k \ge 2$ there exists $\delta > 0$ such that if H is any k-uniform hypergraph with n vertices that contains at most δn^{k+1} simplices, then there is a set J of at most ϵn^k edges of H such that $H \setminus J$ is simplex free.*

The statement for a given k implies Szemerédi's theorem for progressions of length $k+1$. In fact, as observed by József Solymosi, it implies the case for that k of the following multidimensional statement, which itself implies Szemerédi's theorem for progressions of length $k+1$. (The first implication is not trivial but it is not hard either. The second is a simple exercise.) Let us write $e_1, \ldots, e_k$ for the standard basis vectors of $\mathbb{R}^k$.

Theorem 7. *For every $\delta > 0$ and every positive integer k there exists n such every subset A of $\{1, 2, \ldots, n\}^k$ of size at least δn^k has a subset of the form $\{x, x + de_1, \ldots, x + de_k\}$ for some x and some non-zero d.*

Configurations of the form $\{x + de_1, \ldots, x + de_k\}$ are known as *corners*, so this result says that dense sets must contain corners.

The difficult part of the hypergraph approach is to prove the simplex removal lemma. This turns out to require generalizing Szemerédi's regularity lemma appropriately to hypergraphs. The word "appropriately" is important here, since it is not possible in general to split a hypergraph up into a bounded number of pieces that look like random hypergraphs. Instead one has to allow for slightly more complicated pieces: a typical example is to take three disjoint sets X, Y, Z of vertices, to take random-like bipartite graphs, possibly of different densities, joining X to Y, Y to Z, and X to Z, and to pick a random-like subset of the set of all triples xyz that form triangles in the resulting tripartite graph. The simplex removal lemma was proved independently by Nagle, Rödl, Schacht and Skokan, and by me.

A fifth proof of Szemerédi's theorem is actually a proof of a much more general result known as the density Hales-Jewett theorem. I shall discuss it in the next section. And finally there is a sixth proof due to Elek and Szegedy that builds on the theory of graph limits developed by Lovász and Szegedy. They define a notion of a "hypergraphon" that allows them to prove the simplex removal lemma in a clean infinitary way that avoids having to keep careful control of multiple parameters.

Generalizations and strengthenings of Szemerédi's theorem

We now come to a selection of remarkable results that go beyond Szemerédi's theorem.

To begin with, the theorem about k-dimensional corners mentioned in the previous section is a special case of the following multidimensional version of the theorem.

Theorem 8. *For every $\delta > 0$, every positive integer k, and every finite subset $K \subset \mathbb{Z}^k$ there exists n such that every subset $A \subset \{1, 2, \ldots, n\}^k$ of size at least δn^k has a subset of the form $x + dK$ with $d \neq 0$.*

The corners theorem is the case where $K = \{0, e_1, \ldots, e_k\}$. It turns out that the corners theorem implies the multidimensional Szemerédi theorem, so the two are in fact equivalent. (This is a fairly simple exercise to show — one exploits the fact that there is an affine map from a $(|K|-1)$-dimensional simplex to K.)

The multidimensional Szemerédi theorem can itself be generalized a lot further. To explain how, let us briefly return to van der Waerden's theorem. If one examines the original proof of the theorem, one finds that essentially the same argument establishes a result about what one might call multidimensional noughts and crosses. This is played on an n-dimensional grid of width k in each direction. That is, the board is the set $\{1, 2, \ldots, k\}^n$. A *line* in the geometric sense is a set of k points in this set that lie in a straight line. One can check that if $x_1, \ldots, x_k$ is such a line, with $x_i = (x_{i1}, \ldots, x_{in})$, then for each coordinate j, either the x_{ij} are all equal, or $x_{ij} = i$ for all i, or $x_i = k + 1 - i$ for all i. For instance, in a 3×3 grid, the points $(1, 3, 2), (2, 2, 2), (3, 1, 2)$ form a line. A line is called a *combinatorial line* if the third possibility is disallowed: that is, for each coordinate j either the x_{ij} are all equal or $x_{ij} = i$ for all i.

The following theorem is called the Hales-Jewett theorem.

Theorem 9. *For every k and every r there exists n such that if the points of $\{1, 2, \ldots, k\}^n$ are coloured with r colours, then there is a combinatorial line with all its points of the same colour.*

It is straightforward to see that this implies van der Waerden's theorem: it is more convenient to colour the points of $\{0, 1, \ldots, k - 1\}$ instead, and then one can regard them as base-k expansions of the numbers $0, 1, \ldots, k^n - 1$. If one does this, then combinatorial lines map to arithmetic progressions of length k. With only a very slightly harder argument one can also deduce the multidimensional van der Waerden theorem.

Thus, van der Waerden's theorem can be generalized in two directions. One direction gives us Szemerédi's theorem, its density version, while the other gives us the Hales-Jewett theorem. It is natural to ask whether both generalizations can be carried out at the same time. This was an open problem for quite a while until it was solved by Furstenberg and Katznelson, in one of the triumphs of the ergodic-theoretic approach.

Theorem 10 (Density Hales-Jewett theorem). *For every $\delta > 0$ and every k there exists n such that every subset of $\{1, 2, \ldots, k\}^n$ of size at least δk^n contains a combinatorial line.*

The ergodic approach gave no bounds, so it remained a challenge to find a combinatorial proof. This was eventually achieved in 2009 as a result of the first Polymath project — an open collaboration carried out online. Just as van der Waerden's theorem is an easy consequence of the Hales-Jewett theorem, Szemerédi's theorem is an easy consequence of the density Hales-Jewett theorem. Rather unexpectedly, the Polymath proof of the density Hales-Jewett theorem was not significantly harder for progressions of arbitrary length than it was for progressions of length 3, which made it the first proof with this property.

We have just seen two orthogonal directions in which van der Waerden's theorem can be generalized. Remarkably, there is a third direction as well, a story that begins with the following theorem that was proved independently by Sárközy and Furstenberg.

Theorem 11. *For every $\delta > 0$ there exists n such that for every subset $A \subset \{1, 2, ..., n\}$ of size at least δn there exist a and $d \neq 0$ such that a and $a + d^2$ both belong to A.*

This is of course a density statement, but it implies a corresponding colouring statement: if $\{1, 2, ..., n\}$ is coloured with a bounded number of colours then we can find a and $d \neq 0$ such that a and $a + d^2$ have the same colour.

How far can this statement be generalized? Can we replace d^2 by other polynomial functions of d, for instance? A moment's thought shows that we have to be slightly careful, since for example we cannot hope to find a and d such that a and $a + d^2 + 1$ have the same colour: we can colour numbers with three colours according to their value mod 3. However, the strongest imaginable result that takes this kind of simple restriction into account turns out to be true. The following result, which was a conjecture for several years, was proved by Bergelson and Leibman.

Theorem 12. *For every $\delta > 0$ and every finite set $\{P_1, ..., P_k\}$ of polynomials with integer coefficients and no constant term there exists n such that every subset A of $\{1, 2, ..., n\}$ of size at least δn has a subset of the form $\{a, a + P_1(d), ..., a + P_k(d)\}$ with $d \neq 0$.*

Interestingly, before they proved the above theorem, Bergelson and Leibman were able to use the ergodic machinery to establish that it would follow from the corresponding colouring version, which is often referred to as the polynomial van der Waerden theorem. (Given that Szemerédi's theorem was proved long after van der Waerden's theorem, this was a far from trivial observation.) But obtaining the polynomial van der Waerden theorem remained a difficult challenge. To meet it, they used methods of topological dynamics. Later, Mark Walters found a very appealing purely combinatorial proof (which used some of the ideas from Bergelson and Leibman's argument).

It is now difficult not to ask whether there is a simultaneous generalization of the Hales-Jewett theorem and the polynomial van der Waerden theorem. And indeed there is. Since it takes a bit of effort to state, I shall content myself here with a special case, which gives a good flavour of the general result. As is customary in combinatorics, let us write $[n]$ for the set $\{1, 2, ..., n\}$. Note that the set $[k]^{[n]^2}$ consists of functions from $[n]^2$ to $[k]$, which we can think of as $n \times n$ matrices with entries in $[k]$.

Theorem 13. *For every k and every r there exists n such that if the points of $[k]^{[n]^2}$ are coloured with r colours, then there exists $a \in [k]^{[n]^2}$ and a non-empty set $X \subset [n]$ such that all points $b \in [k]^{[n]^2}$ that agree with a outside X^2 and are constant inside X^2 have the same colour.*

Note that this is saying that there is a combinatorial line with the property that the variable coordinates form a set of the form X^2 — that is, a square in the Cartesian-product sense.

If we consider sums of the form $\sum_{i,j} \mu_i \mu_j b_{ij}$ for appropriately chosen integer coefficients μ_i, we see that adding 1 to every b_{ij} with $i, j \in X$ adds $(\sum_{i \in X} \mu_i)^2$ to the sum. Exploiting this, one can show that the above theorem implies the special case of the polynomial van der Waerden theorem where the polynomials are $x^2, 2x^2, ..., (k-1)x^2$, which yields an arithmetic progression of length k with square common difference.

Again, Bergelson and Leibman proved the polynomial Hales-Jewett theorem using topological dynamics, and again Mark Walters found a purely combinatorial proof.

We now have three directions of generalization for van der Waerden's theorem — the density direction, the Hales-Jewett direction, and the polynomial direction. Moreover, any two of these directions can be simultaneously generalized. Can all three be generalized? It is not hard to formulate an appropriate statement — one simply takes the obvious density version of the polynomial Hales-Jewett theorem. But whether or not the statement is true is an open problem. Indeed, even the simplest non-trivial cases of the polynomial Hales-Jewett theorem are (very interesting) unsolved problems.

The alert reader will be remembering that earlier in this article I mentioned yet another direction in which Szemerédi's theorem could be generalized, which was the hypergraph direction. Szemerédi's theorem follows straightforwardly from the simplex removal lemma, but when one examines the simplex removal lemma one realizes that it is a considerably more general statement that has several other consequences besides Szemerédi's theorem. However, there does not seem to be an obvious way of integrating this direction of generalization into the picture just described: I know of no serious attempt to find a simultaneous generalization of this direction with either the polynomial direction or the Hales-Jewett direction. It would be fascinating if even a sensible conjecture could be formulated, but it may be that these are simply incompatible generalizations in some sense.

No account of generalizations of Szemerédi's theorem would be complete without a mention of a famous generalization of a very different kind, which is perhaps better described as a strengthening than a generalization. At the beginning of this article I mentioned that a sufficiently good bound for Szemerédi's theorem would imply that the primes contain arbitrarily long arithmetic progressions. Thanks to Ben Green and Terence Tao, we now know that this consequence is indeed true.

Theorem 14. *For every positive integer k there is an arithmetic progression of length k that consists of prime numbers.*

Green and Tao did not prove this by beating the $1/\log n$ barrier in Szemerédi's theorem — that is still an open problem. Rather, they exploited known results about the distribution of the prime numbers. Very roughly (I shall oversimplify here) the scheme of their proof was to prove that the prime numbers "sit densely inside a pseudorandom set". They then showed that results about dense subsets of $\{1,2,...,n\}$ can be converted in a systematic way into results about dense subsets of pseudorandom subsets of $\{1,2,...,n\}$. Thus, they ended up using Szemerédi's theorem as a black box, though in order to do so they made use of insights from several of the different proofs of the theorem — which provides another answer to the question of why it was worth finding so many different proofs of one result.

The Green-Tao theorem does not itself strengthen Szemerédi's theorem, but their proof applies just as well to dense subsets of the primes. Since a dense set will on average densely intersect a random translate of the primes, it is therefore straightforward to deduce Szemerédi's theorem from their result. (However, this does not give an alternative proof of Szemerédi's theorem given that they used the theorem.)

Their proof did not give an asymptotic for the number of progressions of length k in the first n primes. That turned out to be a significantly harder challenge, which took several years of further research and led to the discovery of a number of deep results, of which the highlight, proved with Tamar Ziegler, is their "inverse theorem for the U^k norms", which gives a complete description of the sets that fail to be quasirandom in the Fourier-analytic proof of Szemerédi's theorem described earlier.

The Green-Tao theorem has also been combined with other directions of generalization. For example, Tao proved a 2-dimensional version for Gaussian primes, and Tao and Ziegler have proved a polynomial version (that is, with the conclusion of the Bergelson-Leibman theorem but for dense sets of primes).

A word about bounds

With the exception of the Bergelson-Leibman theorem, all the results mentioned above have been proved with quantitative bounds. However, in no case do the best known upper and lower bounds match. A particularly striking open problem that has attracted a great deal of attention is to determine the right form of the bound for Roth's theorem. As mentioned earlier, Roth's proof shows that to guarantee a progression of size 3 in a subset of $\{1,2,...,n\}$ it is sufficient for the subset to have density at least $C/\log\log n$. In the other direction, the construction of Behrend mentioned earlier (in connection with the triangle removal lemma) shows that a density of $\exp(-C\sqrt{\log n})$ does not suffice. Between those two bounds there is a huge gap, which has been closed somewhat, but remains huge. The lower bound has remained virtually stationary, but the upper bound has been reduced in a long sequence of papers by Szemerédi, Heath-Brown, Bourgain, Sanders, Bloom, and Bloom and Sisask, each of which introduced very interesting and influential new ideas. The last paper in this sequence, by Thomas Bloom and Olof Sisask, is particularly notable because it breaks the logarithmic barrier, and hence has as a consequence the first non-trivial case of Erdős's famous conjecture. That is, we now know that if A is a set of integers such that $\sum_{a\in A} a^{-1} = \infty$, then A contains an arithmetic progression of length 3.

Another problem that has been resolved but only fairly recently concerns the following variant of Roth's theorem. We write $\mathbb{F}_3$ for the field with three elements.

Theorem 15. *For every $\delta > 0$ there exists n such that every subset of $\mathbb{F}_3^n$ of density at least δ contains an affine line.*

As was first shown by Roy Meshulam, this can be proved by a modification of Roth's proof for subsets of $\{1, 2, \ldots, n\}$. In fact, the technical details are simpler and cleaner for this problem, and one can obtain a bound of C/n for the density required. (This should be regarded as logarithmic, since $\mathbb{F}_3^n$ has size 3^n.)

In the opposite direction, the best known lower bound for the density was of the form c^n for a constant $c < 1$. (A simple lower bound of $(2/3)^n$ is given by the set $\{0, 1\}^n$, but the constant $2/3$ can be improved by taking a Cartesian product of a more complicated small example of a set with no affine line in some small $\mathbb{F}_3^k$.)

