The Baire Category Theorem and the Banach-Mazur game

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Course description:

In 1935, the Polish mathematician Stanislaw Mazur proposed the following game.

There are two players called Player 1 and Player 2. A subset A of the interval $[0,1]$ is fixed beforehand, and the players alternately choose subintervals $I_n \subset [0,1]$ so that $I_{n+1} \subseteq I_n$ for each $n \geqslant 1$. Player 1 wins if the intersection of all I_n intersects A, and Player 2 wins if he can force this intersection to be disjoint from A.

Mazur observed that if A can be covered by a countable union of sets, each with a closure which has empty interior (A is of first category), then the second player wins; while if the complement of A is of first category then the first player wins. Later Banach proved that these conditions are not only necessary for the existing winning strategies but are also sufficient.

The game can be generalised to an arbitrary topological space X. Then in order to decide whether a certain property describes a *typical* object of X it is enough to show that there is a winning strategy for Player 2 with respect to the set of objects satisfying the given property. Remarkably, one can show in this way that a 'typical' continuous function is differentiable at no point!

Recommended texts:

1. Any introductory text on Zorn's Lemma
2. B. Bolobas, Linear Analysis. Cambridge University Press, Cambridge, 1999.
3. J. Oxtoby, Measure and category. A survey of the analogies between topological and measure spaces. Second edition. Graduate Texts in Mathematics, 2. Springer-Verlag, New York-Berlin, 1980.

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The Baire category is a profound triviality which condenses the folk wisdom of a generation of ingenious mathematicians into a single statement.

T.W. Körner, "Linear analysis" Sect.6, p.13.

§1 Baire Category Theorem

Ex. 1.1. (A) [1] Find out how every metric space is a topological space. More precisely, define a topology induced in a *natural* way by the metric.

Ex. 1.2. (B) Let (X, τ) be a topological space and $(U_n)_{n\geqslant 1}$ a collection of open subsets of X. Is it necessarily true that $\bigcap_{n\geqslant 1} U_n \in \tau$ (i.e. is an open set) too? Give a proof if yes, and a counterexample if no.

Ex. 1.3. (A) Let (X, τ) be a topological space and $E \subseteq X$. Prove that $F = \mathrm{Cl}(E)$ is the smallest closed set which contains E. In other words, prove that F is closed, $F \supseteq E$ and for any closed set V such that $V \supseteq E$ we have $V \supseteq F$.

Ex. 1.4. (A) Prove the equivalence of two definitions of a dense subset of a topological space, given in the lecture.

Ex. 1.5. (D) Let X be a topological space which satisfies the following property: for any collection $(G_n)_{n\geqslant 1}$ of open dense subsets of X their intersection $\bigcap_{n\geqslant 1} G_n$ is not empty. Does X need to be a Baire space?

Ex. 1.6. (B) Let X be a topological space, and $A_i \subseteq X$, $B_{i,j} \subseteq A_i$ be such that $\bigcup_i A_i$ is dense in X, and $\bigcup_j B_{i,j}$ is dense in A_i for each fixed i. Prove: $\bigcup_{i,j} B_{i,j}$ is dense in X. Here $i \in I$, $j \in J$, where I and J are arbitrary sets of indices.

Ex. 1.7. (B) Let (X, d) be a complete metric space, and $(F_n)_{n\geqslant 1}$ a *nested* sequence of closed subsets of X (i.e. $F_n \supseteq F_{n+1}$ for each $n \geqslant 1$) such that $\mathrm{diam}(F_n) \to 0$. Prove that $\bigcap_{n\geqslant 1} F_n$ is not empty and is equal to a one-point set: $\bigcap_{n\geqslant 1} F_n = \{x_0\}$ for some $x_0 \in X$.

Is it true that in any complete metric space the intersection of any sequence of nested closed sets is not empty?

Ex. 1.8. (B) Starting with the closed interval $[0, 1]$, the *ternary* Cantor set $C_{1/3}$ is obtained by removing middle thirds countably many times. More precisely, let

$$A_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1], A_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1], \ldots,$$

[1] The letters A,B,C,D refer to the level of difficulty with A being the easiest and D the hardest. This is however very subjective so you might find some of the exercises marked C or D not difficult at all. However I hope that all exercises marked A should be easy to complete.

$$A_{n+1} = \frac{1}{3}A_n \cup (\frac{1}{3}A_n + \frac{2}{3}).$$

Then

$$C_{1/3} := \bigcap_{n \geqslant 1} A_n.$$

Show that $C_{1/3}$ is closed and nowhere dense in $\mathbb{R}$.

Ex. 1.9. (A) Show that a countable union of sets of first category is a set of first category.

Ex. 1.10. (B) Let X be a non-empty topological space. Is it possible that its subset D is both dense and nowhere dense?

Is it possible for a subset E of X to be both dense and of first category?

Ex. 1.11. (B) Let X be a non-empty Baire space. Prove that it is of second category (considered as a subset of itself).

