ON REGULAR NEIGHBOURHOODS

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In (11) J. H. C. Whitehead introduced the theory of regular neighbourhoods, which has become a basic tool in combinatorial topology. We extend the theory in three ways.

First we relativize the concept, and introduce the regular neighbourhood \( N \) of \( X \mod Y \) in \( M \), where \( X \) and \( Y \) are two compact polyhedra in the manifold \( M \), satisfying a certain condition called link-collapsibility. We prove existence and uniqueness theorems. The idea is that \( N \) should be a neighbourhood of \( X - Y \), but should avoid \( Y \) as much as possible. The notion is extremely useful in practice, and is illustrated by the following examples. We assume \( M \) to be closed for the examples.

(i) If \( Y = \emptyset \) then \( N \) is a regular neighbourhood of \( X \). Therefore the relative theory is a generalization of the absolute theory.

(ii) If \( X \) is a manifold with boundary \( Y \), then the interior of \( X \) lies in the interior of \( N \) and the boundary of \( X \) lies in the boundary of \( N \); in other words \( X \) is properly embedded in \( N \).

(iii) Let \( I \) be a cone and suppose that \( X \cap Y \) is contained in the base of the cone. Then \( N \) is a ball containing \( X - Y \) in its interior, \( X \cap Y \) in its boundary, and \( Y - X \) in its exterior.

The last example was used in (12) Lemma 6), and was one of the examples which suggested the need for a relative theory. Other illustrations of the use are to be found in the proofs of Theorem 2, Corollary 8, and Lemmas 7, 8, and 9 below, in the proof of Theorem 3 of (4), and in forthcoming papers by us on isotopy.

Secondly, Whitehead proved a uniqueness theorem that said that any two regular neighbourhoods were (piecewise linearly) homeomorphic. We strengthen this result by showing them to be isotopic, keeping a smaller regular neighbourhood fixed (Theorem 2). In fact they are ambient isotopic provided that they meet the boundary regularly (Theorem 3), which is always the case if \( M \) is unbounded.

Thirdly, Whitehead wrote the theory in the combinatorial category, and we rewrite it in the polyhedral category. The difference is that the combinatorial category consists of simplicial complexes and piecewise-linear maps, whereas the polyhedral category consists of polyhedra and piecewise-linear maps. In this paper by a \textit{polyhedron} we mean a topological...
space together with a maximal non-empty family of piecewise-linearly related triangulations, each triangulation being a countable simplicial complex (for a more general definition of polyhedral space see (14) and (15)). In particular if the polyhedron is compact then each triangulation is a finite complex. The main advantage of using the polyhedral category is to be found in isotopy theory: particular triangulations need never appear in the statements of theorems, only in the proofs.

The paper is divided into three sections. In §1 we give the definitions and state the existence and uniqueness theorems, Theorems 1, 2, and 3. Section 2 is devoted to applications, and we deduce eleven corollaries concerning spines of manifolds, knots of codimension 2, and local knottedness of embeddings and isotopies. Section 3 consists of the proofs of the three theorems.

1. Definitions and results

Notation

\( I \) stands for the unit interval.

\( M \) stands for a connected polyhedral manifold, \( M \) its boundary, and \( M \) its interior. \( M \) may or may not be compact, and may or may not be bounded.

\( X, Y \) stand for compact polyhedra in \( M \). \( X - Y \) stands for the points in \( X \) that are not in \( Y \). We shall always be needing two particular polyhedra obtained unsymmetrically from \( X \) and \( Y \), and so we introduce a special notation for them:

\[
X_k = X - Y,
\]

\[
Y_k = X_k \cap Y.
\]

[Diagram]

We shall only use the symbol \( \xi \) in contexts where it is unambiguous. Notice that if \( X_{\xi_{k}}, Y_{\xi_{k}} \) denote the pair obtained by applying the process \( \xi \) to the pair \( X_k, Y_k \) then \( X_{\xi_{k}} = X_k \) and \( Y_{\xi_{k}} = Y_k \); in other words \( \xi_{\xi} = \xi \).

\( J, K, L \) stand for complexes triangulating \( M, X, Y \). Therefore \( J \) is a countable (finite or denumerable) combinatorial manifold, and \( K, L \) are finite subcomplexes. The modulus sign \( |K| \) stands for the polyhedron
underlying $K$. Thus

$$|J| = M, \quad |K| = X, \quad |L| = Y, \quad |K_t| = X_t, \quad |L_q| = Y_t.$$  

If $A$ is a simplex of $K$, denote by $\text{lk}(A, K)$, $\text{st}(A, K)$, and $\overset{\text{st}}{A}$, respectively, the link, open star, and closed star, of $A$ in $K$.

All maps, homeomorphisms, and isotopies are piecewise linear. We use $\cong$ to denote homeomorphisms onto.

**Collapsing**

A complex $K$ simplicially collapses to a subcomplex $L$ if there exists a sequence of subcomplexes

$$K = K_0 \supseteq K_1 \supseteq \cdots \supseteq K_r = L,$$

such that for each $i$, $K_i - K_{i-1}$ consists of a principal simplex of $K_i$ and a free face.

A complex $K$ collapses to a subcomplex $L$, written $K \searrow L$, if there exist subdivisions $K', L'$ of $K, L$ such that $K'$ simplicially collapses to $L'$. (It is not known whether or not $K$ simplicially collapses to $L$ in these circumstances.) $K$ is collapsible if it collapses to a point. A polyhedron $X$ collapses to a subpolyhedron $Y$, written $X \searrow Y$, if for some triangulation $K, L$ of $X, Y$ we have $K \searrow L$. $X$ is collapsible if it collapses to a point.

Given two subcomplexes $K, L$ of some larger complex, let $K_t, L_q$ be as described above. We say that $K$ is link-collapsible on $L$ if $\text{lk}(A, K_t)$ is collapsible for each simplex $A$ in $L_q$. Given two subpolyhedra $X, Y$ of some larger polyhedron, we say that $X$ is link-collapsible on $Y$ if for some triangulation $K, L$ of $X, Y$ we have $K$ link-collapsible on $L$.

**Remark.** The definitions of collapsibility and link-collapsibility for polyhedra are independent of the triangulations, the first by Theorem 7 of (11), and the second as follows. Suppose $K^*, L^*$ is an arbitrary subdivision of $K, L$. If $A^* \in L_i^*$, then $\text{lk}(A^*, K_t^*)$ is homeomorphic to the $r$-fold suspension of $\text{lk}(A, K_t)$, where $A$ is the unique simplex of $L_q$ whose interior contains the interior of $A^*$, and where $r = \dim A - \dim A^*$. Consequently $K$ is link-collapsible on $L$ if and only if $K^*$ is link-collapsible on $L^*$.

**Examples.**

(i) Any polyhedron is link-collapsible on itself and on the empty set.

(ii) A simplex is link-collapsible on any subcomplex.

(iii) A manifold is link-collapsible on its boundary, and on any subpolyhedron of the boundary.

(iv) A manifold is not link-collapsible on an interior point.
(v) A cone is link-collapsible on its base, and on any subpolyhedron of the base.
(vi) $X$ is link collapsible on $Y$ if and only if $X_\delta$ is link-collapsible on $Y_\delta$.