This again was a tantalizingly large gap. It was narrowed very slightly by Michael Bateman and Nets Katz, who improved the upper bound for the density to bound to a bound of the form $C/n^{1+\epsilon}$ for a fixed (but very small) $\epsilon > 0$, thus decisively breaking the logarithmic barrier for this problem. However, in a very surprising development a few years later, Jordan Ellenberg and Dion Gijswijt, building on an argument of Ernie Croot, Seva Lev, and Peter Pach that used the so-called polynomial method, obtained an upper bound of the form α^n with $\alpha < 1$, thus matching at least the form of the lower bound. Moreover, the proof was very short and easy to understand. This work has led to further very interesting questions: the method works spectacularly well for some problems, but when it fails, it seems to fail completely. For example, nobody has found a way of extending it to prove results about progressions of length greater than 3, or about corners. It is not clear whether that is because it cannot be extended in those directions or whether we are just missing the right idea for how to do so.

Interestingly, in order to obtain their breakthrough on Roth's theorem, Bloom and Sisask returned to the methods of Bateman and Katz, so that proof, though superseded by a much simpler argument that gave a much stronger bound, turned out nevertheless to have introduced very important ideas.

I mentioned above that there is no quantitative proof of the Bergelson-Leibman theorem. However, a great deal of work has been done on special cases. Another very interesting open problem concerns what the right bound is for the Furstenberg-Sárközy theorem. The current best known upper bound for the density needed to guarantee that a subset of $\{1, 2, \ldots, n\}$ contains two integers that differ by a perfect square is $(\log n)^{-c \log\log\log n}$, which is due to Thomas Bloom and James Maynard. However, the best known lower bound is of the form $n^{-\alpha}$ where α is a positive constant. The question is whether there is an upper bound of this form.

Sarah Peluse and Sean Prendiville have obtained several remarkable results (though to explain why they are remarkable I would have to go into more detail about proofs than I can in this article) concerning quantitative bounds for more complicated cases of the Bergelson-Leibman theorem. For example, they have shown that there is a constant $c > 0$ such that a density of $(\log\log n)^{-c}$ is sufficient to guarantee a configuration of the form $\{a, a + d, a + d^2\}$: this was the first quantitative bound for a special case that involved polynomials of different degrees, which causes all sorts of serious difficulties that they managed to circumvent with great ingenuity.

The bound obtained by Polymath for the density Hales-Jewett theorem was of Ackermann type. Another open question is whether one can somehow find an alternative proof that relates to that argument as Shelah's proof of the Hales-Jewett theorem relates to the original proof. Of course, any proof that gave a primitive recursive bound would be interesting.

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The Noether Theorems and their Application to Variational Problems on a Hyperbolic Surface

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Noether's theorem is a theorem in the classical calculus of variations which equates conservation laws with symmetries of the integrand. This article contains a brief history, and some unexpected applications to integrals whose domain is a symmetric space, in particular a hyperbolic surface.

The fundamental theorem of Emmy Noether in the calculus of variations has been important historically. She came to work in Göttingen in 1915 at the invitation of Hilbert and Klein. In the years 1915–1918 there was a healthy competition, particularly between Einstein and Hilbert, as the outlines of general relativity took shape. General relativity is correctly attributed to Einstein, but Hilbert made substantial contributions, particularly in his development of the variational principle [H]. One topic puzzled a number of mathematicians and physicists, Hilbert in particular. Why was there no conserved quantity corresponding to energy in general relativity? Noether was already known for her work on invariants, and was recruited by Hilbert and Klein to work on this problem. Noether surprisingly quickly produced a paper [N] settling the question so thoroughly that there is even today very little to be added. At the time, there was some debate about the importance of the variational approach, and the breadth of the applicability was not at all recognized. However, in the last 50 years her theorem has morphed into a widely recognized principle embedded in the philosophy of mathematics and physics: symmetries are associated with conservation laws. But it is still useful to look at the theorem itself. An excellent and complete reference containing both the history and the mathematics can be found in [K-S].

Her original theorem is strictly a theorem in the calculus of variations and has two distinct parts which are not always differentiated.

Part I: If an integral J in the calculus of variations is invariant under a [group] G_ρ (of dimension ρ), there are ρ linearly independent combinations among the Lagrangian expressions and their derivatives which become divergence free, and conversely ....

Part II: If an integral J in the calculus of variations is invariant under a [group] $G_{(\infty,\rho)}$ depending on ρ arbitrary functions and their derivatives up to order σ, then then there are ρ linearly independent identities among the Lagrangian expressions and their derivatives up to order σ, and conversely ....

In Part I of the theorem each symmetry produces, for every solution of the Euler-Lagrange equations, a local vector field V with $\operatorname{div} V = 0$ in any Euclidean coordinates one might choose to write the integral and functions, provided the symmetry is properly expressed in those coordinates.

Noether was nothing if not a generalist. The integrals under consideration depend on m dependent functions $y^k = f^k(x)$ and their derivatives $y^k_\alpha = \partial_\alpha f^k(x)$ through order $s \geq 1$, defined on a domain of $x = (x^1, \cdots, x^n)$ in $\mathbb{R}^n$. The integrals are multiple integrals with an integrand, known as a Lagrangian density, of the form

$$L\big(x, f(x), \{\partial_\alpha f(x)\}\big)(dx)^n.$$

The importance of the integrals lies in the system of Euler-Lagrange equations $1 \leq k \leq m$ for a function $u = (u^1, \cdots u^k, \cdots, u^m)$ where the variation of J vanishes. In particular, it could be a function at which J takes on a minimum, but most often only the variation of J vanishes:

$$\sum_\beta (-1)^{|\beta|}\partial_\beta[L_{(y^k_\beta)}(x, u, \cdots, \{\partial_\alpha u\})] = 0.$$

The Lagrangian expressions are the partial derivatives of L in the various directions y^k_β, but evaluated at the solution $f = u$ of the Euler-Lagrange equations.

The equations of general relativity require the number of derivatives $s = 2$. The functions on which the Einstein-Hilbert functional is defined are components of a metric tensor g and the Lagrangian density is the scalar curvature times the volume form induced by the metric.

We are not generalists and for the purposes of this note, we will restrict to $s = 1$ and consider only one derivative. Thus the Lagrangian densities depend only on functions and their first derivatives, which simplifies any discussion. For Noether, a symmetry is an infinitesimal symmetry which leaves the Lagrangian density $L\big(x, f(x), \partial f(x)\big)(dx)^n$ invariant (or invariant up to a divergence). It is not necessarily a symmetry preserving the integral.

It is important to realize in the recipe that the symmetries need only be local. For a problem in the calculus of variations based on t in $[0,1]$ that does not contain t explicitly (like $J(f) = \int_0^1 L(f(t), f'(t))\,dt$, translation in t does not preserve $[0,1]$ but it is still a local symmetry and Noether's theorem applies. How it works can be seen in her proof. Insert in the integral as the first variation the infinitesimal symmetry (the Lie algebra) acting on the solution to the Euler-Lagrange equations but multiplied by an arbitrary function ϕ which has local support. This vanishes because we are at a solution of the Euler-Lagrange equations. Because of the symmetry, only terms $\partial\phi$ (and higher derivatives in the case the integral depends on more than one derivative) are in the expression. There is no term with just ϕ. In the case of many derivatives, we freely integrate by parts until an expression like the one present for only one derivative

$$0 = \int \sum_j V^j d_{x^j} \phi \, (dx)^n$$

for all ϕ with support in a ball pops out. Because ϕ is arbitrary, what used to be called the fundamental theorem of the calculus of variations gives $\operatorname{div} V = 0$.

The importance of the divergence free vector fields associated with the solution of the Euler-Lagrange equations is the divergence theorem: if $\Omega \subset \mathbb{R}^n$, $\partial\Omega = \mathcal{S}$, then $\int_{\mathcal{S}} (\nu, V) = \int_\Omega \operatorname{div} V$, where ν is the outward unit normal.

Our modern language interprets this in terms of forms and, since Noether's arguments are entirely local, the theorem easily extends to manifolds. The Lagrangian density is an n-form, the vector field V is transformed into an $(n-1)$-form via $\theta = V \lrcorner (dx)^n$, and the divergence free vector fields V become closed $(n-1)$-forms θ. In turn, $\theta = d\xi$ locally for an $n-2$-form ξ. Later we will be particularly interested in $n = 2$, where ξ is a function.

Conservation laws for energy and momentum were well-known for many geometry and physics problems, in particular those of special relativity. A simple example is an integral in dimension $n = 1$ with the Lagrangian density $L(f(t), f'(t))dt$. As mentioned before, translation in time is a local symmetry. At a solution of the Euler-Lagrange equation u, the quantity

$$E(u) = \sum_k L_{(y_1^k)}(u(t), u'(t))\, u'^k(t) - L(u(t), u'(t))$$

is constant (in t). In higher dimensions, Lagrangian densities which do not depend on x yield for every solution of the Euler-Lagrange equations an energy-momentum tensor S: an $n \times n$ symmetric matrix with divergence free rows and columns. Noether's contribution was to show that the existence of this energy-momentum tensor is due to translation invariance in time for energy and in space for momentum (to say nothing of the additional interpretation of generalized momenta coming from symmetries in the target) and fits into a larger picture of symmetry and conservation laws. It is not clear how well this was understood in the physics community before the revival of interest in her work in the 60's.

The second part of Noether's theorem answers the question on why there are no conserved quantities in general relativity. The usual sources of conservation laws such as space and time translation are all embedded in the larger group of diffeomorphisms. So according to Part II of Noether's theorem, the resulting 'conservation laws' are identities, and hold whether or not one is at a critical point of the integral. Later Schouten and Struik [S-S] identified her identities as applied to general relativity as equivalent to the Bianchi identities (which were already known at the time of her article). This settles the question of conservation laws in general relativity (and for that matter, in gauge theory), although this has taken some time to be recognized. See the article by Rowe [Ro] for a more detailed discussion.

The question of gauge theory is surely an interesting one. A search of the web for *Noether, gauge theory* found an active discussion among physicists. No mathematicians. While some of the references are to the 'Noether identities' of Part II, it is difficult to see how, as these identities hold for all fields, information about specific structures can be derived from these identities. Whatever is happening at the quantum level is a result of the representations and most likely the philosophy arising from the theorem, not the theorem itself. For those curious, the classical identity that arises from applying her theorem to the Yang-Mills functional is either $D * D * F = 0$ or $\mathrm{trace}\,\mathrm{trace}[F, F] = 0$, both of which are self evident even to beginners in the subject. Here and elsewhere D denotes the covariant derivative form of the exterior derivative d on forms and $*$ denotes the operation which takes k-forms to $(n-k)$-forms.

My current interest in the subject arose in my collaboration with George Daskalopoulos in approximating 'best Lipschitz' maps between hyperbolic surfaces $f : M \to N$ studied by Thurston. To obtain approximations to the maps minimizing the Lipschitz constant of f, or equivalently the maximum of partial derivatives of f, we study integrals in variational problems which depend on a Shatten-von Neumann norm. Messing around in a fixed coordinate system on the image N, we found closed 1-forms. These were the derivatives of functions we were seeking. At some point we realized that these closed 1-forms in two dimensions were exactly the duals of the divergence free vector fields promised by Noether from the local symmetric space structure of N. A complete set of three closed 1-forms, one for each symmetry, popped out and we were off to the races. This article will appear shortly [D-U-2].

Here is a simple example. Let $f : M \to S^1$ and $J_p(f) = \int_M |df|^p * 1$, the critical points of which are called p-harmonic functions, be the integral under consideration. The symmetries in question are the rotations of the circle. With even a small amount of knowledge of the calculus of variations, it is not too hard to see that the Euler-Lagrange equations are

$$d * |du|^{p-2} du = 0.$$

One does not even have to refer to Noether's recipe to see that the closed $(n-1)$-form corresponding to rotations of the circle is

$$\theta = d\xi = *|du|^{p-2} du$$

where ξ is a $(\dim M - 2)$-form. It is easy to use this without crediting Noether (as we all do), but the case of N instead of S^1 is not so transparent. We are to make good use of these $n-1 = 1$-forms in two dimensions, but understanding the higher dimensional cases is wide open.

At some point, Daskalopoulos and I idly asked, well, what about the symmetries of the domain M? Now, these are not uncharted waters. For $M = \Omega \subset \mathbb{R}^n$, the translations are all local symmetries and Noether's theorem applies to give what physicists know as conservation of linear momentum (or energy if there is a direction representing time). Without crediting Noether, geometers for the last few decades have used these as the basis for monotonicity theorems, especially for harmonic maps and solutions of the Yang-Mills equations. They apply only approximately on curved manifolds, but that is sufficient for the estimates they are used for. Also, the proof is different from Noether's as there is often insufficient regularity for hers. A discussion of this background can be found in the paper by Bernard [B], which also presents another application in geometry of Noether's first theorem.

We can go back to the integral, now on Ω in $\mathbb{R}^n$,

$$J_p(f) = \int_\Omega |df|^p (dx)^n.$$

There are more modern proofs, but we give the classical argument. The recipe for finding the conservation law corresponding to translation in the x^j direction turns out in this case to be the inner product of the Euler-Lagrange equations with the partial derivative of $f = u$ in the direction x^j which we write $d_{x^j} u$. We get

$$\begin{aligned}(|du|^{p-2}\, du, d(d_{x^j} u)) &= (|du|^{p-2}\, du, d_{x^j} du)\\ &= 1/p\, d_{x^j}\, |du|^p.\end{aligned}$$

This expression is also the same as

$$\begin{aligned}d * (|du|^{p-2}\, du, d_{x^j} u) - (d * (|du|^{p-2}\, du), d_{x^j} u)\\ = d * (|du|^{p-2}\, du, d_{x^j} u)\end{aligned}$$

because we assume the Euler-Lagrange equations. So the divergence free (momentum) vector for the j-th direction is

$$S_{(i,j)} = |du|^{p-2}(d_{x^i} u, d_{x^j} u) - \frac{1}{p}\delta_{(i,j)}|du|^p.$$

It is a bit sticky taking the dual to obtain a closed $(n-1)$-form and to keep the symmetry, and dimensions greater than 2 are more difficult.

