Extremal Graph Theory and Flag Algebra Exercises

1. Prove that a graph G is bipartite if and only if it contains no odd length cycles.

2. Show that if G is a graph with $\chi(G) = k$ then G has at least $\binom{k}{2}$ edges.

3. Prove that for any graph F the *Turán density*,

$$\pi(F) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, F)}{\binom{n}{2}},$$

   is well-defined.

4. Calculate the Turán density of the Petersen graph.

5. If $\mathcal{F} = \{K_6, C_{11}, K_{20,3}\}$ what is $\pi(\mathcal{F})$?

6. Let $G = (V, E)$ be triangle-free with n vertices and suppose $\forall v \in V$ $d(v) > 2n/5$. Show that G is bipartite.

7. Prove that if $G = (V, E)$ is a graph with $n \geq 3$ vertices and $\lfloor n^2/4 \rfloor + 1$ edges, then G contains at least $\lfloor n/2 \rfloor$ triangles.

   Can you give an example to show that this is sharp?

8. Let $G = (V, E)$ be a graph and π be a random ordering of V. By considering the expected size of the set

$$U = \{v \in V \mid v \text{ comes before all of its neighbours in the ordering } \pi\},$$

   show that

$$\alpha(G) \geq \sum_{v\in V} \frac{1}{d(v)+1}. \tag{1}$$

   Deduce Turán's theorem.

9. Let $\mathcal{P} \subset \mathbb{R}^2$ be a set of n points and $\mathcal{L} \subset \mathbb{R}^2$ be a set of m lines. How many pairs can there be $(P, L) \in \mathcal{P} \times \mathcal{L}$ with $P \in L$? (Hint: can you convert this into a forbidden subgraph problem?)

10. Let $S \subset \mathbb{R}^2$ with $|S| = n$ and suppose that $||x-y||_2 \leq 1$ for all $x, y \in S$. Show that

$$T = \left\{(x,y) \in S^2 \mid ||x-y||_2 > \frac{1}{\sqrt{2}}\right\}$$

satisfies $|T| \leq \lfloor n^2/3 \rfloor$. (Hint: can you express this as a forbidden subgraph problem?)

Give an example to show that this bound is sharp.

11. Let

$$\Sigma^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, \ldots, x_n \geq 0, \sum_{i=1}^{n} x_i = 1\}.$$

Given a graph $G = (V, E)$ with $V = [n]$ and $x \in \Sigma^n$, define $\lambda(G, x) = \sum_{ij \in E} x_i x_j$ and $\lambda(G) = \max_{x \in \Sigma^n} \lambda(G, x)$.

Suppose that $y = (y_1, \ldots, y_n) \in \Sigma^n$ satisfies $\lambda(G) = \lambda(G, y)$.

(a) Show that if $y_i, y_j > 0$ then

$$\sum_{k \in \Gamma(i)} y_k = \sum_{k \in \Gamma(j)} y_k.$$

(b) Show that there exists $z \in \Sigma^n$ such that $\lambda(G) = \lambda(G, z)$ and $\{i \mid z_i > 0\}$ is a clique in G.

(c) Prove Turán's theorem using $\lambda(G)$.

12. Let G be a graph with $d(K_3) = d(K_2 + v) = 0$. Show that G is a complete bipartite graph.

13. Let G be a graph with edge density ρ. Show that

$$d(K_3) + d(\bar{K}_3) \geq 3\rho(1 - \rho).$$

14. Let σ be the type consisting of a single labelled edge.

(a) Write down all the σ-flags on 3 vertices. (There are 4 of them.)

(b) Compute all products of pairs of these σ-flags.

(c) Using the averaging argument express these products as real linear combinations of unlabelled graphs on 4 vertices.

You may wish to do parts (b) and (c) as a parallel computation...