**Definition of regular neighbourhood**

We rewrite Whitehead's original definition (11) in terms of polyhedra. Let $X, N$ be compact polyhedra in the manifold $M$. We say that $N$ is a **regular neighbourhood** of $X$ in $M$ if

1. $N$ is an $m$-manifold ($m = \dim M$),
2. $N$ is a topological neighbourhood of $X$ in $M$,
3. $N \setminus X$.

If only conditions (1) and (3) hold we say that $N$ is a **regular enlargement** of $X$ in $M$. We say that $N$ **meets the boundary regularly** if, further,

4. $N \cap \partial M$ is a regular neighbourhood of $X \cap \partial M$ in $M$.

If $N_1$ is another regular neighbourhood of $X$ in $M$, we say that $N_1$ is **smaller** than $N$ if $N$ contains a topological neighbourhood of $N_1$ in $M$.

Now the relativization. Let $X, Y, N$ be compact polyhedra in $M$. We say that $N$ is a **regular neighbourhood** of $X$ mod $Y$ in $M$ if

1. $N$ is an $m$-manifold,
2. $N$ is a topological neighbourhood of $X - Y$ in $M$, and
3. $N \cap Y = \hat{N} \cap Y = Y_\delta$.

We say that $N$ **meets the boundary regularly** if, further,

4. $(N \cap \partial M) - Y$ is a regular neighbourhood of $X \cap \partial M$ mod $Y \cap \partial M$ in $\hat{M}$.

If $N_1$ is another regular neighbourhood of $X$ mod $Y$ in $M$, we say that $N_1$ is **smaller** than $N$ if $N$ contains a topological neighbourhood of $N_1 - Y$ in $\hat{M}$.

<table>
<thead>
<tr>
<th>Absolute regular neighbourhood</th>
<th>Relative regular neighbourhood</th>
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<td><img src="image1" alt="Absolute Regular Neighbourhood" /></td>
<td><img src="image2" alt="Relative Regular Neighbourhood" /></td>
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**Fig. 2**

**Remark 1.** If we put $Y = \emptyset$ in the relative definition, then we recover the absolute definition, and so the relative definition is a generalization.

**Remark 2.** A regular neighbourhood of $X$ mod $Y$ is in particular a regular enlargement of $X_\delta$. 
Remark 3. Any regular neighbourhood of $X \mod Y$ is also a regular neighbourhood of $X_k \mod Y_k$, but not conversely in general, because of condition (2).

Remark 4. If $M$ is unbounded then condition (4) is vacuous, and so trivially true. If $M$ is bounded and $X \subseteq \bar{M}$, then condition (4) is the same as saying $N \subseteq \bar{M}$.

Remark 5. The appearance of $(N \cap M) - Y$ rather than $N \cap M$ in condition (4) of the relative definition looks curious at first sight but is necessary for the following reason. Let $(X \cap M)_h, (Y \cap M)_h$ denote the pair obtained from the pair $X \cap M, Y \cap M$. Then $(X \cap M)_h \subseteq X_h \cap \bar{M}$, but in general they are not equal. For example consider the case when $X$ is a manifold with boundary $Y$, embedded in $M$ so that $X \cap M = Y \cap M = Y$. Then any regular neighbourhood of $X \cap M \mod Y \cap M$ is the empty set; but any regular neighbourhood $N$ of $X \mod Y$ must contain $Y$. Therefore $N \cap M$ cannot be a regular neighbourhood of $X \cap M \mod Y \cap M$. However, we can choose $N$ so that it contains no more of $M$ than $Y$; consequently $(N \cap M) - Y$ is empty, and so this choice of $N$ will satisfy condition (4).

Second-derived neighbourhoods

Let $J$ be a combinatorial manifold and $U$ a subset of $|J|$. The simplicial neighbourhood $N(U,J)$ is defined to be the smallest closed subcomplex of $J$ whose underlying polyhedron contains a topological neighbourhood of $U$ in $|J|$. It consists of all closed simplexes meeting $U$ together with their faces. In particular if $K, L$ are subcomplexes of $J$, we define $N(K - L, J) = N(|K| - |L|, J)$, and deduce that $N(K - L, J) = \bigcup \overline{\triangle}(A, J)$, the union taken over all simplexes $A$ in $K - L$.

Suppose $X, Y$ are polyhedra in the polyhedral manifold $M$. A second-derived neighbourhood $N$ of $X \mod Y$ in $M$ is constructed as follows: choose a triangulation $J$ of $M$ that contains subcomplexes triangulating $X, Y$; then choose† a second derived complex $J''$ of $J$, and define $N = \{N(X-Y, J'')\}$.

Isotopy

We recall the definition of isotopy (see (4)). An isotopy of $N$ in $M$ is a level-preserving embedding $f: N \times I \to M \times I$. Therefore for each $t$ in $I$ there is an embedding $f_t: N \to M$ such that $f(x,t) = (f_t x, t)$ for all $x$ in $N$.

In the special case when $N \subseteq M$ and $f_0$ is the inclusion map and $N_1 = f_1 N$, we call $f$ an isotopy in $M$ moving $N$ onto $N_1$. If $P \subseteq N$ and $f|P \times I$ is the identity then we say that $f$ keeps $P$ fixed.

† To form a derived complex it is not necessary to use barycentres; we can star each simplex at an arbitrary interior point. Therefore a choice is involved.
An ambient isotopy of $M$ is a level-preserving homeomorphism onto, $f : M \times I \to M \times I$, such that $f_0$ is the identity. If $N, P \subseteq M$, $N_1 = f_1 N$, and $f|P \times I$ is the identity, then we say that $f$ is an ambient isotopy of $M$ moving $N$ onto $N_1$ and keeping $P$ fixed.

We can now state the main theorems. Let $X, Y$ be polyhedra in $M$.

**Theorem 1 (Existence).** If $X$ is link-collapsible on $Y$ then any second-derived neighbourhood $N$ of $X$ mod $Y$ in $M$ is regular. If, further, $X \cap M$ is link-collapsible on $Y \cap M$ then $N$ meets the boundary regularly.

**Theorem 2 (Uniqueness).** Suppose that $X$ is link-collapsible on $Y$, and let $N_1, N_2$ be two regular neighbourhoods of $X$ mod $Y$ in $M$. Then there exists a smaller regular neighbourhood $N_3$ and a homeomorphism of $N_1$ onto $N_2$ keeping $N_3$ fixed. Further, the homeomorphism can be realized by an isotopy in $M$ moving $N_1$ onto $N_2$ through a continuous family of regular neighbourhoods and keeping $N_3$ fixed.

![Fig. 3](image)

If the neighbourhoods meet the boundary regularly we can strengthen the isotopy to an ambient isotopy:

**Theorem 3 (Uniqueness).** Suppose that $X$ is link-collapsible on $Y$, and that $X \cap M$ is link-collapsible on $Y \cap M$. Let $N_1, N_2, N_3$ be three regular neighbourhoods of $X$ mod $Y$ in $M$ meeting the boundary regularly, and such that $N_3$ is smaller than $N_1$ and $N_2$. Let $P$ be the closure of the complement of a second-derived neighbourhood of $N_1 \cup N_2$ mod $Y$ in $M$. Then there exists an ambient isotopy of $M$ moving $N_1$ onto $N_2$ and keeping $N_3 \cup P$ fixed. (See Fig. 3.)