In dimension 2, we can take the dual (star) of both indices to obtain $S^*_{(1,1)} = S_{(2,2)}$, $S^*_{(2,2)} = S^*_{(1,1)}$ and $S^*_{(2,1)} = S^*_{(1,2)} = -S_{(2,1)} = -S_{(1,2)}$. Then elementary but somewhat tedious calculations based only on the local calculus fact that in $\mathbb{R}^2$, $d\theta = 0$ if and only if $\theta = d\xi$, results in the following:

Proposition 1. *If $\Omega \subset \mathbb{R}^n$ and J is an integral on $f : \Omega \to N$ well defined on some L^p_1 space and which is invariant under translation, then to every solution of the Euler-Lagrange equations there is an energy momentum tensor S with $d * S = 0$. If $n = 2$, then*

S^* calculated above satisfies $dS^* = 0$. Moreover any symmetric tensor S^* is closed (with respect to either index) if and only if there exists E such that $S^*_{(i,j)} = d_i d_j E$. The function E is unique up to addition by a term $ax^1 + bx^2 + c$.

This theorem is quite general and examples besides the one we give abound. Every solution of the Euler-Lagrange from an integral invariant under translation has an energy-momentum tensor, often even when the solution has minimal regularity.

So what about when the domain is a hyperbolic surface M?

The calculation we did above is well-known to geometers on an arbitrary manifold, symmetric or not, and can be show to yield an equation on an energy-momentum tensor S for many integrals with the domain a Riemannian manifold M. Such calculations are basic steps in studying harmonic maps and Yang-Mills equations, and they come from variations of the parameterization. (In general, S is a symmetric tensor with values in $T^*(M) \circledS T^*(M)$, and may only be in L^1. The equations are interpreted distributionally.) However, the equation is now

$$D^* S = 0.$$

To see the connection with Noether, let ω be an infinitestimal local isometry (Killing vector field). Note that (S, ω) is now a 1-form, and ${}^*(S, \omega)$ is an $(n-1)$-form. The following is an easy calculation.

Proposition 2. *If the domain is a symmetric space then $D^* S = 0$ if and only if $d^*(S, \omega) = 0$ for local Killing fields ω.*

An extension of this to conformally invariant integrals and conformal Killing fields appears in a number of places (without reference to Noether) [B-L], [P].

On a surface, if we construct the symmetric tensor S^* from S as above, this becomes

$$DS^* = 0.$$

The covariant derivative form of the exterior differentiation of course spoils the above calculations. On a hyperbolic surface, we have three linearly independent vector fields ω_α corresponding to the local symmetries.

The three 1-forms promised by Noether are ${}^*(S, \omega_\alpha)$. It should be noted, however, that there is a linear relation among these three forms. They are NOT linearly independent. Noether's theorem predates an understanding of Lie algebras. In another elementary but tedious calculation similar to the one in $\mathbb{R}^2$ in coordinates in the upper half plane we find the following:

Theorem 16. *On a hyperbolic surface M, the equation $D^* S = 0$ is valid if and only if locally $S^* = \nabla dE + RgE$ where ∇ is the covariant derivative, R is the constant curvature and g is the metric tensor. Moreover, the kernel K of the operator $\nabla d + Rg$ corresponds to functions E which are the Hamiltonians for the action of the symmetry group $\mathrm{SO}(2,1)$ with respect to the natural symplectic form on M.*

Note that non-constant curvature would completely spoil the 'if' part, which is easily checked once the operator is found.

We add a brief description of this kernel K in coordinates $(x^1, x^2) = (x, y)$ of the upper half space H^2. Three linearly independent Killing vector fields on H^2 are

$$\begin{aligned}
\omega_0 &= x d_x + y d_y \\
\omega_1 &= d_y \\
\omega_2 &= (x^2 - y^2)\, d_x + 2xy\, d_y.
\end{aligned}$$

The kernel K of the operator $\nabla d + Rg$ is generated by the three functions

$$\begin{aligned}
k_0 &= x/y \\
k_1 &= 1/y \\
k_2 &= (x^2 + y^2)/y.
\end{aligned}$$

To see that k_j is the Hamiltonian for $-\omega_j$ we compute in coordinates. The computation for k_0 goes like this:

$$\begin{aligned}
*dk_0 &= 1/y * dx - x/y^2 * dy \\
&= -x/y^2\, dx - 1/y\, dy \to -x d_x - y d_y.
\end{aligned}$$

The last step identifying a 1-form with a tangent vector uses the fact that the metric is conformal to the Euclidean metric with conformal factor $1/y$. The computation is similar for k_1 and k_2.

The kernel K is a 3-dimensional space of functions on $\tilde{M} = H^2$ whose elements are $\sum_j a_j k_j$. If γ is a closed curve in M, we lift to $\gamma : [0,1] \to H^2$. Solve $S^* = \nabla dE + RgE$ as a solution in the cover. We can check that $E(\gamma X)$ is also a solution, so $\nu(X) = E(\gamma X) - E(X)$ is in K. This and the homotopy invariance give us a theorem that can be used to construct an affine bundle.

Corollary 1. *Let M be a hyperbolic surface. To every S^* in $T^*(M) \circledS T^*(M)$ which is closed with respect to either index, $\nu : \pi_1(M) \to K$ is well-defined on $\pi_1(M)$ and satisfies*

$$\nu(\gamma_1\gamma_2) = \nu(\gamma_1) + \nu(\gamma_2).$$

We are specifically interested in the case where S^* is a measure with support on a lamination [D-U-1]. Applications to this case as well as detailed proofs of the results outlined here can be found in a forthcoming paper.

The references include an article on Noether's role in the relativity revolution [Ro] and the references on applications of Noether's theorem [B-G-G, O]. I particularly recommend the reference by Kosmann-Schwarzbach for historical background and an extensive description of related articles in both physics and mathematics [K-S].

We have no idea of either the use or real meaning of any of these observations. Or an exact statement of Theorem 2 in higher dimensions. Or what applications they may have,

Sometimes, however, the pursuit of useless knowledge pays off.

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Karen Uhlenbeck

Karen is retired from the University of Texas in Austin, and holds a position at as distinguished visiting professor at the Institute for Advanced Study in Princeton. She was awarded the Abel Prize in 2019. Over the years, she has worked in a number of areas of partial differential equations. Right now she is working with a former student George Daskalopoulos in trying to see what the calculus of variations can contribute to the theory of transverse measures on surfaces, an idea originally due to Bill Thurston. On the side, her interest in opening up the world of research mathematics so as to include women has expanded into doing the same for underrepresented minorities. Photo credit: Andrea Kane/Insitute for Advanced Study.

The Combinatorics of Hopping Particles and Positivity in Markov Chains

LAUREN K. WILLIAMS

The *asymmetric simple exclusion process* (ASEP) is a model for translation in protein synthesis and traffic flow; it can be defined as a Markov chain describing particles hopping on a one-dimensional lattice. In this article I explain how the stationary distribution of the ASEP has beautiful connections to combinatorics (tableaux and multiline queues) and special functions (Askey-Wilson polynomials, Macdonald polynomials, and Schubert polynomials). I also make some general observations about positivity in Markov chains.

Introduction

The goal of this article is to illustrate some of the elegant connections between combinatorics and probability that arise when one studies Markov chains. We will focus in particular on several variations of the *asymmetric simple exclusion process*, illustrating combinatorial formulas for its stationary distribution and connections to special functions. The last section of this article makes some general observations about positivity in Markov chains, in the context of the Markov Chain Tree Theorem.

The asymmetric simple exclusion process (ASEP) is a model for particles hopping on a one-dimensional lattice (e.g. a *line* or a *ring*) such that each site contains at most one particle. The ASEP was introduced independently in biology [MGP68] and in mathematics [Spi70] around 1970, see also [Lig85]. It exhibits boundary-induced phase transitions, and has been cited as a model for translation in protein synthesis, sequence alignment, the nuclear pore complex, and traffic flow.

Though we will not discuss them here, the ASEP has remarkable connections to a number of topics, including the XXZ model [San94*], vertex models [BP18*, BW18*], the Tracy-Widom distribution [Joh00*, TW09*], and the KPZ equation [BG97*, CST18*, CS18*, CK21*]. The ASEP is often viewed as a prototypical example of a random growth model from the *KPZ universality class* in $(1+1)$-dimensions, see [KPZ86*, Cor12*, Qua12*].

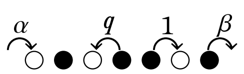

Figure 1. The (three-parameter) open boundary ASEP.

The ASEP with open boundaries, staircase tableaux, and Askey-Wilson polynomials

In the ASEP with open boundaries (see Figure 1), we have a one-dimensional lattice of n sites such that each site is either empty or occupied by a particle. At most one particle may occupy a given site. During each infinitesimal time interval dt, each particle at a site $1 \leq i \leq n-1$ has a probability dt of jumping to the next site on its right, provided it is empty, and each particle at a site $2 \leq i \leq n$ has a probability $q\,dt$ of jumping to the next site on its left, provided it is empty. Furthermore, a particle is added at site $i = 1$ with probability $\alpha\,dt$ if site 1 is empty and a particle is removed from site n with probability $\beta\,dt$ if this site is occupied. This model can be equivalently formulated as a discrete-time Markov chain.

The ASEP with open boundaries

Let α, β, and q be constants between 0 and 1. Let B_n be the set of all 2^n words of length n in $\{\circ, \bullet\}$. The open boundary ASEP is the Markov chain on B_n with transition probabilities:

- If $\tau = A\bullet\circ B$ and $\sigma = A\circ\bullet B$ (where A and B are words in $\{\circ, \bullet\}$), then we have that $\Pr(\tau \to \sigma) = \frac{1}{n+1}$ and $\Pr(\sigma \to \tau) = \frac{q}{n+1}$ (particle hops right or left).
- If $\tau = \circ B$ and $\sigma = \bullet B$ then $\Pr(\tau \to \sigma) = \frac{\alpha}{n+1}$ (particle enters the lattice from left).
- If $\tau = B\bullet$ and $\sigma = B\circ$ then $\Pr(\tau \to \sigma) = \frac{\beta}{n+1}$ (particle exits the lattice to the right).
- Otherwise $\Pr(\tau \to \sigma) = 0$ for $\sigma \neq \tau$ and $\Pr(\tau \to \tau) = 1 - \sum_{\sigma \neq \tau} \Pr(\tau \to \sigma)$.

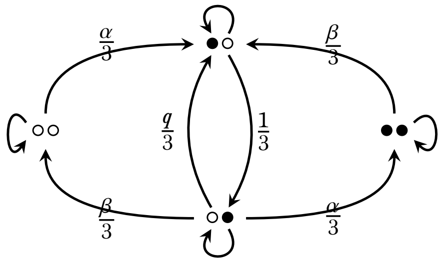

Figure 2. The state diagram of the open-boundary ASEP on a lattice of 2 sites.

In the long time limit, the system reaches a steady state where all the probabilities $\pi(\tau)$ of finding the system in configuration τ are stationary, i.e. satisfy $\frac{d}{dt}\pi(\tau) = 0$. Moreover, the stationary distribution is unique. We can compute it by solving for the left eigenvector of the transition matrix with eigenvalue 1, or equivalently, by solving the *global balance* equations: for all states $\tau \in B_n$, we have

$$\pi(\tau)\sum_{\sigma\neq\tau}\Pr(\tau\to\sigma) = \sum_{\sigma\neq\tau}\pi(\sigma)\Pr(\sigma\to\tau),$$

where both sums are over all states $\sigma \neq \tau$.

The steady state probabilities are rational expressions in α, β and q. For convenience, we clear denominators, obtaining "unnormalized probabilities" $\Psi(\tau)$ which are equal to the $\pi(\tau)$ up to a constant: that is, $\pi(\tau) = \frac{\Psi(\tau)}{Z_n}$, where $Z_n = Z_n(\alpha,\beta,q)$ is the *partition function* $\sum_{\tau\in B_n}\Psi(\tau)$.

State τ	Unnormalized probability $\Psi(\tau)$
●●	α^2
●○	$\alpha\beta(\alpha+\beta+q)$
○●	$\alpha\beta$
○○	β^2

Example 1. Figure 2 shows the state diagram of the open-boundary ASEP when $n = 2$, and the table above gives the corresponding unnormalized probabilities. Therefore we have $\pi(●●) = \frac{\alpha^2}{Z_2}$, $\pi(●○) = \frac{\alpha\beta(\alpha+\beta+q)}{Z_2}$, $\pi(○●) = \frac{\alpha\beta}{Z_2}$, and $\pi(○○) = \frac{\beta^2}{Z_2}$, where $Z_2 = \alpha^2 + \alpha\beta(\alpha+\beta+q) + \alpha\beta + \beta^2$.

For $n = 3$, if we again write each probability $\pi(\tau) = \frac{\Psi(\tau)}{Z_3}$, we find that $Z_3(\alpha,\beta,q)$ is a polynomial which is *manifestly positive* – that it, it has only positive coefficients. Also, Z_3 has 24 terms (counted with multiplicity): $Z_3(1,1,1) = 24$. Computing more examples quickly leads to the conjecture that the partition function $Z_n = Z_n(\alpha,\beta,q)$ is a (manifestly) positive polynomial with $(n-1)!$ terms.

In algebraic combinatorics, if a quantity of interest is known or believed to be a positive integer or a polynomial with positive coefficients, then one seeks an interpretation of this quantity as counting some combinatorial objects. For example, one seeks to express such a polynomial as a generating function for certain tableaux or graphs or permutations, etc. A prototypical example is the Schur polynomial $s_\lambda(x_1,\ldots,x_n)$ [Sta99]: there are several formulas for it, including the bialternant formula and the Jacobi-Trudi formula, but neither makes it obvious that the Schur polynomial has positive coefficients. However, one can express the Schur polynomial as the generating function for semistandard tableaux of shape λ, and this formula makes manifest the positivity of coefficients [Sta99].

Given the above, and our observations on the positivity of the partition function $Z_n(\alpha,\beta,q)$, the natural question is: can we express each probability as a (manifestly positive) sum over some set of combinatorial objects? We will explain how to answer this question using (a special case of) the staircase tableaux of [CW11].

In what follows, we will depict Young diagrams in Russian notation (with the corner at the bottom).

An *$\alpha\beta$-staircase tableau* T of size n is a Young diagram of shape $(n, n-1, \ldots, 2, 1)$ (drawn in Russian notation) such that each box is either empty or contains an α or β, such that:

(1) no box in the top row is empty

(2) each box southeast of a β and in the same diagonal as that β is empty.

(3) each box southwest of an α and in the same diagonal as that α is empty.

See Figure 3. We encourage the reader to check that there are $(n+1)!$ $\alpha\beta$-staircase tableaux of size n.

