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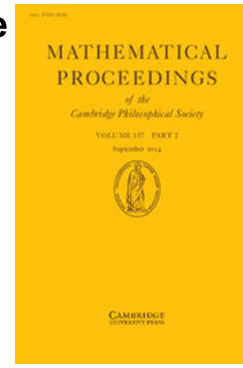

The lack of universal coefficient theorems for spectral sequences

E. C. Zeeman

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THE LACK OF UNIVERSAL COEFFICIENT THEOREMS FOR SPECTRAL SEQUENCES

By E. C. ZEEMAN

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Let $K = \Sigma K_n$ be a chain complex of free abelian groups K_n, with homology groups $H_n(K)$. If G is a coefficient group, the homology groups $H_n(K; G)$ and $H^n(K; G)$ are defined to be those of the chain complexes $K_n \otimes G$ and† $K_n \pitchfork G$, and may be calculated by the *Universal Coefficient Theorems*:

$$(1) \qquad \begin{aligned} H_n(K; G) &\cong H_n(K) \otimes G + \operatorname{Tor}(H_{n-1}(K), G), \\ H^n(K; G) &\cong H_n(K) \pitchfork G + \operatorname{Ext}(H_{n-1}(K), G). \end{aligned}$$

Since these are functorial equations, we have as a corollary:

(2) *If $f\colon K \to \tilde{K}$ is a chain map between two such complexes inducing homology isomorphisms $f_*\colon H_n(K) \cong H_n(\tilde{K})$, then f also induces isomorphisms*

$$f_*\colon H_n(K; G) \cong H_n(\tilde{K}; G),$$
$$f^*\colon H^n(\tilde{K}; G) \cong H^n(K; G).$$

We shall give a simple example to show that (1) and (2) do not generalize to spectral sequences. In other words, the knowledge of a homology spectral sequence of some topological situation does not necessarily imply knowledge of the corresponding cohomology spectral sequence, nor of that over arbitrary coefficients.

Let $A = \Sigma A_{p,q}$ be a double complex of free abelian groups $A_{p,q}$ (the summation running from $-\infty$ to $+\infty$), with commuting boundary operators

$$\partial\colon A_{p,q} \to A_{p-1,q}, \quad \partial'\colon A_{p,q} \to A_{p,q-1}.$$

Filtering with respect to p, we may define a spectral sequence

$$E^r(A) = \Sigma E^r_{p,q}(A) \quad (0 \leqslant r \leqslant \infty).$$

Similarly, we may define the sequences $E^r(A; G)$ and $E_r(A; G)$ to be those obtained from the double complexes $\Sigma A_{p,q} \otimes G$ and $\Sigma A_{p,q} \pitchfork G$.

The questions we ask are

(3) *For fixed r, can $E^r_{p,q}(A; G)$ and $E^{p,q}_r(A; G)$ be written functorially in terms of the groups $E^r_{p,q}(A)$, $-\infty < p, q < \infty$, and the group G?*

(4) *For fixed r, if $f\colon A \to \tilde{A}$ is a chain map between two such double complexes inducing an isomorphism $f_*\colon E^r(A) \cong E^r(\tilde{A})$, then are the homomorphisms*

$$f_*\colon E^r(A; G) \to E^r(\tilde{A}; G),$$
$$f^*\colon E_r(\tilde{A}; G) \to E_r(A; G)$$

necessarily isomorphisms?

† We use $F \pitchfork G$ to denote the group of homomorphisms of F into G, usually written as $\operatorname{Hom}(F, G)$.

The answers to both questions are *yes* for $r = 0, 1$, and in general *no* for $r \geqslant 2$.

Proof. We first observe that, for a given r, (3) implies (4). Thus, it is sufficient to prove (3) for $r = 0, 1$, and give an example contradicting (4) for $r \geqslant 2$. Now (3) holds for $r = 0$ since by definition

$$E^0_{p,q}(A; G) = E^0_{p,q}(A) \otimes G,$$
$$E_0^{p,q}(A; G) = E^0_{p,q}(A) \pitchfork G.$$

Moreover, it follows at once that (3) holds for $r = 1$, by applying (1) to the formulae $E^1 = H(E^0)$ and $E_1 = H(E_0)$.