Ex. 1.12. (B) Let X be a non-empty Baire space, and $(G_n)_{n\geqslant 1}$ a sequence of open dense sets. If $A \subset X \setminus \bigcap_{n\geqslant 1} G_n$, then show that A is of first category.

Is it true that for each $D \subseteq X$ dense, $X \setminus D$ is of first category?

Ex. 1.13. (B) Consider $X = \mathbb{R}$ with the usual metric. Find a subset $D \subseteq \mathbb{R}$ of first category which is of second category when considered a metric space itself (with respect to the induced metric).

Ex. 1.14. (B) Assume (X, τ) is a Baire space and $G \subseteq X$ is a non-empty open subset of X. Show that (G, τ_G) is a Baire space too.

Give an example of a non-complete metric space which is Baire.

Ex. 1.15. (D) Show that Sorgenfrey Line $\mathbb{R}_S$ is a Baire space but it is not metrisable:

$$X = \mathbb{R}, \qquad B_\tau = \Big\{[a, b), a < b \text{ and } a, b \in \mathbb{R}\Big\}$$

is the *base* for topology on $X = \mathbb{R}_S$.

Ex. 1.16. (D) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, such that $f(nx) \underset{n\to\infty}{\to} 0$ for each fixed $x > 0$. Prove that $f(x) \underset{x\to\infty}{\to} 0$.

Hint: Use Baire category theorem here, for a fixed $\varepsilon > 0$ consider sets

$$F_n = \{x > 0 : |f(kx)| \leqslant \varepsilon \text{ for all } k \geqslant n\}.$$

Ex. 1.17. (D) Let an infinitely differentiable function $f : [0, 1] \to \mathbb{R}$ be such that for each fixed $x \in \mathbb{R}$ there exists a natural index $n \geqslant 1$ (depending on x) such that $f^{(n)}(x) = 0$. Prove that there is a non-empty open interval $(a, b) \subseteq [0, 1]$ such that $f|_{(a,b)}$ is a polynomial.

Hint: Use Baire category theorem here, consider sets $F_n = \{x \in \mathbb{R} : f^{(n)}(x) = 0\}$.

§2 The Banach-Mazur Game

Ex. 2.1. (A) Let A be a finite subset of $[0,1]$. Describe a winning strategy for Player 2 in the Banach-Mazur game with target A.

Ex. 2.2. (B) Let $A = \mathbb{Q} \cap [0,1]$. Describe a winning strategy for Player 2 in the Banach-Mazur game with target A.

Ex. 2.3. (C) Let $A \subseteq [0,1]$ be any set of first category. Describe a winning strategy for Player 2 in the Banach-Mazur game with target A.

Ex. 2.4. (C) Prove Statement 2 from the proof of Theorem 2.1.

Ex. 2.5. (B) Prove that if in the Banach-Mazur game with target set A Player 2 has a winning strategy, it is possible to modify this strategy in such a way that the strategy remains a winning strategy but in addition the intersection of all intervals is equal to a one-point set $\{x_0\}$ which does not coincide with any of the endpoints of intervals chosen by Player 2.

Ex. 2.6. (B) Prove that if in the definition of the game, instead of choosing closed intervals of positive length, we say that Players 1 and 2 always choose non-empty open intervals, we get an equivalent game. That is, if $A \subseteq [0,1]$ is the target set, then Player n has a winning strategy in the "closed intervals" game if and only if Player n has a winning strategy in the "open intervals" game.

Ex. 2.7. (B) Prove that if in the definition of the game, instead of choosing non-empty open intervals, we say that each player chooses a non-empty open subset of the set chosen by their opponent, we get an equivalent game.

Ex. 2.8. (B) Explain why the conclusion of Theorem 2.1 remains true if we replace the game within $[0,1]$ by the same game on subsets of $\mathbb{R}$.

Ex. 2.9. (B) Consider a generalised Banach-Mazur game on a topological space X. Prove that if A is of first category, then Player 2 has a winning strategy.

Ex. 2.10. (C) Let (X,τ) be a topological space and $f : \tau \setminus \{\emptyset\} \to \tau \setminus \{\emptyset\}$ a mapping such that $f(U) \subseteq U$ for each non-empty $U \in \tau$.

Consider a family $\mathcal{F}$ of collections $\mathcal{A}_\alpha \subseteq \tau \setminus \{\emptyset\}$[2] such that for every two distinct elements $U_1, U_2 \in \mathcal{A}_\alpha$ we have $f(U_1) \cap f(U_2) = \emptyset$.

Verify that $(\mathcal{F}, \subseteq)$ satisfies the conditions of the Zorn's lemma. Let further $\mathcal{V}$ be a maximal element of $\mathcal{F}$. Verify that $\bigcup_{U \in \mathcal{V}} f(U)$ is dense in X and $f(U_1) \cap f(U_2) = \emptyset$ for any $U_1, U_2 \in \mathcal{V}$, $U_1 \neq U_2$.

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[2]That is, each $\mathcal{A}_\alpha$ is a collection of non-empty open subsets of X.