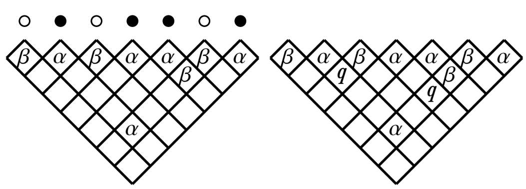

Figure 3. At left: an $\alpha\beta$-staircase tableau T of type (○ ● ○ ● ● ○ ●). At right: T with a q in each unrestricted box. We have $\mathrm{wt}(T) = \alpha^5\beta^4q^2$.

Definition 2. Some boxes in a tableau are forced to be empty because of conditions (2) or (3) above; we refer to all other empty boxes as *unrestricted*. (The unrestricted boxes are precisely those whose nearest neighbors on the diagonals to the northwest and northeast, respectively, are an α and β.)

After placing a q in each unrestricted box, we define the *weight* $\mathrm{wt}(T)$ of T to be $\alpha^i\beta^j q^k$ where i, j and k are the numbers of α's, β's, and q's in T.

The *type* of T is the word obtained by reading the letters in the top row of T and replacing each α by $\bullet$ and β by $\circ$, see Figure 3.

The following result [CW07b, CW07a, CW11] gives a combinatorial formula for the steady state probabilities of the ASEP. (Note that [CW07b, CW07a] used *permutation tableaux*, which are in bijection with staircase tableaux, and are closely connected to the *positive Grassmannian*.) The $q = 0$ case was previously studied in [DS05].

Theorem 17. *Consider the ASEP with open boundaries on a lattice of n sites. Let $\tau = (\tau_1, \ldots, \tau_n) \in \{\bullet, \circ\}^n$ be a state. Then the unnormalized steady state probability $\Psi(\tau)$ is equal to $\sum_T \mathrm{wt}(T)$, where the sum is over the $\alpha\beta$-staircase tableaux of type τ.*

Equivalently, if we let $\mathcal{T}_n$ be the set of $\alpha\beta$-staircase tableaux of size n, and $Z_n := \sum_{T\in\mathcal{T}_n} \mathrm{wt}(T)$ be the weight generating function for these tableaux, then the steady state probability $\pi(\tau)$ is $\frac{\sum_T \mathrm{wt}(T)}{Z_n}$, where the sum is over the $\alpha\beta$-staircase tableaux of type τ.

In the case $n = 2$, there are six tableaux of size 2, shown in Figure 4 and arranged by type. Computing the weights of the tableaux of the various types reproduces the results from Example 1.

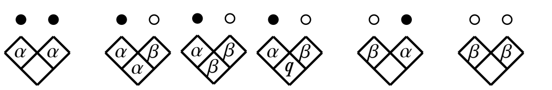

Figure 4. The six $\alpha\beta$-staircase tableau T of size 2.

How can we prove such a formula? One option is to realize the ASEP as a *lumping* (or *projection*) of a Markov chain on tableaux [CW07a]. (See also [DS05] for the case $q = 0$.) Recall that we have a surjection $f : \mathcal{T}_n \to B_n$, which maps an $\alpha\beta$-staircase tableau to its *type*. We'd like to construct a Markov chain on tableaux whose projection via f recovers the ASEP. If we can do so, and moreover show that the steady state probability $\pi(T)$ is proportional to $\mathrm{wt}(T)$, then we will have proved Theorem 17.

Markov chain lumpings [KS60*, Section 6]

Let $\{X_t\}$ be a Markov chain on state space Ω_X with transition matrix P, and let $f : \Omega_X \to \Omega_Y$ be a surjective map. Suppose there is an $|\Omega_Y| \times |\Omega_Y|$ matrix Q such that for all $y_0, y_1 \in \Omega_Y$, if $f(x_0) = y_0$, then

$$\sum_{x: f(x)=y_1} P(x_0, x) = Q(y_0, y_1). \qquad (1)$$

Then $\{f(X_t)\}$ is a Markov chain on Ω_Y with transition matrix Q. We say that $\{f(X_t)\}$ is a *(strong) lumping* of $\{X_t\}$ and $\{X_t\}$ is a *(strong) lift* of $\{f(X_t)\}$.

Suppose π is a stationary distribution for $\{X_t\}$, and let π_f be the measure on Ω_Y defined by $\pi_f(y) = \sum_{x:f(x)=y} \pi(x)$. Then π_f is a stationary distribution for $\{f(X_t)\}$.

See [KS60*, Pan19*] for more details on lumping.

The ASEP can be lifted to a Markov chain on $\alpha\beta$-staircase tableaux [CW07a], see Figure 5. In each diagram, the grey boxes represent boxes that must be empty. Note that the remaining empty boxes on the left and right side of a "$\mapsto$" are in bijection with each other; they must be filled the same way. The lifted chain has the nice property that the left hand side of (1) always has at most one nonzero term.

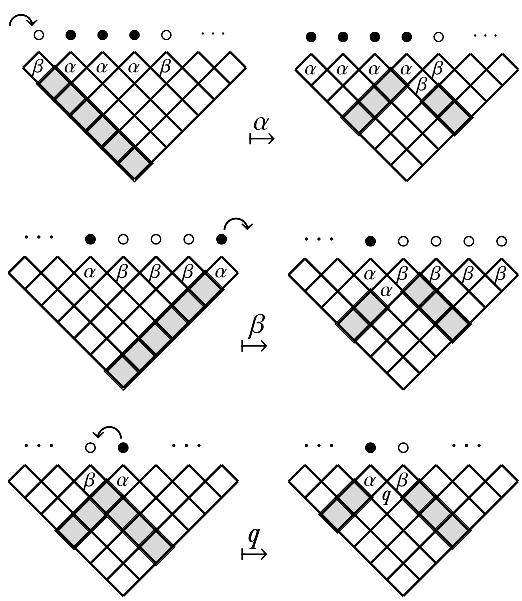

Figure 5a. "Particle enters from the left; particle exits to the right; particle hops left."

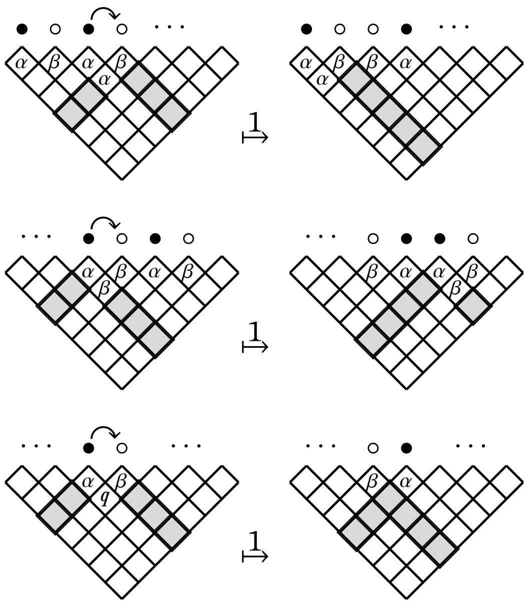

Figure 5b. "Particle hops right" (three cases).

If we identify α's and β's with particles and holes, then the chain on tableaux reveals a circulation of particles and holes in the second row of the tableaux; this is similar to a phenomenon observed in [DS05].

Another way to prove Theorem 17 is to use the *Matrix Ansatz* of Derrida–Evans–Hakim–Pasquier [DEHP93].[1]

The Matrix Ansatz

Theorem 18 (Derrida-Evans-Hakim-Pasquier). *Consider the ASEP with open boundaries on a lattice of n sites. Suppose that D and E are matrices, $|V\rangle$ is a column vector, $\langle W|$ is a row vector, and c is a constant, such that:*

$$DE - qED = c(D+E) \tag{2}$$

$$\beta D|V\rangle = c|V\rangle \tag{3}$$

$$\alpha\langle W|E = c\langle W| \tag{4}$$

If we identify $\tau = (\tau_1, \dots, \tau_n) \in \{0,1\}^n$ with a state (by mapping 1 and 0 to $\bullet$ and $\circ$, respectively), then the steady state probability $\pi(\tau)$ is equal to

$$\pi(\tau) = \frac{\langle W|(\prod_{i=1}^{n}(\tau_i D + (1-\tau_i)E))|V\rangle}{\langle W|(D+E)^n|V\rangle}.$$

[1][DEHP93] stated this result with $c = 1$.

For example, the steady state probability of state $\circ\bullet\circ\bullet\bullet\circ\bullet$ is $\frac{\langle W|EDEDDED|V\rangle}{\langle W|(D+E)^7|V\rangle}$.

We note that Theorem 18 does not imply that a solution $D, E, |V\rangle, \langle W|$ exists nor that it is unique. Indeed there are multiple solutions, which in general involve infinite-dimensional matrices.

To prove Theorem 17 using the Matrix Ansatz, we let $D_1 = (d_{ij})$ be the (infinite) upper-triangular matrix with rows and columns indexed by $\mathbb{Z}^+$, defined by $d_{i,i+1} = \alpha$ and $d_{ij} = 0$ for $j \neq i+1$. That is, D_1 is

$$\begin{pmatrix} 0 & \alpha & 0 & 0 & \dots \\ 0 & 0 & \alpha & 0 & \dots \\ 0 & 0 & 0 & \alpha & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

Let $E_1 = (e_{ij})$ be the (infinite) lower-triangular matrix defined by $e_{ij} = 0$ for $j > i$ and

$$e_{ij} = \beta^{i-j+1}\left(q^{j-1}\binom{i-1}{j-1} + \alpha\sum_{r=0}^{j-2}\binom{i-j+r}{r}q^r\right)$$

for $j \leq i$. That is, E_1 is the matrix

$$\begin{pmatrix} \beta & 0 & 0 & \dots \\ \beta^2 & \beta(\alpha+q) & 0 & \dots \\ \beta^3 & \beta^2(\alpha+2q) & \beta(\alpha+\alpha q+q^2) & \dots \\ \beta^4 & \beta^3(\alpha+3q) & \beta^2(\alpha+2\alpha q+3q^2) & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

We also define the (infinite) row and column vectors $\langle W_1| = (1,0,0,\dots)$ and $|V_1\rangle = (1,1,1,\dots)^t$. Then one can check that $D_1, E_1, \langle W_1|, |V_1\rangle$ satisfy (2), (3), and (4), with $c = \alpha\beta$. One can also show D_1 and E_1 are *transfer matrices* whose products enumerate $\alpha\beta$-staircase tableaux. For example, $\langle W_1|E_1D_1E_1D_1D_1E_1D_1|V_1\rangle$ enumerates the staircase tableaux of type $\circ\bullet\circ\bullet\bullet\circ\bullet$. Now Theorem 18 implies Theorem 17.

The five-parameter open boundary ASEP

More generally, we would like to understand a generalized ASEP in which particles can both enter and exit the lattice at the left (at rates α, γ), and exit and enter the lattice at the right (at rates β, δ). There is a version of the Matrix Ansatz for this setting [DEHP93], as well as suitable tableaux filled with α, β, γ and δ's [CW11].

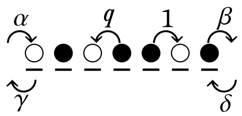

Figure 6. The (five-parameter) open boundary ASEP.

A staircase tableau T of size n is a Young diagram of shape $(n, n-1, \ldots, 2, 1)$ such that each box is either empty or contains an α, β, γ, or δ such that:

(1) no box in the top row is empty

(2) each box southeast of a β or δ and in the same diagonal as that β or δ is empty.

(3) each box southwest of an α or γ and in the same diagonal as that α or γ is empty.

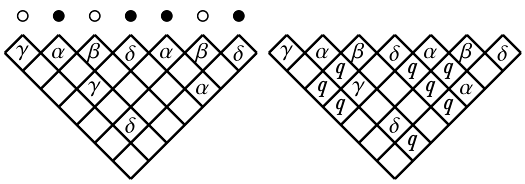

Figure 7. At left: a staircase tableau T of type $(\circ\bullet\circ\bullet\bullet\circ\bullet)$. At right: T with a q in the appropriate boxes. We have $\mathrm{wt}(T) = \alpha^3\beta^2\gamma^2\delta^3 q^8$.

See Figure 7 for an example. There are exactly $4^n n!$ staircase tableaux of size n.

Definition 3. We call an empty box of a staircase tableau T *distinguished* if either:

- its nearest neighbor on the diagonal to the northwest is a δ, or
- its nearest neighbor on the diagonal to the northwest is an α or γ, and its nearest neighbor on the diagonal to the northeast is a β or γ.

After placing a q in each distinguished box, we define the *weight* $\mathrm{wt}(T)$ of T to be the product of all letters in the boxes of T.

The *type* of T is the word obtained by reading the letters in the top row of T and replacing each α or δ by $\bullet$, and each β or γ by $\circ$, see Figure 7.

The following result from [CW11] subsumes Theorem 17. It can be proved using a suitable generalization of the Matrix Ansatz.

Theorem 19. *Consider the ASEP with open boundaries on a lattice of n sites as in Figure 6. Let $\tau \in \{\bullet, \circ\}^n$ be a state. Then the unnormalized steady state probability $\Psi(\tau)$ is equal to $\sum_T \mathrm{wt}(T)$, where the sum is over the staircase tableaux of type τ.*

Remarkably, there is another solution to the Matrix Ansatz which involves orthogonal polynomials. More specifically, one can find a solution where D and E are tridiagonal matrices, such that the rows of $D + E$ encode the three-term recurrence relation characterizing the *Askey-Wilson polynomials* [USW04]; these are a family of polynomials $p_n(x; a, b, c, d|q)$ at the top of the hierarchy of classical one-variable orthogonal polynomials (including the others as special or limiting cases) [AW85*].

The connection of Askey-Wilson polynomials with the ASEP [USW04] leads to applications on both sides. On the one hand, it facilitates the computation of physical quantities in the ASEP such as the phase diagram [USW04]; it also leads to a relation between the ASEP and the *Askey-Wilson stochastic process* [BWo17*]. On the other hand, this connection has applications to the combinatorics of Askey-Wilson moments. Since the 1980's there has been a great deal of work on the combinatorics of orthogonal polynomials (e.g. Hermite, Charlier, Laguerre) [Vie85*, ISV87*, CKS16*]; the connection of staircase tableaux to ASEP, and of ASEP to Askey-Wilson polynomials, led to the first combinatorial formula for Askey-Wilson moments [CW11], [CSSW*].

Even more generally, one can an ASEP with open boundaries with different species of particles. This version is closely connected [Can17*, CW18*, CGdGW16*] to *Koornwinder polynomials* [Koo92*], a family of multivariate orthogonal polynomials which generalize Askey-Wilson polynomials.