The example. Let Z denote a free cyclic group, and Z_2 a cyclic group of order 2. Let $A_{0,0}$, $A_{1,0}$ be free cyclic with generators a, b, respectively, and let $A_{p,q} = 0$, otherwise. The boundaries ∂ and ∂' are defined by $\partial b = 2a$ and $\partial' = 0$. Let $\tilde{A}_{0,0}$, $\tilde{A}_{1,0}$, $\tilde{A}_{0,1}$, $\tilde{A}_{1,1}$ be free cyclic with generators $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$, respectively, and let $\tilde{A}_{p,q} = 0$, otherwise. Let $\partial\tilde{b} = 2\tilde{a}$, $\partial\tilde{d} = \tilde{c}$, $\partial'\tilde{c} = 2\tilde{a}$, $\partial'\tilde{d} = \tilde{b}$. Define $f\colon A \to \tilde{A}$ by $fa = \tilde{a}, fb = \tilde{b}$.

Then $$f_*\colon E^2_{p,q}(A) \cong E^2_{p,q}(\tilde{A}), \quad \text{where} \quad E^2_{p,q}(A) \cong \begin{cases} Z_2, & p = q = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore $$f_*\colon E^r(A) \cong E^r(\tilde{A}) \quad (2 \leqslant r \leqslant \infty).$$

However, applying the functor $\otimes Z_2$ we have

$$E^2_{p,q}(A; Z_2) \cong \begin{cases} Z_2, & p = 0, 1, \quad q = 0, \\ 0, & \text{otherwise.} \end{cases} \quad E^2_{p,q}(\tilde{A}; Z_2) = \begin{cases} Z_2, & p = 0, \quad q = 0, 1, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore $$f_*\colon E^r(A; Z_2) \ncong E^r(\tilde{A}; Z_2) \quad (2 \leqslant r \leqslant \infty).$$

Again, applying the functor $\pitchfork Z$, we have

$$E_2^{p,q}(A; Z) \cong \begin{cases} Z_2, & p = 1, \quad q = 0, \\ 0, & \text{otherwise.} \end{cases} \quad E_2^{p,q}(\tilde{A}; Z) \cong \begin{cases} Z_2, & p = 0, \quad q = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore $$f^*\colon E_r(\tilde{A}; Z) \ncong E_r(A; Z) \quad (2 \leqslant r \leqslant \infty).$$

This completes the proof.

Remark 1. The non-isomorphisms we have established above are non-isomorphisms of *bigraded* groups. If graded structure $E^r = \Sigma E^r_{p,q}$ is dropped, and E^r is considered as a group only, then the homomorphisms concerned do in fact become isomorphisms in the example. I do not know whether this is always true.

Remark 2. The importance of the grading is again emphasized in the limit term ($r = \infty$). Suppose $A_{p,q} = 0$ unless $p, q \geqslant 0$. Then $E^r(A)$ is convergent to $E^\infty(A)$, and $E^\infty(A) = \operatorname{Gr} H_*(A)$, the graded group associated with the homology group of A with respect to the total differential. Suppose that the same is true for $\tilde{A}$. If

$$f_*\colon E^r(A) \cong E^r(\tilde{A})$$

for some r, then by the 'five' lemma and by (2) we deduce that

$$f_*\colon H_*(A; G) \cong H_*(\tilde{A}; G),$$
$$f^*\colon H^*(\tilde{A}; G) \cong H^*(A; G).$$

However, this does not imply an isomorphism of the corresponding E^∞ and E_∞ terms, since the graded structures may differ, as in the example.

Remark 3. In the special case that G is the reals modulo 1, then $E_r^{p,q}(A; G)$ is the character group of $E_{p,q}^r(A)$, so that the cohomology parts of both (3) and (4) hold for all r.

Alternatively, if each $A_{p,q}$ is finitely generated, and G is a field, then $E_{p,q}^r(A; G)$ and $E_r^{p,q}(A; G)$ are dual vector spaces, so that one may be calculated from the other. In other words modified forms of (3) and (4) hold for all r.

The proof of these statements follows from the fact that the homology functor commutes with the taking of character groups or of dual vector spaces.

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