**Remarks.**

(i) In Theorem 3, since the isotopy is ambient and keeps $X \cup Y$ fixed, it is a corollary that it moves $N_1$ onto $N_3$ through a continuous family of regular neighbourhoods that meet the boundary regularly.
(ii) Our proof of Theorem 3 below shows that in fact the ambient isotopy is by linear moves (see (4)).

(iii) It is necessary to have \( N_1 \) and \( N_2 \) meeting the boundary regularly for Theorem 3 to be true. For example suppose \( Y = \emptyset \) and \( X \subseteq N \subseteq \hat{M} \), and suppose \( N_2 \) meets the boundary \( \hat{M} \). Then \( N_1 \) meets the boundary regularly (i.e. not at all) but \( N_2 \) does not, and it is impossible to ambient-isotope \( N_1 \) onto \( N_2 \).

(iv) It is necessary to have the smaller neighbourhood \( N_3 \) in the thesis of Theorem 2 rather than in the hypothesis (as it is in Theorem 3). For consider the following example. Let \( Y = \emptyset \), and let \( X \) be a point inside a 3-ball \( M \). Let \( N_1 = M \) itself. Let \( N_2, N_3 \) be second- and third-derived neighbourhoods of a knotted arc in \( M \), that contains \( X \) and has its end-points in \( \hat{M} \). Then \( N_1, N_2, N_3 \) are regular neighbourhoods of \( X \) in \( M \), and \( N_3 \) is smaller than \( N_1 \) and \( N_2 \). But there is no homeomorphism of \( N_1 \) onto \( N_2 \) keeping \( N_3 \) fixed, because \( N_1 - N_3 \) is not homeomorphic to \( N_2 - N_3 \). It is true that we have a situation in which \( N_1 \) and \( N_2 \) do not meet the boundary regularly, but then Theorem 2 is tailored for just such a situation. If we rechoose \( N_3 \) to be a little ball about \( X \), then, keeping this new \( N_3 \) fixed, we can isotope \( N_2 \) off the boundary, unknot it, and then push it out onto \( N_1 \).

2. Applications

We postpone the proofs of Theorems 1, 2, and 3 until the next section, and devote this section to applications, in the form of eleven corollaries. The first seven depend only on the corresponding absolute theorems (when \( Y = \emptyset \)) and are concerned primarily with spines of manifolds. The last four corollaries depend essentially upon the relative theorems, and are concerned with knots of codimension 2, and with local knottedness of embeddings and isotopies. We also ask six questions concerning knots.

**Corollary 1. The regular-neighbourhood annulus theorem.**

Let \( N, N_1 \) be two regular neighbourhoods of \( X \) in \( M \). Suppose that \( N \) is smaller than \( N_1 \), and that \( N \) meets the boundary regularly. Then \( N_1 - N \) is homeomorphic to \( \text{Fr}(N) \times I \), where \( \text{Fr}(N) \) denotes the frontier of \( N \) in \( M \). In particular, if \( X \subseteq \hat{M} \) then \( N_1 - N \) is homeomorphic to \( \hat{N} \times I \).

**Proof.** Choose a triangulation \( J \) of \( M \) containing \( X \) as a subcomplex. Let \( J' \) be the first barycentric derived of \( J \). Let \( f : J' \to I \) be the simplicial map that maps a vertex to 0 or 1 according to whether or not it lies in \( X \). Given \( \epsilon, 0 < \epsilon < \frac{1}{2} \), we can choose second- and third-derived complexes of
J such that

\[ N(X,J^e) = f^{-1}[0,\frac{1}{2}] = N_2, \quad \text{say,} \]
\[ N(X,J^e) = f^{-1}[0,\varepsilon] = N_3, \quad \text{say.} \]

Then \( N_2 - N_3 \) is homeomorphic to \( \text{Fr}(N_3) \times I \) as follows. Let \( A \) run over the simplexes of \( J' \) meeting \( X \) but not contained in \( X \), in some order of increasing dimension. The homeomorphism is constructed inductively on \( A \cap N_2 - N_3 \), which is a 'skew' prism, with walls \( A \cap N_2 - N_3 \), top \( A \cap \text{Fr}(N_2) \), and bottom \( A \cap \text{Fr}(N_3) \). By induction the homeomorphism has already been defined on the walls, so extend it to the top, bottom, and interior, each of which is a convex linear cell, by mapping some interior point arbitrarily and joining linearly to the boundary.

By Theorem 2 there is a homeomorphism \( N_2 \to N_1 \) keeping a smaller regular neighbourhood fixed, and therefore keeping \( N_3 \) fixed if we choose \( \varepsilon \) sufficiently small. Since \( N \) and \( N_3 \) both meet the boundary regularly, and are both smaller than \( N_1 \), we can ambient-isotope \( N_3 \) onto \( N \) keeping outside—\( N_1 \) fixed by Theorem 3. Therefore there are homeomorphisms

\[ N_1 - N \cong N_2 - N_3 \cong \text{Fr}(N_3) \times I \cong \text{Fr}(N) \times I, \]

the last because the ambient isotopy moves \( \text{Fr}(N_3) \) onto \( \text{Fr}(N) \).

If \( X \subseteq \hat{M} \) then the hypothesis that \( N \) meets the boundary regularly implies that \( N \subseteq \hat{M} \) also, and so \( \text{Fr}(N) = \hat{N} \).

Notice that in our proof we can, further, choose the homeomorphism \( h : N_1 - N \to \text{Fr}(N) \times I \) so that \( hx = (x,0) \) for all \( x \) in \( \text{Fr}(N) \).

**Corollary 2. The Annulus Theorem (Newman (7)).**

*If \( B^n \) is an \( n \)-ball in the interior of the \( n \)-ball \( B_1^n \) then \( B_1^n - B^n \cong S^{n-1} \times I. \)*

*Proof.* If \( x \) is an interior point of \( B^n \), then both balls are regular neighbourhoods of \( x \) in \( B_1^n \), and so the result follows from Corollary 1.

**Corollary 3. A ball collapses onto any collapsible polyhedron in its interior.**

*Proof.* Let \( B_1^n \) be the ball and \( X \) the collapsible polyhedron, and let \( B^n \) be a regular neighbourhood of \( X \) in the interior of \( B_1^n \). Then \( B^n \) is a ball by Lemma 1 below. Therefore by Corollary 2, \( B_1^n \setminus B^n \setminus X \). Notice that it is necessary that \( X \) be in the interior of the ball, otherwise the corollary is not true; for consider a knotted arc in a 3-ball with its ends in the boundary.

**Spines**

We want to generalize the last two corollaries from balls to manifolds. Let \( M \) be a compact bounded manifold, and let \( X \) be a polyhedron in the interior of \( M \). We call \( X \) a spine of \( M \) if \( M \setminus X \). The interest lies in
finding a spine that is as ‘minimal’ and simple as possible. There is no loss of generality in assuming (for technical convenience) that a spine is in the interior of \( M \), because we can first collapse away a collar from the boundary. There always exist spines of dimension less than \( M \), because we can then collapse away all top-dimensional simplexes of some triangulation. The minimum dimension of a spine is an invariant of \( M \); for example this dimension is zero if and only if \( M \) is a ball.

**Corollary 4.** Let \( X \) be a spine of \( M \) and \( N \) a regular neighbourhood of \( X \) in \( M \). Then \( \overline{M \setminus N} \cong M \times I \).