The (multispecies) ASEP on a ring

It is also natural to consider the ASEP on a lattice of sites arranged in a ring, of which some sites are occupied by a particle. Each particle in the system can jump to the next site either clockwise or counterclockwise, provided that this site is empty. In this model, the resulting stationary distribution is always the uniform distribution. This motivates considering a *multispecies* generalization of the ASEP, in which particles come with different weights, which in turn influence the hopping rates.

The multispecies ASEP, multiline queues, and Macdonald polynomials

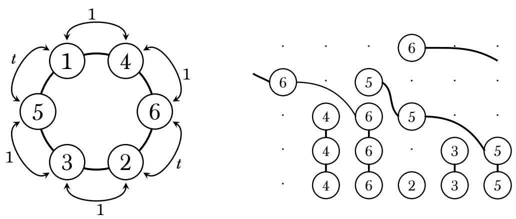

Figure 8. The mASEP on a ring with $\lambda = (6,5,4,3,2,1)$, and a multiline queue of type $(1,4,6,2,3,5)$.

In the multispecies ASEP (mASEP) on a ring, two neighboring particles exchange places at rates 1 or t, depending on whether the heavier particle is clockwise or counterclockwise from the lighter one.

The multispecies ASEP on a ring

Let t be a constant such that $0 \le t \le 1$, and let $\lambda = \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0$ be a partition. We think of the parts of λ as representing various types of particles of different weights. Let $B_n(\lambda)$ be the set of all words of length n obtained by permuting the parts of λ. The *multispecies ASEP* on a ring is the Markov chain on $B_n(\lambda)$ with transition probabilities:

- If $\mu = (\mu_1, \ldots, \mu_n)$ and ν are in $B_n(\lambda)$, and ν is obtained from μ by swapping μ_i and μ_{i+1} for some i (indices considered modulo n), then $\Pr(\mu \to \nu) = \frac{t}{n}$ if $\mu_i > \mu_{i+1}$ and $\Pr(\mu \to \nu) = \frac{1}{n}$ if $\mu_i < \mu_{i+1}$.
- Otherwise $\Pr(\mu \to \nu) = 0$ for $\nu \neq \mu$ and $\Pr(\mu \to \mu) = 1 - \sum_{\nu \neq \mu} \Pr(\mu \to \nu)$.

As before, one would like to find an expression for each steady state probability as a manifestly positive sum over some combinatorial objects. One may give such a formula in terms of Ferrari-Martin's *multiline queues* shown in Figure 8, see [Mar20, CMW22].

One fascinating aspect of the multispecies ASEP on a ring is its close relation [CdGW15] to *Macdonald polynomials* $P_\lambda(x_1, \ldots, x_n; q, t)$ [Mac95], a remarkable family of polynomials that generalize Schur polynomials, Hall-Littlewood polynomials, and Jack polynomials. The next result follows from [CdGW15] and [CMW22].

Theorem 20. *Let $\mu \in B_n(\lambda)$ be a state of the mASEP on a ring. Then the steady state probability $\pi(\mu)$ is*

$$\pi(\mu) = \frac{\Psi(\mu)}{Z_\lambda},$$

where $\Psi(\mu)$ is obtained from a permuted basement Macdonald polynomial *and Z_λ is obtained from the* Macdonald polynomial P_λ *by specializing $q = 1$ and $x_1 = x_2 = \cdots = x_n = 1$.*

The following table shows the probabilities of the mASEP when $\lambda = (4,3,2,1)$. Note that because of the circular symmetry in the mASEP, e.g. $\pi(1,2,3,4) = \pi(2,3,4,1) = \pi(3,4,1,2) = \pi(4,1,2,3)$, it suffices to list the probabilities for the states w with $w_1 = 1$.

State w	Unnormalized probability $\Psi(w)$
1234	$9t^3 + 7t^2 + 7t + 1$
1243	$3(t^3 + 3t^2 + 3t + 1)$
1324	$3t^3 + 11t^2 + 5t + 5$
1342	$3(t^3 + 3t^2 + 3t + 1)$
1423	$5t^3 + 5t^2 + 11t + 3$
1432	$t^3 + 7t^2 + 7t + 9$

In light of Theorem 20 and the connection to multiline queues, it is natural to ask if one can give a formula for Macdonald polynomials in terms of multiline queues. This is indeed possible, see [CMW22].

We remark that there is a family of Macdonald polynomials associated to any affine root system; the "ordinary" Macdonald polynomials discussed in this section are those of type $\tilde{A}$. It is interesting that they are related to particles hopping on a ring (which resembles a Dynkin diagram of type $\tilde{A}$). Meanwhile, the Koornwinder polynomials from the previous section are the Macdonald polynomials attached to the non-reduced affine root system of type $\tilde{C}_n^\vee$. It is interesting that they are related to particles hopping on a line with open boundaries (which resembles a Dynkin diagram of type $\tilde{C}_n^\vee$).

We note that there are other connections between probability and Macdonald polynomials, including *Macdonald processes* [BC14*], and a Markov chain on partitions whose eigenfunctions are coefficients of Macdonald polynomials [DR12*]. There is also a variation of the exclusion process called the *multispecies zero range process*, whose stationary distribution is related to *modified Macdonald polynomials* [AMM20*].

The inhomogeneous TASEP, multiline queues, and Schubert polynomials

Another multispecies generalization of the exclusion process on a ring is the *inhomogeneous totally asymmetric exclusion process* (TASEP). In this model, two adjacent particles with weights i and j with $i < j$ can swap places only if the heavier one is clockwise of the lighter one, and in this case, they exchange places at rate $x_i - y_j$, see Figure 9.

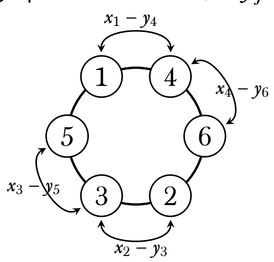

Figure 9. The inhomogeneous multispecies TASEP on a ring, with $\lambda = (6,5,4,3,2,1)$.

Inhomogeneous TASEP on a ring

Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be constants such that $0 < x_i - y_j \le 1$ for all i, j, and let $\lambda = \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0$ be a partition. Let $B_n(\lambda)$ be the set of all words of length n obtained by permuting the parts of λ. The *inhomogeneous TASEP* on a ring is the Markov chain on $B_n(\lambda)$ with transition probabilities:

- If $\mu = (\mu_1, \ldots, \mu_n)$ and ν are in $B_n(\lambda)$, and ν is obtained from μ by swapping μ_i and μ_{i+1} for some i (indices considered mod n), then $\Pr(\mu \to \nu) = \frac{x_{\mu_i} - y_{\mu_{i+1}}}{n}$ if $\mu_i < \mu_{i+1}$.
- Otherwise $\Pr(\mu \to \nu) = 0$ for $\nu \ne \mu$ and $\Pr(\mu \to \mu) = 1 - \sum_{\nu \ne \mu} \Pr(\mu \to \nu)$.

When $y_i = 0$ for all i, there is a formula for the stationary distribution of the inhomogeneous TASEP in terms of multiline queues; this can be proved using a version of the Matrix Ansatz [AM13].

Recall that the mASEP on a ring is closely connected to Macdonald polynomials. Curiously, when $y_i = 0$ the inhomogeneous TASEP on a ring is related to *Schubert polynomials*, a family of polynomials which give polynomial representatives for the Schubert classes in the cohomology ring of the flag variety. For example, many steady state probabilities are proportional to products of Schubert polynomials [Can16, KW21], and all of them are conjecturally positive sums of Schubert polynomials [LW12*].

Given $w = (w_1, \ldots, w_n)$ a permutation in the symmetric group S_n and $p = (p_1, \ldots, p_m) \in S_m$ with $m < n$, we say that w *contains* p if w has a subsequence of length m whose letters are in the same relative order as those of p. For example, the permutation $(\mathbf{3}, 2, \mathbf{6}, 5, \mathbf{1}, \mathbf{4})$ contains the pattern $(2,4,1,3)$ because its letters $3,6,1,4$ have the same relative order as those of $(2,4,1,3)$. If w does not contain p we say that w *avoids* p. We say that $w \in S_n$ is *evil-avoiding* if w avoids the patterns $(2,4,1,3), (4,1,3,2), (4,2,1,3)$ and $(3,2,1,4)$.[2]

We have the following result, see [KW21] for details.

Theorem 21. *Let $\lambda = (n, n-1, \ldots, 1)$ so that the inhomogeneous TASEP can be viewed as a Markov chain on the $n!$ permutations of the set $\{1, 2, \ldots, n\}$. Let $w \in S_n$ be a permutation with $w_1 = 1$ which is* evil-avoiding, *and let k be the number of descents of w^{-1}. Then the steady state probability $\pi(w)$ equals*

$$\pi(w) = \frac{\Psi(w)}{Z_n},$$

where $\Psi(w)$ is a monomial in $x_1, \ldots, x_{n-1}$ times a product of k Schubert polynomials, and $Z_n = \prod_{i=1}^{n} h_{n-i}(x_1, x_2, \ldots, x_{i-1}, x_i, x_i)$ with h_i the complete homogeneous symmetric polynomial.

The following table shows the probabilities of the inhomogeneous TASEP when $\lambda = (4,3,2,1)$.

State w	Unnormalized probability $\Psi(w)$
1234	$x_1^3 x_2$
1243	$x_1^2(x_1x_2 + x_1x_3 + x_2x_3) = x_1^2 \operatorname{Sym}_{1342}$
1324	$x_1(x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_2x_3 + x_2^2x_3) = x_1 \operatorname{Sym}_{1432}$
1342	$x_1x_2(x_1^2 + x_1x_2 + x_2^2) = x_1x_2 \operatorname{Sym}_{1423}$
1423	$x_1^2x_2(x_1 + x_2 + x_3) = x_1^2x_2 \operatorname{Sym}_{1243}$
1432	$(x_1^2 + x_1x_2 + x_2^2)(x_1x_2 + x_1x_3 + x_2x_3) = \operatorname{Sym}_{1423} \operatorname{Sym}_{1342}$

For general y_i, there is a version of Theorem 21 involving *double* Schubert polynomials [KW21].

Very often, beautiful combinatorial properties go hand-in-hand with *integrability* of a model. While this topic goes beyond the scope of this article, the reader can learn about integrability and the exclusion process from [Can16], [CRV14*], or more generally about *integrable probability* from [BG16*].

[2] We call these permutations *evil-avoiding* because if one replaces i by 1, e by 2, l by 3, and v by 4, then *evil* and its anagrams *vile*, *veil* and *leiv* become the four patterns 2413, 4132, 4213 and 3214. (Leiv is a name of Norwegian origin meaning "heir.")

Positivity in Markov chains

The reader may at this point wonder how general is the phenomenon of positivity in Markov chains? That is, how often can one express the steady state probabilities of a Markov chain in terms of polynomials with all coefficients positive (ideally as a sum over combinatorial objects)?

In some sense, the answer to this question is *all the time*: the *Markov Chain Tree Theorem* gives a formula for the stationary distribution of a finite-state irreducible Markov chain as a positive sum indexed by rooted trees of the state diagram. However, the number of terms of this formula grows fast very quickly! (By Cayley's formula, the complete graph on n vertices has n^{n-2} spanning trees.) Moreover, for many Markov chains, there is a common factor which can be removed from the above formula for the stationary distribution, resulting in a more compact formula. Sometimes the more compact formula involves polynomials with negative coefficients.

Let G be the *state diagram* of a finite-state irreducible Markov chain whose set of states is V. That is, G is a weighted directed graph with vertices V, with an edge e from i and j weighted $\Pr(e) := \Pr(i,j)$ whenever the probability $\Pr(i,j)$ of going from state i to j is positive. We call a connected subgraph T a *spanning tree* rooted at r if T includes every vertex of V, T has no cycle, and all edges of T point towards the root r. (Irreducibility of the Markov chain implies that for each vertex r, there is a spanning tree rooted at r.) Given a spanning tree T, we define its *weight* as $\mathrm{wt}(T) := \prod_{e\in T}\Pr(e)$.

Theorem 22 (Markov Chain Tree Theorem). *The stationary distribution of a finite-state irreducible Markov chain is proportional to the measure that assigns the state τ the "unnormalized probability"*

$$\Psi(\tau) := \sum_{\mathrm{root}(T)=\tau} \mathrm{wt}(T).$$

That is, the steady state probability $\pi(\tau)$ equals $\pi(\tau) = \frac{\Psi(\tau)}{\mathbf{Z}}$, where $\mathbf{Z} = \sum_\tau \Psi(\tau)$.

Theorem 22 first appeared in [Hil66*] and was proved for general Markov chains in [LR83]. It has many proofs, one of which involves lifting the Markov chain to a chain on the trees themselves; the result then follows from *Kirchhoff's Matrix Tree Theorem*. See [AT90*, LP16*] and [PT18], plus references therein.

Example 2. Consider the Markov chain with five states $1,\dots,5$, whose transition matrix is as follows:

$$\begin{bmatrix} \frac{2-q}{3} & 0 & \frac{1}{3} & \frac{q}{3} & 0 \\ 0 & \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ \frac{q}{3} & \frac{1}{3} & \frac{1-q}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{q}{3} & 0 & \frac{2-2q}{3} & \frac{q}{3} \\ 0 & 0 & \frac{q}{3} & \frac{1}{3} & \frac{2-q}{3} \end{bmatrix} \tag{5}$$

The state diagram of the Markov chain is shown in Figure 10. Note that we have omitted the factor of $\frac{1}{3}$ from each transition probability (this does not affect the eigenvector of the transition matrix). For simplicity, we also omitted the loops at each state.

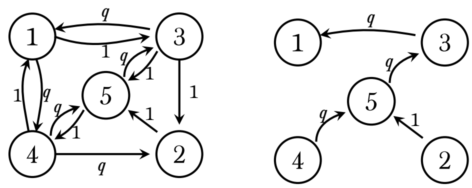

Figure 10. The state diagram from Example 2, plus a spanning tree rooted at 1 with weight q^3.

If one applies Theorem 22, one finds e.g. that there are six spanning trees rooted at state 1, with weights $q^3, q^3, q^2, q, 1$, and 1. Adding up these contributions gives $\Psi(1) = 2q^3+q^2+q+2$. Computing the spanning trees rooted at the other states gives rise to the following unnormalized probabilities $\Psi(\tau)$ for the stationary distribution.

