# Local-Global Principles in Number Theory

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Diophantine equations are polynomial equations with integer coefficients, where one typically looks for solutions in the integers or rational numbers. One of the problems with studying such equations is that analytic techniques are not directly applicable: for example, if we are just looking for the existence of real solutions then we might be able to use the intermediate value theorem but the fact that the field of rational numbers is not complete precludes this line of attack.

Analysis can, on the other hand, give another idea by looking at an analogy between $\mathbb{Q}$ and the field $\mathbb{C}(x)$ of rational functions with complex coefficients. Any function in $\mathbb{C}(x)$ defines a meromorphic function on the complex plane so, from complex analysis, is determined by its "local behaviour" – that is, from its Laurent series expansion $\sum_{n\geq n_0}^{\infty} a_n(x-z_0)^n$ in a neighbourhood of any $z_0 \in \mathbb{C}$. Now $(x-z_0)$ generates a maximal ideal in the polynomial ring $\mathbb{C}[x]$ and, by truncation, the coefficients in the Laurent series expansion are describing the behaviour of the function "modulo $(x-z_0)^n$".

The analogue of this for the rational numbers is to take a maximal ideal in $\mathbb{Z}$, which is generated by a prime number $p$, and to look modulo powers of $p$. Thus the idea of the local-global principle in number theory is to try to derive implications on (the existence of) solutions to a diophantine equation from (the existence of) solutions modulo $p^n$, for each prime $p$ and positive integer $n$. One might also ask whether one can have an analogue of a Laurent series expansion and, if so, where this would live – the answer being a suitable completion of $\mathbb{Q}$.

In these lectures we will first look at the local-global principle in number theory first at the level of congruences (if a diophantine equation has solutions modulo $n$ for all $n$ then does it have an integral/rational solution), then introduce the $p$-adic numbers as a way of understanding the local behaviour "at a prime number $p$", and finally look at some examples of the success/failure of the local-global principle in number theory.

## Recommended literature

Cassels J., *Local Fields*, LMS Student Texts 3, Cambridge University Press (1986).

Gouvêa F.Q., *$p$-adic Numbers - an introduction*, Springer-Verlag (1997).