**Proof.** The result follows from Corollary 1, because both \( M \) and \( N \) are regular neighbourhoods of \( X \) in \( M \).

**Corollary 5.** Suppose that \( X, Y \subseteq M \), and \( X \subsetneq Y \). Then \( X \) is a spine of \( M \) if and only if \( Y \) is a spine of \( M \).

**Proof.** If \( X \) is a spine then trivially \( Y \) is also because \( M \setminus X \setminus Y \). Conversely, suppose that \( Y \) is a spine. Let \( N \) be a regular neighbourhood of \( X \) in \( M \). Then \( N \) is also a regular neighbourhood of \( Y \) because \( N \setminus X \setminus Y \). Therefore by Corollary 4, \( M \setminus N \), and so \( X \) is a spine of \( M \) because \( M \setminus N \setminus X \).

**Ambient simple homotopy type**

Two polyhedra \( X, Y \) in \( M \) are of the same ambient simple homotopy type, or, more briefly, of the same type, if there exists a sequence of polyhedra \( X = X_0, X_1, \ldots, X_k = Y \) in \( M \) such that

\[
X_0 \nearrow X_1 \searrow X_2 \searrow X_3 \searrow \ldots \searrow X_k;
\]

i.e. each \( X_i \) is obtained from its predecessor by collapse or expansion. If \( X, Y \) lie in the interior of \( M \) we can without loss of generality assume that all the \( X_i \) also lie in the interior of \( M \); for if not, choose a collar of \( M \) not meeting \( X \) or \( Y \) (a collar is an embedding \( M \times I \to M \) such that \((x, 0)\to x\) for each \( x \) in \( M \)), and let \( M \to M' \) be the embedding that leaves the inside of the collar fixed and shrinks the collar to half its length. Then the images of the \( X_i \) give a sequence running from \( X \) to \( Y \) in the interior of \( M \).

**Corollary 6.** Any \( X \) in the interior of \( M \) is a spine if and only if it is of the type of a spine.

**Proof.** One way is a fortiori. For the other way suppose that \( X \) is of the type of a spine \( X_0 \). In other words there is a sequence \( X_i, 0 \leq i \leq k \), in the interior of \( M_i \), of collapses and expansions running from \( X_0 \) to \( X = X_k \). Corollary 5 shows, by induction on \( i \), that each \( X_i \) is a spine; therefore in particular \( X \) is a spine.
Remark 1. Corollary 6 is useful for simplifying spines. For example the spine of a connected bounded 3-manifold $M^3$ can be normalized in the following sense. We say a 0-dimensional complex is normal if it is a point; a 1-dimensional complex is normal if each vertex locally bounds exactly three 1-cells (the word ‘locally’ is to be interpreted by the convention that if a 1-cell has both ends at the vertex then this counts as the vertex locally bounding two 1-cells). A 2-dimensional complex is normal if each 1-cell locally bounds exactly three 2-cells, and each vertex locally bounds exactly four 1-cells and six 2-cells.

Given $M^3$ we can find a normal spine as follows. Choose a minimal spine $K$, that is to say one which is of minimal dimension, $d$ say, and which cannot be collapsed any further. If $d = 0$ ($M^3$ is a ball) then $K$ is a point. If $d = 1$ ($M^3$ is a handlebody) then expand each vertex of $K$ into a little disk and collapse the disk from one face. If $d = 2$, expand each 1-cell of $K$ like a banana and collapse from one side, and then expand each vertex like a pineapple and collapse from one face. In each case the corollary ensures that we are left with a spine, and the process described makes it normal.

Remark 2. Further theorems about spines can be obtained by using Smale’s handle theory (9). For example it is shown in ((15) Chapter 9) that if $M$ is simply connected and $Y$ is a spine of $M$ of codimension $> 3$, and if $X \simeq Y$ is a homotopy equivalence, then $X$ is also a spine of $M$.

Remark 3. In the next corollary we generalize the result from spines to arbitrary polyhedra in $M$. For simplicity we assume that $M$ has no boundary, although a similar result holds for bounded manifolds.

**Corollary 7.** Suppose that $M$ is without boundary. Two compact polyhedra in $M$ are of the same type if and only if their regular neighbourhoods are ambient isotopic.

**Proof.** Suppose that $X_0, X_k$ are of the same type, $X_0 \not\simeq X_1 \setminus X_2 \setminus \ldots \setminus X_k$. Let $N_i$ be a regular neighbourhood of $X_i$. We show that $N_i$ is ambient isotopic to $N_0$ by induction on $i$, the induction starting trivially with $i = 0$. Therefore assume $i > 0$. If $i$ is even then $N_{i-1} \setminus X_{i-1} \setminus X_i$, and so $N_{i-1}, N_i$ are both regular neighbourhoods of $X_i$. If $i$ is odd then $N_i \setminus X_i \setminus X_{i-1}$, and so $N_{i-1}, N_i$ are both regular neighbourhoods of $X_{i-1}$. In either case $N_i$ is ambient isotopic to $N_{i-1}$ by Theorem 3, and hence to $N_0$ by induction.

Conversely, suppose that $X_0, X_1$ have ambient-isotopic regular neighbourhoods $N_0, N_1$. To show that $X_0, X_1$ are of the same type it suffices to show that $N_0, N_1$ are of the same type, because $X_i$ is of the same type as $N_i$, $i = 0, 1$. Let $f : M \times I \to M \times I$ be the given ambient isotopy, and let $N_i = f(N_0 \times \epsilon)$, $0 \leq \epsilon \leq 1$. Let $N$ be a regular neighbourhood of $N_0$ in $M$. For sufficiently small $\epsilon > 0$, $f(N_0 \times [0, \epsilon]) \subseteq \hat{N}$, and so by
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((4) Addendum 1.2) there is another ambient isotopy moving \( N_0 \) to \( N_e \) supported by \( N \). Therefore the pair \((N, N_e)\) is homeomorphic to the pair \((N, N_0)\), and so \( N \setminus N_e \) because \( N \setminus N_0 \). Therefore \( N_0 \) is of the same type as \( N_e \) by \( N_0 \sim N \setminus N_e \). By the compactness of \( I \), \( N_0 \) is of the same type as \( N_1 \).

**Knotted balls and spheres**

Recall the notation of (13). Suppose \( q > m \). A sphere pair \( S^q \uparrow S^m \) is a pair of spheres \( S^q \supseteq S^m \). A ball pair \( B^q \uparrow B^m \) is a pair of balls \( B^q \supseteq B^m \), such that \( B^q \cap B^m = B^m \). Call \( q - m \) the codimension of the pair. The standard ball pair \( \Delta^q \uparrow = (\Sigma \Delta^m, \Delta^m) \), where \( \Delta^m \) is the standard \( m \)-simplex and \( \Sigma \) denotes \((q - m)\)-fold suspension. The standard sphere pair is the boundary of the standard ball pair one dimension higher. A pair is unknotted if it is homeomorphic to a standard pair. In (13) it was proved that any pair of codimension \( \geq 3 \) is unknotted. Knots can occur in codimension 2, but

**Question 1.** Is any pair of codimension 1 unknotted?

The answer is 'yes' for \( q \leq 3 \) by (1); for \( q > 3 \) we know there is a topological unknotted by (2) and (6), but we do not know whether there is a piecewise linear unknotted. The answer to Question 1 depends upon

**Question 2.** If \( M^m \) is a combinatorial manifold triangulating a topological \( m \)-ball, then is \( M^m \) a combinatorial \( m \)-ball?