State τ	Unnormalized probability $\Psi(\tau)$
1	$2q^3+q^2+q+2$
2	$q^4+3q^3+4q^2+3q+1$
3	$2q^3+2q^2+q+1$
4	q^3+q^2+2q+2
5	$2q^3+4q^2+4q+2$

Note that all of the above unnormalized probabilities share a common factor of $(q+1)$. Dividing by this common factor gives the following (more compact) unnormalized probabilities $\overline{\Psi}(\tau)$.

State τ	Unnormalized probability $\overline{\Psi}(\tau)$
1	$2q^2-q+2$
2	q^3+2q^2+2q+1
3	$2q^2+1$
4	q^2+2
5	$2q^2+2q+2$

We see that when we write the stationary distribution in "lowest terms," we obtain a vector of polynomials which do *not* have only nonnegative coefficients.

This example motivates the following definitions.

Definition 4. Consider an unnormalized measure $(\Psi_1,\dots,\Psi_n)$ on the set $\{1,2,\dots,n\}$ in which each component $\Psi_i(q_1,\dots,q_N)$ is a polynomial in $\mathbb{Z}[q_1,\dots,q_N]$. We say that $(\Psi_1,\dots,\Psi_n)$ is *manifestly positive* if the coefficients of Ψ_i are positive for all i. We say that $(\Psi_1,\dots,\Psi_n)$ is *compact* if there is no polynomial $\phi(q_1,\dots,q_N) \neq 1$ which divides all the components Ψ_i.

Theorem 22 shows that every finite-state Markov chain has a manifestly positive formula for the stationary distribution. Meanwhile, Example 2 shows that in general this formula is not compact, and that there are Markov chains whose compact formula for the stationary distribution is not manifestly positive.

In light of Theorem 22, it is interesting to revisit e.g. the stationary distribution of the open boundary ASEP with parameters α, β, and q. One can use Theorem 17 to express the components $\Psi_{\mathrm{tab}}(\tau)$ of the stationary measure as a sum over the tableaux of type τ. On the other hand, one can use Theorem 22 to express the components $\Psi_{\mathrm{tree}}(\tau)$ of the stationary measure as a sum over spanning trees rooted at τ of the state diagram. Both $\Psi_{\mathrm{tab}}(\tau)$ and $\Psi_{\mathrm{tree}}(\tau)$ are polynomials in α,β,q with positive coefficients; however, the former is compact, and has many fewer terms than the latter. Because the stationary measure is unique (up to an overall scalar), for each n there is a polynomial $Q_n(\alpha,\beta,q)$ such that $\frac{\Psi_{\mathrm{tree}}(\tau)}{\Psi_{\mathrm{tab}}(\tau)} = Q_n(\alpha,\beta,q)$ for all $\tau \in \{0,1\}^n$. The number of terms in Q_n appears in the table below.

n	$Q_n(1,1,1)$
2	1
3	4
4	840
5	2285015040
6	11335132600511975880768000

It would be interesting to reprove e.g. Theorem 17 using the Markov Chain Tree Theorem.

We note that the analysis of the ASEP and its variants would be easier if these Markov chains were *reversible*; in general they are not (except for special cases of the parameters). Nevertheless there has been progress on the mixing time of the ASEP, see [GNS21*] and references therein.

Besides the ASEP, there are other interesting Markov chains arising in statistical mechanics whose stationary distributions admit manifestly positive formulas as sums over combinatorial objects (which are often compact). These include the *Razumov-Stroganov correspondence* [DF04*, dGR04*, CS14*], the *Tsetlin library* [Tse63*, Hen72*], and many other models of interacting particles [AM10*, AN21*], see also [Ayy22*].

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Selected references are listed below. All other references cited in the text (those marked with a $*$) can be found in the arXiv preprint [Wil22].

FURTHER READING

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Lauren K. Williams

Lauren K. Williams is the Robinson Professor of Mathematics at Harvard University and the Seaver Professor at the Radcliffe Institute. She was previously a professor at UC Berkeley. Her main research interests are in algebraic combinatorics, including total positivity and cluster algebras, but she gets considerable inspiration from mathematical physics. Lauren grew up in California, the oldest of four girls; she now lives in Cambridge with her family.

Notes of a Numerical Analyst

A picture worth 2000 words

NICK TREFETHEN FRS

When I saw this image at Dave Hewett's home page at UCL, it brought an instant smile to my face. It's just one picture, but it illustrates two famous and important phenomena.

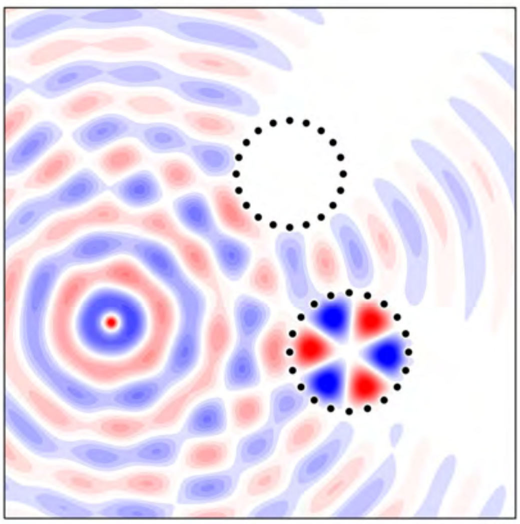

Figure 1. Electromagnetic waves around cages of slightly different radii.

Clearly it's an image of 2D wave scattering (computed by Hewett's coauthor Ian Hewitt). The PDE is the Helmholtz equation

$$\Delta u + k^2 u = 0 \qquad (1)$$

for a wave number k, and the solution $u(z)$ represents the spatial dependence of a wave at a fixed time-frequency. The context is electromagnetic radiation, with u representing the out-of-plane component of the electric field. Following a standard mathematical simplification, we take advantage of linearity to let u be complex, and the plot shows its oscillatory real part. The wave is driven by a point source on the left, a Hankel function $H_0(k|z - z_0|)$, and the Sommerfeld radiation condition is imposed at infinity.

The action is in the two cages on the right, which have radii that differ by 10% and very different behaviours. Each black dot is a disk of finite radius, and the boundary condition is $u = 0$ on all the disks. We can think of these as cross-sections of parallel wires in the third dimension, which are all connected and thus at the same fixed potential.

The story in the upper cage is *Faraday shielding*. Obviously the wave has not penetrated much inside, and this effect has been exploited since Faraday's discovery to shield people and instruments from electrostatic and electromagnetic fields. We all have a Faraday cage in our kitchens, namely the microwave oven, whose front door has a metal screen with holes big enough to let light out but small enough to keep microwaves in. As the holes get smaller, or the wires get closer together in our 2D model, the shielding only improves algebraically, not exponentially as has often been supposed [2].

The story in the lower cage is *resonance*, and we can think of this as a model of an AM radio. These thick wires exclude most wave energy, but if the radius is tuned just right, so that k corresponds to an eigenmode of a disk of this radius, then the response can be very great. As the wires get thicker and closer together, the tuning gets ever sharper and the potential amplification ever greater. In the limit where the wires touch, we have perfect shielding and perfect tuning—with infinite amplification, if only it could be excited.

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Nick Trefethen

Trefethen is Professor of Numerical Analysis and head of the Numerical Analysis Group at the University of Oxford.

Mathematics News Flash

Jonathan Fraser reports on some recent breakthroughs in mathematics.

Attaining the exponent 5/4 for the sum-product problem in finite fields

AUTHORS: Ali Mohammadi and Sophie Stevens
ACCESS: https://arxiv.org/abs/2103.08252

For a finite set $A \subseteq \mathbb{N}$ to have a small sumset, that is, for the cardinality of $A + A$ to be 'not much bigger' than the cardinality of A itself, A should possess a lot of arithmetic structure. However, if A possesses a lot of arithmetic structure, then the product set $A \cdot A$ should have cardinality much larger than that of A. Quantifying such heuristics is often known as a 'sum-product problem'. This paper, appearing on arXiv last year, makes novel contributions to sum-product problems in finite fields. It is shown that if $A \subseteq \mathbb{F}_p$ has cardinality $|A| \ll p^{1/2}$, then

$$\max\{|A+A|, |A \cdot A|\} \gtrsim |A|^{5/4}.$$

This is stated using Vinogradov notation to suppress constants and logarithmic terms. The exponent $5/4$ improves upon $11/9$ obtained previously by Rudnev-Shakan-Shkredov. It is conjectured that the exponent can be made arbitrarily close to 2.

Squaring the circle

AUTHORS: András Máthé, Jonathan A. Noel and Oleg Pikhurko
ACCESS: https://arxiv.org/abs/2202.01412

Tarski posed the following question in 1925: is it possible to partition a circle and reassemble the pieces (via isometry) to form a partition of a square of the same area. This is not to be confused with the Banach-Tarski paradox which seeks to increase volume by rearranging a sphere in 3-dimensions into two spheres identical to the original. Such Banach-Tarski rearrangements are not possible in the plane since the group $E(2)$ is amenable, but that is another story.

Laczkovich answered Tarski's question in the affirmative in 1990 via a complicated argument relying on the axiom of choice. As a result, it was not possible to say anything about the regularity of the pieces. Subsequently, several papers have shown that one can answer Tarski's question using pieces with increasing regularity. For example, the pieces can be Borel by a result of Marks and Unger. Note that the Banach-Tarski paradox necessarily involves non-Borel pieces!

This paper, appearing on arXiv in February 2022, proved that the pieces can simultaneously have positive Lebesgue measure, be Jordan measurable, and be Boolean combinations of $\mathscr{F}_\sigma$ sets.

Integral Factorial Ratios

AUTHORS: Kannan Soundararajan
ACCESS: https://arxiv.org/abs/1901.05133

Can we classify tuples of natural numbers $(a_1, \ldots, a_K)$ and $(b_1, \ldots, b_L)$ with $\sum_i a_i = \sum_i b_i$ such that

$$\frac{(a_1 n)!(a_2 n)! \cdots (a_K n)!}{(b_1 n)!(b_2 n)! \cdots (b_L n)!}$$

is an integer for all $n \in \mathbb{N}$? This problem sounds fiendishly difficult – and it is – but there is a precise classification due to Bober in the case $L - K = 1$. Reducing to primitive examples, there are three (simple to express) infinite families together with 52 sporadic examples! This problem goes back to Chebyshev's work on counting primes up to a given N where he made use of the fact that the pair $(30, 1)$ and $(15, 10, 6)$ are an (imprimitive) example. Bober's argument builds on work of Rodriguez-Villegas, which established that a given pair of tuples satisfying the above is equivalent to an associated hypergeometric function being algebraic.

This paper, published in *Duke Mathematical Journal* in 2022, provides a new and simpler proof of Bober's result. The method also sheds light on related problems, in particular, the case when $L - K > 1$.

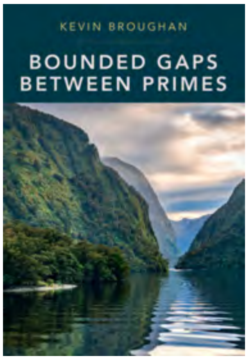
Jonathan Fraser is a Professor at the University of St Andrews and an Editor of this Newsletter. He likes fractals and is pictured here with Reuben.

Bounded gaps between primes

by Kevin Broughan, Cambridge University Press, 2021, £40,

ISBN 978-1108799201

Review by Sam Chow



Prime numbers, the building blocks of the integers. We know so much but yet so little about them. By the prime number theorem, the average gap between consecutive primes up to x is roughly $\log x$. This motivates the *Cramér random model*, that a positive integer n has an independent probability $1/(\log n)$ of being prime. Based on this, one might guess that

$$\#\{p_1, p_2 \leq x \text{ primes} : p_1 - p_2 = 2\} \approx \frac{x}{(\log x)^2}.$$

Hardy and Littlewood refined this heuristic by considering divisibility by small primes, and empirical data support their conjecture that

$$\#\{p_1, p_2 \leq x \text{ primes} : p_1 - p_2 = 2\} \sim 2C_2 \frac{x}{(\log x)^2},$$

where

$$C_2 = \prod_{p \neq 2 \text{ prime}} \left(1 - \frac{1}{(p-1)^2}\right) \approx 0.66$$

is the twin prime constant. Even before that, de Polignac had predicted that there are infinitely many twin primes, that is, pairs of primes differing by two—the twin prime conjecture. The twin prime conjecture is one of Landau's four problems presented at the 1912 International Congress of Mathematicians, and is one of the most coveted open problems in number theory.

I first encountered James Maynard at a graduate student conference in Bristol, in May 2013. In his presentation, he described his attempts to prove *bounded gaps between primes*, i.e., that

$$\liminf_{n \to \infty} (p_{n+1} - p_n) < \infty,$$

where $p_1 < p_2 < \ldots$ are the primes. As we were aware, the media had reported just days earlier that unheralded mathematician Yitang Zhang had proved this [4], specifically that

$$\liminf_{n \to \infty} (p_{n+1} - p_n) \leq 70\,000\,000.$$

Maynard was not discouraged, however, and later that year released proofs [2] that

$$\liminf_{n \to \infty} (p_{n+1} - p_n) \leq 600$$

and

$$\liminf_{n \to \infty} (p_{n+m} - p_n) < \infty \qquad (m \in \mathbb{N}).$$

The mathematical community was naturally curious as to the extent to which these bounds could be sharpened using essentially the same methods. The Polymath8 project, led by Terence Tao, was set up for this purpose, with a second goal of understanding and clarifying Zhang's argument. Polymath8a reduced Zhang's bound to 4 680. Polymath8b improved Maynard's bound from 600 to 246, where it remains to this date [3].

Kevin Broughan is an Emeritus Professor at the University of Waikito, who has researched in analytic number theory and is the author of the two-volume work *Equivalents of the Riemann Hypothesis*. In the extensive book under review, he chronicles the full story behind these developments. After an introduction and some background on sieve theory, the book presents early work, the groundbreaking method of Goldston, Pintz and Yildirim [1] upon which subsequent papers build, the aforementioned breakthroughs of Zhang and Maynard, and the meticulous refinements of Polymath8. Computational inputs are also discussed at length, as are related topics.

As well as supplying the mathematical details, Broughan embraces the human aspects of the saga. It is, after all, a wonderful tale of how two lesser-known mathematicians worked extremely hard to solve an intriguing, long-standing open problem that so many leading experts could not. The author draws from many sources, and writes with unbridled passion. He has an unusual but nonetheless effective way of writing proofs, breaking the arguments up into numbered steps, each of which is fairly short. I must caution that the book itself is very long, so the reader would not necessarily want to read every chapter or every proof.

The material is presented at a serious level and is not intended for the general public. *Bounded Gaps Between Primes* is suitable for graduate students in analytic number theory, but others may also find it interesting and informative.

