Graph Theory Basics

A *graph* is a pair $G = (V, E)$, consisting of a set of *vertices* V and a set of unordered pairs of vertices $E \subseteq V^{(2)}$ called *edges*.

If $v \in V(G)$ then the *neighbourhood* of v is $\Gamma(v) = \{w : vw \in E(G)\}$. The size of this neighbourhood is the *degree* of v denoted by $d(v)$.

Theorem 1 *If $G = (V, E)$ is a graph then*

$$\sum_{v \in V} d(v) = 2|E|.$$

Proof: Consider how many times each edge is counted in the LHS of this equation. $\square$

Important examples of graphs include K_n, the *complete graph of order n*,

$$V(K_n) = [n] := \{1, 2, \ldots, n\}, \quad E(K_n) = [n]^{(2)} = \{ij : 1 \leq i < j \leq n\}$$

and the *cycle of length n*, C_n:

$$V(C_n) = [n], \quad E(C_n) = \{i(i+1) : 1 \leq i \leq n-1\} \cup \{1n\}$$

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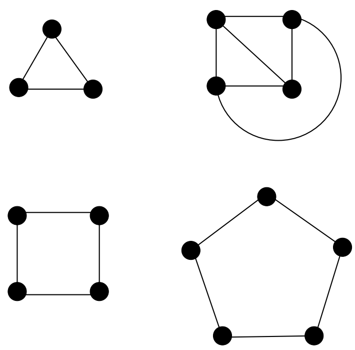

Figure 1: K_3, K_4, C_4 and C_5

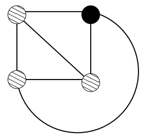

Figure 2: K_3 as a subgraph of K_4

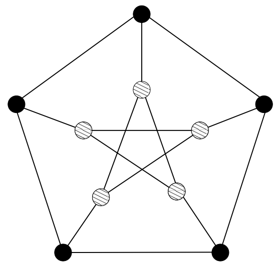

Figure 3: C_5 as a subgraph of the *Petersen graph*

A graph F is a *subgraph* of a graph G if there is an injective mapping $h : V(F) \to V(G)$ such that for all $uv \in E(F)$ we have $h(u)h(v) \in E(G)$. Moreover if h satisfies $uv \in E(F) \iff h(u)h(v) \in E(G)$ then we say that F is an *induced* subgraph.

If G has no subgraph that is isomorphic to F then we say G is F*-free*.

One of the main objectives of extremal graph theory is to calculate how many edges an F-free graph of order n may contain and so we define:

$$\mathrm{ex}(n, F) = \max\{|E(G)| : G \text{ is an } F\text{-free graph of order } n\}.$$

Often this is too difficult to compute so we instead aim to find the *Turán density*

$$\pi(F) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, F)}{\binom{n}{2}}.$$

A k*-colouring* of a graph G is $c : V(G) \to [k]$ satisfying $uv \in E(G) \implies c(u) \neq c(v)$.

If a k-colouring of G exists we say that G is *k-partite*. A 2-partite graph is said to be *bipartite*. A special example of a bipartite graph is $K_{r,s}$, the complete bipartite graph with classes of size r and s:

$$V(K_{r,s}) = [r+s], \quad E(K_{r,s}) = \{ij : 1 \leq i \leq r, r+1 \leq j \leq r+s\}.$$

The *chromatic number* of G is $\chi(G) = \min\{k : G \text{ is } k\text{-partite}\}$. For example, $\chi(K_t) = t$, while $\chi(C_t) = 2$ if t is even and $\chi(C_t) = 3$ if t is odd.

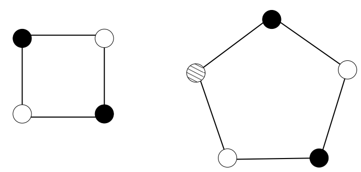

Figure 4: Colouring cycles

A *complete k-partite* graph is a k-partite graph in which all possible edges are present between the k colour classes.

If the k colour classes are as equal as possible in size then we say it is *balanced*.

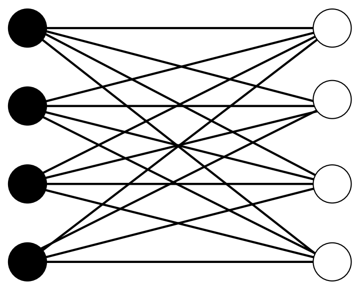

Figure 5: The balanced complete bipartite graph $K_{4,4}$