We know the answer to Question 2 is 'yes' if \( m \leq 3 \) by the Hauptvermutung, and if \( m \geq 6 \) by (9), but our ignorance of the dimensions 4 and 5 prevents an inductive proof of Question 1.

Write \( B^q \uparrow \subseteq S^q \uparrow \) if \( B^q \subseteq S^q \) and \( B^m = B^q \cap S^m \).

**Question 3.** If \( B^q \uparrow \subseteq S^q \uparrow \) and \( S^q \uparrow \) is unknotted, then is \( B^q \uparrow \) unknotted?

In codimension \( \geq 3 \) the answer is 'yes' by (13). In codimension 1 the question is the same as Question 1; for a 'yes' to Question 1 trivially implies a 'yes' to Question 3. Conversely a 'no' to Question 1 implies the existence of a knotted ball pair with unknotted boundary, and glueing two copies of this together by the involution on the boundary embeds this knotted ball pair in an unknotted sphere pair.

In codimension 2 the answer is 'yes' for \( q = 3 \) by the unique-factorization theorem of classical knot theory (see ((3) 140)); in other words an unknotted curve is not the sum of two knots. But for \( q > 3 \) the question is unsolved, and the various proofs for \( q = 3 \) break down in higher dimensions for the following reasons. (i) Schubert's unique-factorization proof ((3) 140) depends upon the genus of a knot, which has no sufficiently strong higher-dimensional analogue. (ii) Mazur's proof ((3) 142) works only if we know that the boundary of \( B^q \uparrow \) is unknotted, and then gives
only a topological unknotting rather than a piecewise-linear one. (iii) The algebraic proof using van Kampen's theorem on the complement breaks down because, for $q > 3$, we do not know the answer to

**Question 4.** Given a ball pair $B = (B^q, B^{q-2})$ such that $\pi_1(B^q - B^{q-2}) \cong \mathbb{Z}$, then is $B$ unknotted?

If $q = 3$ the answer is 'yes' by a theorem of Papakyriakopoulos (8), but if $q > 3$ the answer is not known.† If $q \geq 5$ and we add the additional hypothesis that $B$ is locally unknotted then Stallings (10) gives a topological unknotting, but not necessarily a piecewise-linear one.

We now proceed to prove a partial answer to Question 3, which is useful for applications, for example in the three subsequent corollaries.

**Corollary 8.** If $B^{a,m} \subseteq S^{a,m}$ are both unknotted pairs then the complementary ball pair $B^{*a,m} = S^{a,m} - B^{a,m}$ is also unknotted.

**Proof.** The proof that we give works for all codimensions, although for codimension $\geq 3$ the result follows from (13), and for codimension 1 the result follows from the foundational lemma: if a ball contains another ball of the same dimension, and their boundaries meet in a common face, the closure of the complement is a ball. In codimension 2 we run into potential trouble near the boundary, but this kind of trouble is exactly what the concept of the relative regular neighbourhood is designed to cope with, as follows. Choose an unknotting homeomorphism $h : S^{a,m} \rightarrow (\Sigma \Delta^m, \Delta^m)$, where $\Sigma$ denotes $(q - m)$-fold suspension. Let $B^a_0 = h^{-1}(\Sigma(hB^m_0))$. Then $B^a$ and $B^q_0$ are both regular neighbourhoods of $B^m$ mod $B^q_0$ in $S^q$. Therefore by Theorem 3 there is an ambient isotopy of $S^q$ moving $B^a$ onto $B_0^a$ keeping $B^m \cup B^a_0 = S^m$ fixed. The end of the isotopy throws $B^a_0$ onto $h^{-1}(\Sigma(hB^a_0), hB^a_0)$, which is unknotted. Hence $B^{*a,m}$ is unknotted.

**Local unknottedness**

We recall the definitions of (13). Suppose that $M, Q$ are manifolds and that $M$ is compact. An embedding $f : M \rightarrow Q$ is called proper if $f^{-1}Q = M$. If $f$ is proper denote by $\hat{f}$ the restriction of $f$ to the boundaries, $\hat{f} : \partial M \rightarrow \partial Q$.

Let $\varphi$ be a triangulation of $f$; that is to say, choose triangulations $K, L$ of $M, Q$ with respect to which $f$ is simplicial, and call the simplicial map $\varphi$. In other words the diagram

$$
\begin{array}{ccc}
K & \xrightarrow{\varphi} & L \\
\downarrow & & \downarrow \\
M & \xrightarrow{f} & Q
\end{array}
$$

† Added in proof. J. Levine has shown that, for locally unknotted ball pairs, with $q \geq 6$, if $B^q - B^{q-2} \cong \mathbb{Z}$ then the pair is unknotted; but knots do exist with $\pi_1(B^q - B^{q-2}) \cong \mathbb{Z}$.
is commutative. Denote by $\tilde{\varphi}$ the corresponding triangulation of $f$. If $v$ is a vertex of $K$, denote by $\tilde{\varphi}(v, \varphi)$ and $\text{lk}(v, \varphi)$ the pairs
\[
\tilde{\varphi}(v, \varphi) = (\tilde{\varphi}(\varphi v, L), \varphi(\tilde{\varphi}(v, K)));
\]
\[
\text{lk}(v, \varphi) = (\text{lk}(\varphi v, L), \varphi(\text{lk}(v, K))).
\]
We call $f$ a locally unknotted embedding if for each $v$ in $K$, $\tilde{\varphi}(v, \varphi)$ is an unknotted ball pair.

Remark 1. The definition is independent of the triangulation $\varphi$, because if all the vertex stars are unknotted then the same is true for any subdivision of $K, L$, and hence also true for any other triangulation.

Remark 2. By (13) local knotting can only occur in codimension 2 and possibly codimension 1.

Remark 3. 'Locally unknotted' implies 'proper'. For if $v$ is an interior vertex of $K$, then $v$ is interior to $\tilde{\varphi}(v, K)$, and so $\varphi v$ is interior to $\tilde{\varphi}(\varphi v, L)$, by the definition of ball pair. Similarly if $v$ is a boundary vertex of $K$, then $\varphi v$ must be on the boundary of $\tilde{\varphi}(\varphi v, L)$ and so on the boundary of $L$. Therefore $\varphi^{-1} L = K$.

Remark 4. A sphere or ball pair is called locally unknotted if the inclusion map of the smaller in the larger is locally unknotted. If a pair is unknotted then it is locally unknotted, because we can triangulate with a standard pair. On the other hand if a pair is locally unknotted it may be (globally) knotted; for example consider the knots of classical knot theory.

**Corollary 9.**

(i) If $f$ is locally unknotted then so is $f$.

(ii) $f$ is locally unknotted if and only if, for some triangulation $\varphi$ of $f$, all the links $\text{lk}(v, \varphi)$, $v \in \text{dom} \varphi$, are unknotted.

**Proof.** (i) Let $\varphi : K \rightarrow L$ triangulate $f$, and, given $v$ in $K$, let $B = \tilde{\varphi}(v, \varphi)$. If $f$ is locally unknotted then $B$ is unknotted, and so $\hat{B}$ is an unknotted sphere pair. Therefore $\hat{B}$ is locally unknotted. If $v \in \hat{K}$ then $v \in \hat{B}$ and so $\tilde{\varphi}(v, \hat{B})$ is unknotted. But $\tilde{\varphi}(v, \hat{B}) = \tilde{\varphi}(v, \hat{\varphi})$. Therefore $f$ is locally unknotted.