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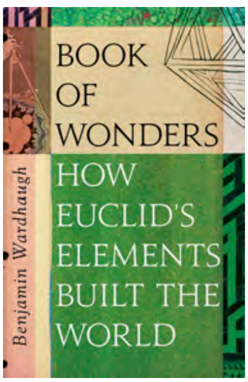
Sam Chow

Sam Chow is a lecturer of mathematics at the University of Warwick. He is a number theorist working on diophantine approximation and diophantine equations. He enjoys playing chess and other games.

The Book of Wonders: The Many Lives of Euclid's *Elements*

by Benjamin Wardhaugh, William Collins, 2020, £25, ISBN: 978-0008299903

Review by Vicky Neale



There aren't many books in history that merit a biography, but then there aren't many books in history that have the longevity, reach and impact of Euclid's *Elements*. I think that 'biography' really is the best description of Benjamin Wardhaugh's new book. It tells the story of the *Elements* through around 2300 years of history, and back and forth across the world. A book like this might seem daunting (especially to a mathematician, rather than historian of mathematics, like me): it's hardback, and quite thick, and scholarly. But in reality it is not daunting at all: it is readable, digestible, and enjoyable. The chapters are quite short, and each feels like a self-contained satisfying read in itself, while part of the sweeping narrative, and that makes it easy to read in sections.

The book starts by taking us from Euclid himself through to the 17th century. It turns out that this is the first of four sections, but because I wasn't concentrating when I read the contents page, I was disconcerted to find that section two jumps right back to Plato (before Euclid), before heading forwards through time again. I'm not sure that I can clearly articulate the difference between these two sections; I'm sure there is one, but both felt to me like tracking the book through time. I was less surprised when section three arrived, and I think this one was more about the practical uses to which the ideas of the *Elements* have been put. The final, fourth, section, addresses the limitations of the *Elements*, and the ways in which societies have fallen out of love with Euclid. My slight confusion about the different sections didn't detract from my engagement with the book, but did leave me wondering whether it might be helpful to have a timeline (or to have created my own) showing the sequence of events addressed in overlapping chronologies.

I enjoyed the broad sweep through time and between cultures. But I also, at least as much, enjoyed the individual chapters, each feeling like a snapshot of a moment in time and space. There were many new insights and historical details here for me. I was intrigued by the characterisation that "Greek geometry was essentially a performance, consisting of drawing a diagram and talking about it, to oneself or to an audience". The pandemic-imposed need to move to lecture videos in recent months has prompted me to reconsider many aspects of my own teaching practice, including the 'performance' angle, which feels to me different but just as important at my kitchen table as at a whiteboard in a lecture theatre. Probably my absolute favourite nugget in the book was about the first attempts to typeset the *Elements* for printing, in the late 15th century. Remember that this was a world of physical blocks for characters, which had to be arrayed on a frame before being inked for printing. Geometrical diagrams, with disproportionately many references to points A, B, C, D, meant that compositors would literally run out of certain characters!

Alongside the relatively mainstream narrative about the *Elements*, I enjoyed the cameo appearances by numerous people I'm sorry to say I hadn't heard of before, or where I was unaware of their interest in Euclid. No doubt many readers of the *LMS Newsletter* will know of the work of Hroswitha of Gandersheim, but I didn't. She was a Benedictine canoness in Saxony in the late 10th century, very well educated, and an author of numerous works, including a play called *Wisdom*, which included some number theory. Another example: Anne Lister's extraordinary life was recently brought to public attention when her diaries inspired the BBC television drama series *Gentleman Jack*. As Wardhaugh tells us, Lister chose in her twenties to resume her study of the *Elements*, and got further "frankly, than most university graduates".

Wardhaugh understandably places much emphasis on tracing the progress of the *Elements* as it moves between cultures and indeed between languages, with each new editor/author/translator giving their own new take on it. In a chapter named after Xu Guangqi, he explores the difficulties of translating words such as 'definition', 'proof' and 'axiom' into Chinese in a way that would capture their significance in the Greek tradition. I found this thought-provoking: having been raised in a certain tradition, seeing another culture's viewpoint can prompt us to reassess the meaning of ideas we otherwise take as self-evident.

This biography explores the impact of Euclid's great work on mathematics, of course, but also on other areas. The chapter on Piero della Francesco describes the influence of the *Elements* on the style of his pioneering writing on the theory of perspective. An earlier chapter on Muhammad Abu al-Wafa al-Buzjani has a fascinating discussion of the creation of a certain number of squares from the dissection and rearrangement of smaller squares — a question with interesting geometrical significance, but also a practical one for artisans creating mosaic designs. A chapter late in the book is named after two fictional characters, Maggie and Tom Tulliver, from George Eliot's *The Mill on the Floss*. Marian Evans (the real name of Eliot) elected to study geometry in later life, but her creation Tom Tulliver was no fan of Euclid after his bad experiences at school, and Wardhaugh uses this to exemplify "the failure of Euclidean education in the nineteenth century: or at least of the failure of that education once it had become detached from humanity, and indeed from common sense about the needs of different students and their differences from one another". A sobering reminder for us all.

I could keep going with tasty morsels from the book, but frankly Wardhaugh tells the stories better than I do. So instead, let me consider: who will want to read this book? I am sure that historians of mathematics will find much of value in this book, helped by the extensive notes and references at the end. But it deserves to have a wider readership too, amongst mathematicians and aspiring mathematicians, who will enjoy and learn from dipping in or indeed reading from cover to cover, and getting another glimpse into the development of our subject, and the understanding and insight to be gained through the exchange, refinement and revision of ideas as they pass between cultures. A book of wonders indeed.

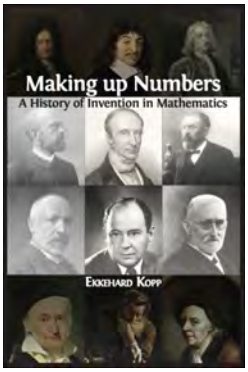
Vicky Neale

Vicky Neale is the Whitehead Lecturer at the Mathematical Institute and Balliol College, University of Oxford. Her job is to be enthusiastic about maths with undergraduates, school students and the wider public. She enjoys mathematical knitting and crochet, with the 'help' of her two cats.

Making up Numbers: A History of Invention in Mathematics

by Ekkehard Kopp, Open Book Publishers, 2020, ISBN 978-1-80064-097-9

Review by Adrian Rice



God created the natural numbers, all else is the work of man. — attributed to Leopold Kronecker.

Is mathematics invented or discovered? Do mathematical concepts spring into existence only when first imagined by a human mind or are they eternal? The famous quotation above makes Kronecker's opinion on the matter crystal clear, encapsulating his constructivist philosophy that, aside from the positive integers, the entirety of mathematics is a human invention. In its strongest form, this view maintains that nothing in mathematics can really be said to exist unless a human has seen it, thought of it, or written it down. This has always seemed to me to be a little extreme: for example, even if no one else in the history of humanity has had any reason to write down the complex number $57.49544982368932873252537 + 0.00027265238742335340712662i$ before me, my claim to have invented it is somewhat dubious. On the other hand, the case for invention in mathematics is still strong: for instance, nothing like the concept of a Galois group can be said to have existed in any form before the 19th century—it is purely a creation of the human mind. In fact, as the history of mathematics reveals, the subject's development has been a combination of both invention *and* discovery: concepts and techniques are invented to solve certain problems, the solution of which often leads to the discovery of new ideas, that themselves pose new questions, often requiring the creation of new concepts and techniques, and the process continues.

Making up Numbers is written from the standpoint of mathematics as a human invention. More specifically, since the notion of number has been essential to human civilization for millennia, as well as possibly the oldest concept in mathematics, this book presents a broadly chronological treatment of the history of the use of numbers in pure mathematics from antiquity to the 20th century. From Mesopotamia and the ancient Greek world to more recent developments, the author presents an engaging account of how the concept of number was gradually extended to include negatives, irrationals and complex numbers. The first four chapters concentrate on these extensions and the motivations behind them, from the Pythagorean obsession with positive integers to Gauss's proof of the fundamental theorem of algebra. We then come to a two-chapter interlude on issues arising from the concepts of infinity and infinitesimals, obviously concerning the creation of calculus and analysis. This leads inevitably to questions concerning continuity, with the final four chapters concerning the rigorisation of the real number system, the growth of axiomatization in the 19th century, Cantor's transfinite numbers, and the infamous paradoxes of the early 20th century. In this way, readers are guided through a variety of themes, from solving the earliest known equations to the role of axioms in the foundations of mathematics.

It's a well-known story and one that is told very well here. Those for whom this book is their first encounter with the history of mathematics will find it a reliable and informative introduction. But the prospective reader should perhaps also be aware that this book is fundamentally a history of real numbers. This is by no means a criticism—and perhaps says more about the contrasting mathematical tastes of author and reviewer—but it might be helpful to briefly highlight what this book does not contain. While there is certainly plenty on complex numbers, the discussion of further extensions in this direction is halted with just a passing reference to Hamilton's creation of quaternions. This is a pity, not only because the birth of quaternions and subsequent systems of hypercomplex numbers revolutionized the whole notion of what a 'number' could be, but also because the introduction of new imaginary quantities such that $i^2 = j^2 = k^2 = ijk = -1$ is one of the most remarkable instances of invention in the history of mathematics. (Even the first LMS President, Augustus De Morgan, was taken aback by the idea of '*imagining* imaginaries', as he put it.)

There is a good discussion of primes up to the fundamental theorem of arithmetic, and an introduction to algebraic numbers later on; but there are no allusions to further possibilities, such as Gaussian primes, or other generalizations of the idea of a prime number. Staying within the real domain, mention could have been made of Mersenne primes, along with their intriguing connection to perfect numbers (another concept not mentioned). A reference to Germain primes might also have given the author a chance to highlight the work of a female mathematician — difficult to do in a work of this kind. Another figure not mentioned, but whose results alone would be sure to astonish any reader, is Ramanujan, who produced some stunning rational approximations of irrational numbers, and of whom Littlewood famously remarked 'every positive integer was one of his personal friends'. Perhaps also surprising is that, although the final chapter deals with a host of foundational issues and contributors, including Cantor, Hilbert, Russell, Brouwer and Gödel, one contribution is not included, namely that of Alan Turing, whose monumental LMS paper of 1936, as well as answering Hilbert's Entscheidungsproblem via the introduction of Turing machines, also invented a new theoretical concept of relevance to a history of real numbers: the computable number.

None of this should, however, detract from the book's significant strengths. While not so much concerned with algebra and number theory, it is nevertheless crammed with some very interesting mathematics, particularly in the realms of analysis and set theory. It does not profess to be a learned volume on the history of mathematics, since its intended audience is A-level and undergraduate students and their teachers; but it was clearly written by someone with great knowledge and appreciation of the discipline. More importantly, the author clearly conveys a deep love and understanding of the mathematics he discusses, combining lively readable prose with a clear presentation of the material to give just enough technical detail without overwhelming the relative beginner. The lay reader will definitely come away with an idea of what it is to *do* mathematics and above all, with the knowledge that mathematics is far from being a static discipline, but an ever-evolving creative process. In short, the author seeks to convince the reader that mathematics is fundamentally a human endeavour — and in that, this book succeeds admirably.

Adrian Rice

Adrian Rice is the Dorothy and Muscoe Garnett Professor of Mathematics at Randolph-Macon College in Virginia, USA, where his research focuses on 19th- and early 20th-century British mathematics. He is nostalgic for the days when his photograph made him look younger than he is.

Obituaries of Members

Bruce D. Craven: 1931 – 2022

Dr Bruce D. Craven, who was elected a member of the London Mathematical Society on 16 January 1964, died on 26 January 2022, aged 90.

Barney Glover writes: Bruce Desmond Craven was born in Melbourne in 1931 and educated at the University of Melbourne, completing a BSc in 1951, MSc in 1953 (both with honours) and BA (Honours) in 1959. He was awarded his DSc degree in 1973 for published and unpublished works across a variety of fields including optimisation, mathematical analysis, linear algebra and number theory, reflecting his broad interest in pure and applied mathematics.

Bruce's working life began in the UK with General Electric, working on early analog computers in 1954. He then returned to Australia and worked for a period at Australian Paper Manufacturers as a Senior Research Physicist again focusing on applications of computers and the emerging field of operations research and linear programming. He was appointed as a Senior Lecturer in the Department of Mathematics at the University of Melbourne in 1962 eventually retiring as a Reader in 1997 and then continuing as a Principal Research Fellow.

Bruce was a gifted mathematician with broad and deep interests in pure and applied mathematics, publishing extensively across operations research, mathematical programming, optimisation theory, generalised and abstract convexity, as well as nonsmooth and classical mathematical analysis. Bruce published over 200 research articles and seven books during a career spanning almost 50 years. His contributions were characterised by extensive collaborations with researchers in Europe, Asia and North America. Bruce enjoyed travel throughout his career, visiting colleagues and attending conferences.

Bruce's books included monographs on mathematical programming, Lebesgue measure and integration, control theory, functions of several variables and fractional programming. His first book, *Mathematical Programming and Control Theory*, published by Chapman and Hall in the late 1970s, was a significant contribution to graduate programs in optimisation theory and applications. It provided a unified approach to nonlinear optimisation and control theory using abstract cone-constrained mathematical programming as the underlying structure. Bruce developed the general constraint qualification of local solvability and extended the notion of invariant convexity to establish necessary and sufficient optimality and duality conditions for a range of mathematical programming problems including in multicriteria optimization. This theme continued through much of his published work until ill health restricted his collaboration in recent years.

Bruce was awarded the Ren Potts Medal by the Australian Society of Operations Research in 1997 for his contribution to and excellence in Operations Research.

Throughout his life Bruce enjoyed a passion for the Apple Macintosh computer and for many years provided a regular column, amounting to over 150 articles, to the Apple Users' Society of Melbourne (AUSOM).

Bruce was a larger than life man, truly compassionate and friendly to everyone he met. He enjoyed travelling, learning languages, playing the church organ, supporting many charities and living a simple life immersed in mathematics and computing. He was frustrated at times by the challenges of what appeared to be a never ending administrative burden on academics, something which led him to pen a satirical poem at the time of his retirement in 1997. He will be dearly missed by all who knew him, experienced his wonderful lectures and struggled with his somewhat eccentric humour.

Death Notices

We regret to announce the following death:

- Roger W. Carter, formerly of the University of Warwick, who was elected a member of the London Mathematical Society on 18 June 1959, died on 21 February 2022, aged 87.

Noncommutative Algebra and its Applications

Location: Online
Date: 9–12 May 2022
Website: icnca.modares.ac.ir

The work of the conference concerns, on one hand, classical aspects of ring and module theory and, on the other, some applications of this theory, focusing on coding theory and cryptography in which rings and modules play a substantial role. The conference will present new results and future challenges, in a series of virtual keynote lectures and virtual contributed short talks. The invited speakers will present their recent works and many participants will contribute to enriching further the themes addressed by the conference. To register by 7 May, email moussavi.a@modares.ac.ir.

Wales Mathematics Colloquium 2022

Location: Gregynog Hall, Tregynon
Date: 23–25 May 2022
Website: https://gregynogwmc.github.io

The Colloquium is a forum for the promotion and discussion of current research in Mathematics in Wales. The invited speakers are Chris Breward (University of Oxford), Brita Nucinkis (Royal Holloway, London) and Stefan Weigert (University of York). Attendance and contributed talks in any area of mathematics are very welcome. Supported by an LMS Conference grant.

Integrability and Nonlinear Waves at Northumbria

Location: Northumbria University, Newcastle
Date: 27–28 May 2022
Website: tinyurl.com/4ajvr4v6

This workshop aims at exploring the interconnections between integrable systems and nonlinear waves, and to foster new links between these two research communities. Talks will be given by Costanza Benassi (Northumbria University), Thibault Congy (Northumbria University), Jenya Ferapontov (Loughborough University), Claire Gilson (University of Glasgow) and Noel Smyth (University of Edinburgh). The workshop is funded by two LMS Scheme 9 awards and supported by Northumbria University.

Colloquia in Combinatorics 2022

Location: Queen Mary and LSE
Date: 11–12 May 2022
Website: tiny.cc/2dayCC

This is the 15th edition of the Colloquia in Combinatorics. The talks will be of interest to all those working in combinatorics and related fields. Confirmed speakers are F. Frick (Berlin), A. Heckel (Uppsala), A. Holroyd (Bristol), M. Paterson (Birkbeck), J. Saharasbude (Cambridge), L. Sanità (Eindhoven), M. Bouvel (LORIA), A. Coja-Oghla (Dortmund), M. Kwan (IST), A. Steger (Zurich) and V. Traub (Zurich). Visit the website for more details. Research students are invited to present their research in a poster session. The meeting is supported by an LMS Scheme 1 Conference grant.

Inference for Expensive Systems in Mathematical Biology

Location: Oxford University
Date: 23–24 May 2022
Website: tinyurl.com/nh9n7j5j

The purpose of this conference is to bring together mathematical biologists and statisticians to share ideas about best practices for computationally expensive inference problems encountered in biological applications. The conference is open to participants from all career stages. Invited speakers include Professor Ruth Baker (University of Oxford) and Professor Heikke Haario (Lappeenranta-Lahti University of Technology, Finland). The meeting is supported by an LMS Scheme 1 Conference grant.

BMC73

Location: King's College, London
Date: 6–9 June 2022
Website: bmc2022.co.uk

This year's British Mathematical Colloquium (BMC73), which is supported by the LMS, will take place on 6–9 June 2022, and will be hosted by King's College London. The LMS Lecture will take place on 7 June. The colloquium will provide an invaluable opportunity for mathematicians across the spectrum of pure and applicable mathematics to meet, hear about new developments from international and national experts in both proximate and more distant areas, and to discuss their work with each other. Registration deadline: 31 May.

Finite Groups

Location: Cambridge/Hybrid
Date: 10 June 2022
Website: tinyurl.com/2ej55h2e

This meeting will build on the LMS online lecture on the theory of crowns in finite groups, delivered by Gareth Tracey in 2020. It will take place in hybrid format at the Isaac Newton Institute during the programme Groups, Representations and Applications: New Perspectives and is devoted to some original techniques pertaining to the study of finite groups. Speakers: Lucia Morotti, Alice Niemeyer, Gareth Tracey. The meeting forms part of the Functor Categories for Groups Joint Research Group supported by an LMS Scheme 3 Grant.

Defects and Symmetry

Location: King's College London
Date: 23–24 June 2022
Website: tinyurl.com/2p8uhucn

There is a deep connection between the topological defects that arise in diverse models of mathematical physics and recent generalisations of symmetry. In this short meeting we want to bring together researchers working with higher categorical structures and those whose research involves defects and generalised symmetries in quantum field theories or lattice models, in order to present new results and explore new connections. This meeting is supported by an LMS Conference grant.

Operator Algebras: Subfactors, K-theory, Conformal Field Theory

Location: Gregynog Hall, Wales
Date: 27 July – 2 August 2022
Website: oa-gregynog-2021.weebly.com

This conference, held in honour of David Evans' 70th birthday, aims to bring together international experts and young researchers in operator algebras to discuss problems in subfactor theory, K-theory and their application in conformal quantum field theory. Full cost (including accommodation at Gregynog Hall, meals and conference dinner) is £680, which must be paid in advance. To book, and for details of the cancellation policy, please visit the webpage.

Oxford's Sedleian Professors of Natural Philosophy: The first 400 Years

Location: Weston Library, Oxford
Date: 18 June 2022
Website: tinyurl.com/yckj5fvb

The Sedleian Professorship of Natural Philosophy, founded in 1619, is one of Oxford's oldest Chairs. Although it is now a post devoted to applied mathematics, in previous centuries it was held variously by physicians, theologians, and an astronomer. At this one-day meeting, the contributors to a forthcoming multidisciplinary volume on the history of the professorship will give short talks on the subjects of their chapters. The meeting is supported by an LMS Scheme 1 Conference grant.

ICM 2022

Location: Online
Date: 6–14 July 2022
Website: mathunion.org/icm/virtual-icm-2022

The International Congress of Mathematicians 2022 (ICM 2022) will this year take place as a fully virtual event. Lectures are scheduled to take place between 9:00 and 19:00 CEST every day. There are also efforts within the mathematical community to organise in-person and online events to complement the virtual ICM. Participation is free of charge, but registration will be required. Prior to ICM 2022, the IMU will host its 19th General Assembly in Helsinki, Finland, on 3–4 July 2022. The IMU Award Ceremony will also be held as a live event in Helsinki on 5 July; the event will be streamed.

Groups St Andrews 2022 in Newcastle

Location: Newcastle University
Date: 30 July – 7 August 2022
Website: tinyurl.com/2p9czjzk

This conference covers all aspects of group theory. The programme will include minicourses given by each of the four principal speakers: Michel Brion (Institut Fourier, Université Grenoble Alpes), Fanny Kassel (Institut des Hautes Études Scientifiques), Denis Osin (Vanderbilt University) Clay Lecturer, Pham Huu Tiep (Rutgers University) Clay Lecturer, and 1-hour talks by a further five invited speakers. All delegates will be able to offer a short talk. This meeting is supported by an LMS Scheme 1 Conference grant.

Society Meetings and Events

May 2022

6 Society Meeting and Hirst Lecture, De Morgan House, London
10 IMA/LMS David Crighton Lectures, Royal Society, London
25 LMS/Gresham Lecture: The Maths of Gyroscopes and Boomerangs: Museum of London and online
24 Northern Regional Meeting, Leeds

June 2022

7 Society Meeting at the BMC–BAMC

July 2022

1 Society Meeting and Aitken Lecture, BMA House, London
18-22 LMS–INI–Bath Symposium: K-Theory and Representation Theory, University of Bath
18-22 LMS Invited Lectures, Equations in Groups and Complexity, Newcastle University

Calendar of Events

This calendar lists Society meetings and other mathematical events. Further information may be obtained from the appropriate LMS Newsletter whose number is given in brackets. A fuller list is given on the Society's website (www.lms.ac.uk/content/calendar). Please send updates and corrections to calendar@lms.ac.uk.

May 2022

2-6 Adaptive Methods and Model Reduction for PDEs Research School, Nottingham (496)
9-12 Noncommutative Algebra and its Applications, Tarbiat Modares University/hybrid (500)
11-12 Colloquia in Combinatorics 2022, Queen Mary and LSE (500)
23-25 Inference for Expensive Systems in Mathematical Biology, Oxford University
18-20 Mathematics in Signal Processing, Aston, Birmingham (495)
23-24 Inference for Expensive Systems in Mathematical Biology, Oxford University (500)
23-25 Wales Mathematics Colloquium 2022, Gregynog Hall, Tregynon (500)
27-28 Integrability and Nonlinear Waves at Northumbria, Northumbria University, Newcastle (500)

June 2022

6-9 BMC73, King's College London (500)
8-10 Mathematics of Finance and Climate Risk Conference, Holiday Inn, Liverpool (498)
10 Finite Groups, Cambridge/Hybrid (500)
18 Oxford's Sedleian Professors of Natural Philosophy: The first 400 Years, Weston Library, Oxford (500)
20-24 Fast Solvers for Frequency-Domain Wave-Scattering Problems and Applications, University of Strathclyde (500)
23-24 Defects and Symmetry, King's College London (500)
27-1 Jul Point Configurations: Deformations and Rigidity Graduate Research School, University College London (498)
29-1 Jul 7th IMA Conference on Numerical Linear Algebra and Optimization (498)

July 2022

6-14 ICM 2022, online (500)
11-15 British Combinatorial Conference, Lancaster University (499)
13-15 Maths in Music Conference, Royal College of Music, London (498)
18-22 New Challenges in Operator Semigroups, St John's College, Oxford (498)
18-22 Rigidity, Flexibility and Applications LMS Research School, Lancaster (497)

20-22 Microlocal Analysis and PDEs, University College London (498)

24-26 7th IMA Conference on Numerical Linear Algebra and Optimization, Birmingham (487)

27-2 Aug Operator Algebras: Subfactors, K-theory, Conformal Field Theory, Gregynog Hall, Wales (500)

25-29 New Trends in Moduli Spaces and Vector Bundles, University of Warwick (499)

30-7 Aug Groups St Andrews 2022 in Newcastle (500)

August 2022

22-26 Unlikely Intersections in Diophantine Geometry, University of Oxford (500)

September 2022

1-2 Applied Mathematical Challenges and Recent Advances in the Micro-Mechanics of Matter 2022, University of Bristol (500)

5-9 COMB in CAMB: Combinatorial Methods in Algebraic Geometry in Cambridge (500)

19-20 Mathematical Challenges of Big Data IMA Conference, University of Oxford/Hybrid (500)

PHILOSOPHICAL
TRANSACTIONS A

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by Maria Patrizia Pera and Marco Spadini.
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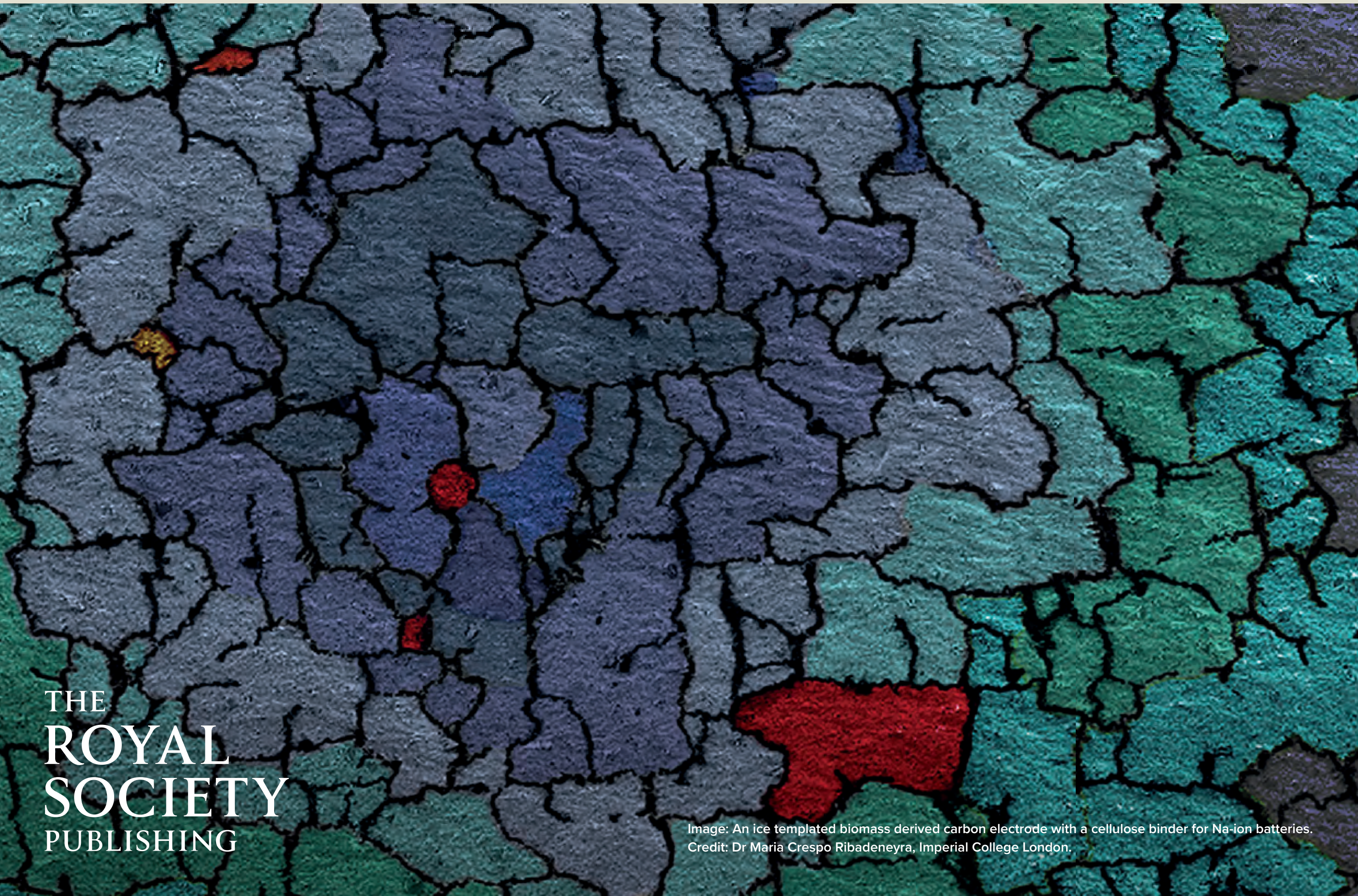

THE ROYAL SOCIETY PUBLISHING

Image: An ice templated biomass derived carbon electrode with a cellulose binder for Na-ion batteries.
Credit: Dr Maria Crespo Ribadeneyra, Imperial College London.