(ii) Suppose the links are unknotted; then the stars are unknotted because they are cones on the links. Conversely, suppose the stars are unknotted; then to prove that the links are unknotted it is necessary to use Corollary 8 as follows. With the notation of part (i), if $v \in \hat{K}$ then $\text{lk}(v, \varphi) = \hat{B}$ is an unknotted sphere pair. If $v \in \hat{K}$, then $\text{lk}(v, \varphi) = \hat{B} - \text{st}(v, \hat{B})$, which is an unknotted ball pair by Corollary 8.

**Corollary 10.** A locally unknotted ball is unknotted in its regular neighbourhood. More precisely, let $(B^q, B^m)$ be a locally unknotted ball pair of codimension 1 or 2, which may be globally knotted. Let $N^q$ be a regular neighbourhood of $B^m$ in $B^q$. Then $(N^q, B^m)$ is unknotted.
Proof. The proof is similar to the proof of ((13) Lemmas 3 and 6), and we sketch it as follows. The essential new ingredient is Corollary 8, and we use this to prove, as in ((13) Lemma 3), that if two unknotted ball pairs meet in a common unknotted face, then their union is also unknotted. Now we proceed as in ((13) Lemma 6): triangulate $B^q$ so that $B^m$ collapses simplicially to a point, $x$ say. Then clothe the expansion $x \not
rightarrow B^m$ by pairs of second-derived neighbourhoods in $B^q$ and $B^m$. In the expanding sequence of pairs of neighbourhoods, each step is equivalent to glueing on two little ball pairs each by a face. The local unknottedness of $B^m \subseteq B^q$ ensures that each little ball pair is unknotted, and by using Theorem 3 we can show that the face by which it is glued on is also unknotted: for if $S$ is the boundary of the little ball pair and $F$ the face, then $F^{q-1}$ is a regular neighbourhood of $F^{m-1} \mod F^{m-1}$ in $S^{q-1}$, and since $S$ is unknotted, we can unknot $F$ by isotoping $F^{q-1}$ onto a suspension of $F^{m-1}$ fixed by Theorem 3 (as in the proof of Corollary 8). We begin with a little unknotted ball pair, namely the star of $x$, and each time we glue on a little ball pair the result remains unknotted. Therefore, by induction on the number of steps, we finish with $(N^q, B^m)$ unknotted, where $N^q$ is the second-derived neighbourhood of $B^m$ in $B^q$. By Theorem 2 there is a homeomorphism $N^q \rightarrow N^m$ keeping $B^m$ fixed, and so $(N^q, B^m)$ is also unknotted.

Locally unknotted isotopies

As above, let $M, Q$ be manifolds, with $M$ compact. Let $f : M \times I \rightarrow Q \times I$ be an isotopy, that is to say a level-preserving embedding. Call $f$ a proper isotopy if it is a proper embedding. If $f$ is a proper isotopy then each level $f_t : M \rightarrow Q$ is a proper embedding. If $f$ is a proper isotopy denote by $\partial f$ the restriction of $f$ to the boundaries, $\partial f : M \times I \rightarrow Q \times I$.

Remark. We have to use a different symbol $\partial$ for the boundary, because the boundary of $f$, qua isotopy, is smaller than the boundary of $f$, qua embedding; in other words $\partial f \neq f'$, although at each level $(\partial f)_t = (f_t)'$.

We define $f$ to be a locally unknotted isotopy if

(i) at each level $f_t : M \rightarrow Q$ is a locally unknotted embedding, and

(ii) for each subinterval $J$ of $I$ the restriction $f_J : M \times J \rightarrow Q \times J$ of $f$ is a locally unknotted embedding.

As in the case of embeddings we can deduce that any locally unknotted isotopy is proper.

Corollary 11. If $f$ is a locally unknotted isotopy then so is $\partial f$.

Proof. Condition (i) for $\partial f$ follows from Corollary 9(i) because $(\partial f)_t = (f_t)'$. To prove condition (ii), let $K_1, K_2, K_3$ be triangulations of
ON REGULAR NEIGHBOURHOODS

Let \( (M \times J)' \), \( \hat{M} \times J \), \( M \times J \), and let \( L_1, L_2, L_3 \) be triangulations of \( (Q \times J)' \), \( \hat{Q} \times J \), \( Q \times J \), respectively, such that the restrictions \( \varphi_i : K_i \to L_i \), \( i = 1, 2, 3 \), of \( f \) are simplicial. By condition (ii) for \( f \) and Corollary 9(i), \( \varphi_1 \) is a locally unknotted embedding, and by condition (i) for \( f \) so is \( \varphi_3 \). We want to prove that \( \varphi_2 \) is a locally unknotted embedding. Let \( v \) be a vertex of \( K_2 \). If \( v \) is an interior vertex of \( K_2 \) then \( \overline{\text{st}}(v, \varphi_2) = \overline{\text{st}}(v, \varphi_1) \), which is unknotted. If \( v \) is a boundary vertex then

\[
\text{lk}(v, \varphi_2) = \text{lk}(v, \varphi_1) - \text{lk}(v, \varphi_3) = \text{unknotted sphere pair} - \text{unknotted ball pair} = \text{unknotted ball pair}, \text{ by Corollary 8.}
\]

Therefore \( \overline{\text{st}}(v, \varphi_2) \) is unknotted. Therefore \( \partial f \) is a locally unknotted isotopy.

The definition of locally unknotted isotopy that we have given immediately raises two questions:

**Question 5.** Does condition (ii) imply condition (i)?

**Question 6.** If \( f \) is an isotopy and a locally unknotted embedding, then is \( f \) a locally unknotted isotopy?

If \( M \) is closed then the answer to Question 5 is 'yes' by Corollary 9(i); but if \( M \) is bounded then we may run into Question-3-type trouble at the boundary, as can be seen from the way we had to use Corollary 8 in the proof of Corollary 11. In fact if \( M \) is bounded then Question 5 is the same as Question 3; and if \( M \) is bounded or closed then Question 6 is the same as Question 3. For if the answer to Question 3 is 'yes' then it is easy to show that the answer to Questions 5 and 6 is 'yes'. Conversely, if there is a counterexample to Question 3 then we can use this counterexample to manufacture counterexamples to Questions 5 and 6. For instance, suppose \( B, C \) are knotted \((q, m)\) ball pairs with a common boundary \( B = C \), such
that $B \cup C$ is an unknotted sphere pair. Identify $B^a, C^a$ with the northern and southern hemispheres of the prism $\Delta^a \times I$ (i.e. those subsets of the boundary above and below the equatorial plane $t = \frac{1}{2}$; see Fig. 4). If $V$ is the centre of the prism then $V(\overline{B \cup C})$ is unknotted, but each of $V B, V C$ is locally knotted at $V$. We can regard $V(\overline{B \cup C})$ as an embedding $f : \Delta^m \times I \to \Delta^a \times I$, and by the trick of ((4) Lemma 4) we can make $f$ level-preserving in a neighbourhood of the equator. The restriction of $f$ to this neighbourhood gives a locally knotted isotopy that is a locally unknotted embedding.

The justification for the above definition of locally unknotted isotopy is that it is a sufficient condition for an isotopy to be covered by an ambient isotopy ((4) Theorem 2); and if $f_0$ is locally unknotted then it is also a necessary condition.

3. Proofs of the fundamental theorems

This last section consists of the proofs of Theorems 1, 2, and 3. Some of the lemmas come straight from (11), and in others we follow Whitehead’s style of proof closely.

Lemma 1. Given $X \subseteq M$, if $X$ is collapsible then any regular enlargement of $X$ in $M$ is a ball.

Proof. Let $N$ be the regular enlargement. By ((11) Theorems 4 and 7) we can choose triangulations $J, K, L$ of $M, X, N$ such that $L$ collapses simplicially to $K$, and $K$ collapses simplicially to a point. Then by ((11) Theorem 23, Corollary 1) $L$ is a combinatorial ball, and so $N$ is a polyhedral ball.

Full subcomplexes

Let $L$ be a subcomplex of $K$. We call $L$ a full subcomplex if no simplex of $K - L$ has all its vertices in $L$. As a consequence, any simplex of $K - L$ meets $L$ either in a face or in the empty set.

Example (i). If $L$ is a subcomplex of $K$, and $K'$ a first-derived complex of $K$, then $L'$ is full in $K'$ (see ((11) Lemma 4)).

Example (ii). If $L$ is a full subcomplex of $K$, and $K^*$ an arbitrary subdivision of $K$, then $L^*$ is full in $K^*$.

Well situated

We introduce a technical term for the convenience of the proof of Theorem 1. Let $J$ be a combinatorial manifold and $K, L$ finite subcomplexes. We say that the pair $K, L$ is well situated in $J$ if the following
three conditions hold:

1. \(K\) is link-collapsible on \(L\);
2. \(K \cup L\) and \(K_3\) are full subcomplexes of \(J\);
3. if \(A\) is a simplex in \(N(K - L, J) - K\), then \(\text{lk}(A, J)\) meets \(K\) in a simplex (possibly the empty simplex).

We say that \(K, L\) is well situated at the boundary if, in addition,

4. the pair \(K \cap J, L \cap J\) is well situated in \(J\),
5. \(K \cup J\) is a full subcomplex of \(J\).

**Lemma 2.** Suppose that \(K, L \subseteq J\), and let \(N = N(K - L, J)\). If \(K, L\) is well situated in \(J\) then \(N \setminus K_3\). If, further, \(K, L\) is well situated at the boundary then \(N \setminus N \cap J \cup K_3 \setminus K_3\); in other words \(N\) collapses to \(K_3\) admissibly in the sense of Irwin (5).

**Proof.** The first part follows from conditions (2) and (3) of well-situatedness by (11) Theorem 2. We obtain the second part by refining Whitehead’s proof slightly. His proof goes as follows. Order the simplexes \(A_1, ..., A_r\) of \(N\) that do not meet \(K_3\) in some order of decreasing dimension; by condition (3), for each \(i\), \(K_3 \cap \text{lk}(A_i, N)\) is a (non-empty) simplex, \(B_i\) say. The collapse \(N \setminus K_3\) is achieved by collapsing \(A_i B_i \setminus A_i B_i\) in turn, \(i = 1, 2, ..., r\). Condition (2) ensures that we eventually arrive at \(K_3\).

For the proof to work it is only necessary that the ordering be such that each \(A_i\) precedes its faces. Therefore we can re-order so that \(A_1, ..., A_q\) are in \(J\) and \(A_{q+1}, ..., A_r\) are in \(J\). Let

\[
F = K_3 \cup \bigcup_{q+1}^{r} A_i B_i.
\]

Then \(N \setminus F \setminus K_3\). The lemma will be proved if we show that \(F = N \cap J \cup K_3\).

If \(C \in N \cap J\) then \(C\) is contained in some \(A_i B_i\), \(i > q\), and so \(N \cap J \cup K_3 \subseteq F\). Conversely, if \(i > q\) then \(A_i B_i\) has all its vertices in \(K \cup J\), which is full in \(J\) by condition (5), and so \(A_i B_i \in K \cup J\). But \(A_i \notin K\), and so \(A_i B_i \in J\). Hence \(F \subseteq N \cap J \cup K_3\).

**Lemma 3.** Suppose \(K, L\) well situated in \(J\), and let \(N = N(K - L, J)\). Then \(|N|\) is a regular neighbourhood of \(|K| \mod |L|\) in \(|J|\). If, further, \(K, L\) is well situated at the boundary, then \(|N|\) meets the boundary regularly.

**Proof.** The proof is by induction on \(m = \dim J\), and is analogous to the proof of (11) Theorem 22) and (12) Lemma 6). The induction begins trivially with \(m = 0\). Assume the lemma for \(m - 1\), and suppose that \(J\) is of dimension \(m\).
First we show that \( N \) is a combinatorial \( m \)-manifold, i.e. that the link of every vertex \( x \) in \( N \) is an \((m - 1)\)-sphere or ball. There are three cases, depending upon whether \( x \) lies in \( K - L, L, \) or \( N - K \).

\textit{Case} (i). \( x \in K - L \). Then \( \text{lk}(x, N) = \text{lk}(x, J) \) is sphere or ball.

\textit{Case} (ii). \( x \in L \). Let \( J^* = \text{lk}(x, J), K^* = K \cap J^*, L^* = L \cap J^* \). It is straightforward to verify that \((K^*)_a = K_a \cap J^* = \text{lk}(x, K_a)\), and hence that \( K^*, L^* \) is well situated in \( J^* \). Therefore \( \text{lk}(x, N) = N(K^* - L^*, J^*) \), and so, by induction on \( m \),

\[
|\text{lk}(x, N)| = \text{a regular neighbourhood of } |K^*| \mod |L^*| \text{ in } |J^*|
\]

\[
= \text{a regular enlargement of } |(K^*)_a| \text{ in } |J^*|
\]

\[
= \text{a ball, by Lemma 1,}
\]

because \((K^*)_a = \text{lk}(x, K_a)\), which is collapsible by condition (1), since \( x \in L_a \).

\textit{Case} (iii). \( x \in N - K \). Let \( J^*, K^*, L^* \) be as in the last case. Then \( K^* \) is a simplex by condition (3), and so \( K^* \) is link-collapsible on \( L^* \), because a simplex is link-collapsible on any subcomplex. Also \( K^* \) meets \( K - L \) because \( x \in N \), and so \( K^* \neq L^* \). Therefore \((K^*)_a = K^* \), and hence \((K^*)_a \) is collapsible. We can verify as in the last case that the pair \( K^*, L^* \) is well situated in \( J^* \); therefore as before \( \text{lk}(x, N) = N(K^* - L^*, J^*) \), and so \( \text{lk}(x, N) \) is a ball by induction.

Next we prove that \( |N| \) satisfies the second condition for being a regular neighbourhood. \( |N| \) is a topological neighbourhood of \( |K| - |L| \) in \( |J| \), because every point of \( |K| - |L| \) is in the open star in \( J \) of some vertex in \( K - L \), which is an open set of \( |J| \) contained in \( |N| \). Next we prove that \( N \cap L = L_a \). For \( L_a = K_a \cap L \subseteq N \cap L \). Conversely, suppose that \( A \) is a simplex in \( N \cap L \). Since \( A \subseteq L \), \( A \) does not meet \( K - L \). Therefore \( Ax \subseteq J \) for some vertex \( x \) in \( K - L \). Since \( Ax \) has all its vertices in \( K \cup L \), and \( K \cup L \) is full in \( J \) by condition (2), we have \( Ax \in K \cup L \). But \( x \notin L \). Hence \( Ax \notin L \), and so \( Ax \subseteq K - L \). Therefore \( A \subseteq K_a \). Therefore \( A \in K_a \cap L = L_a \). Now we shall prove \( L_a \subseteq N \). For given \( A \) in \( L_a \), write \( A = xA^* \), where \( x \) is some vertex of \( A \). In the notation of case (ii) above, \( \text{lk}(A, N) = \text{lk}(A^*, N^*) \), where \( N^* = N(K^* - L^*, J^*) \). Now \( A^* \subseteq L_a \cap J^* = (L^*)_a \), and so, by induction on \( m \), we have \( A^* \in \text{the boundary of } N^* \). Therefore \( \text{lk}(A^*, N^*) \) is a ball, and so \( A \subseteq N \). Hence \( L_a \subseteq N \). Therefore \( L_a \subseteq N \cap L \subseteq N \cap L = L_a \). Therefore there is equality \( L_a = N \cap L = N \cap L \), and condition (2) is proved.

The third and last condition for \( |N| \) to be a regular neighbourhood is that \( N \setminus K_a \), which comes from Lemma 2.

For the second part of Lemma 3, we assume, further, that \( K, L \) are well situated at the boundary. By what we have already proved, we
deduce that if \( N_1 = N((K \cap J) - (L \cap J), J) \) then \(| N_1 |\) is a regular neighbourhood of \(| K \cap J | \mod | L \cap J | \) in \(| J |\). The proof of the lemma will be completed if we show that
\[
(N \cap J) - L = N_1.
\]
If \( A \in N_1 \) then \( A \in N(x, J) \) for some \( x \) in \((K - L) \cap J\), and
\[
N(x, J) \subseteq N(x, J) \subseteq N.
\]
Therefore \( N_1 \subseteq N \cap J \). Conversely, suppose that \( A \in N \cap J \). Then \( A \in N(x, J) \) for some \( x \) in \((K - L) \cap J\). Therefore \( xA \) has all its vertices in \( K \cup J \), which is full in \( J \) by condition (5), and so \( xA \in K \cup J \). Therefore \( xA \) lies in either \( J \) or \( K \). If \( xA \in J \) then \( A \in N_1 \). Alternatively, if \( xA \in K \) then \( A \in N_1 \), and so either \( A \in K - L \), whence \( A \in (K - L) \cap J \subseteq N_1 \), or else \( A \in L \). We have shown that
\[
N_1 \subseteq N \cap J \subseteq N_1 \cup L.
\]
Therefore \( N_1 - L = (N \cap J) - L \). But since \(| N_1 |\) is a regular neighbourhood of \(| K \cap J | \mod | L \cap J |\), the interior of \( N_1 \) does not meet \( L \). Consequently \( N_1 = N_1 - L = (N \cap J) - L \), as desired. The proof of Lemma 3 is complete.

**Proof of Theorem 1.** We are given a second-derived neighbourhood \( N \) of \( X \mod Y \) in \( M \), which we want to show is regular. \( N \) is formed by choosing a triangulation \( J, K, L \) of \( M, X, Y \), choosing a second-derived \( J^* \) of \( J \), and defining
\[
N = | N(X - Y, J^*) |.
\]
Let \( J' \) be the first derived of \( J \). Let \( J^* \) be the first derived of \( J' \mod K' \cup L' \); that is to say we form \( J^* \) by starring in some order of decreasing dimension all the simplexes of \( J' - (K' \cup L') \), at the same points that were used to form \( J^* \). The theorem will then follow from Lemmas 3 and 4 below.

**Lemma 4.** (i) \( N = | N(X - Y, J^*) |. \)

(ii) If \( X \) is link-collapsible on \( Y \) then \( K^*, L^* \) is well situated in \( J^* \).

(iii) If, further, \( X \cap M \) is link-collapsible on \( Y \cap M \) then \( K^*, L^* \) is well situated at the boundary.

**Proof.** First form \( J^{**} \) from \( J^* \) by starring all the simplexes in \( L^* - L^*_q = L' - L^*_q \). This process leaves \( N(X - Y, J^*) \) untouched, because if \( A \in L' - L^*_q \) then \( A \notin N(X - Y, J^*) \) for the following reason.

Suppose not: then there is a vertex \( x \) in \( K' - L' \) such that \( xA \in J^* \). The simplex \( xA \) has all its vertices in \( K' \cup L' \), which is full in \( J' \), because we have taken first deriveds, and hence is also full in \( J^* \). But \( x \notin L' \), and so \( xA \in K' - L' \). Therefore \( A \in K'_q \). Therefore \( A \in K'_q \cap L' = L^*_q \), which is a contradiction. We have shown that

(iv) \( N(X - Y, J^*) = N(X - Y, J^{**}) \).
Now star the simplexes in $K_0$, and form $J''$ from $J^{**}$. Let $B$ denote a typical simplex in $K' - L'$, and $\hat{B}$ the point at which it is starred. Then

$$|N(X-Y,J^{**})| = \bigcup_{B} |N(B-\hat{B},J^{**})|$$

$$= \bigcup_{B} |N(\hat{B},J'')|$$

$$= N.$$

This combined with (iv) proves (i).

Proof of (ii). Condition (1) of well-situatedness is true by hypothesis. Condition (2) is true in $J'$ and remains true in $J^*$. Condition (3) is true in $J^{**}$ by ((11) Lemma 4), and remains true in $J^*$ by (iv).

Proof of (iii). Since $(J^*)^* = (J)^*$, condition (4) follows from (ii) applied to the boundary. Condition (5) is already true in $J'$ and remains true in $J^*$. The proof of Lemma 4 and Theorem 1 is complete.

For the proofs of Theorems 2 and 3 we shall need a lemma about isotopies of balls.

**Lemma 5.** Let $P^\subseteq Q^\subseteq R$ be a nest of three $m$-balls, whose boundaries all meet in a common $(m-1)$-face $F$. Then there exist

(i) an isotopy in $Q$ moving $P$ onto $Q$ and keeping $F$ fixed, and

(ii) an ambient isotopy of $R$ moving $P$ onto $Q$ keeping $R$ fixed.

**Proof.** The set-up is homeomorphic to a standard set-up in which $P^\subseteq Q^\subseteq R$ are $m$-simplexes with a common $(m-1)$-face $F$, with the vertex $p$ of $P$ opposite $F$ at the barycentre of $Q$, and the vertex $q$ of $Q$ opposite $F$ at the barycentre of $R$. Therefore it suffices to prove the lemma for the standard set-up.

In the standard set-up there are obvious isotopies by straight-line paths, but these are not in the category in which we are working, since they are piecewise algebraic rather than piecewise linear (see the footnote to the proof of ((4) Lemma 5)). However, we can define piecewise-linear isotopies as follows. In case (i), represent $P \times I$ as a cone with vertex $p \times 1$ and base $P \times 0 \cup F \times I$, and in case (ii) represent $R \times I$ as a cone with vertex $p \times 1$ and base $R \times 0 \cup R \times I$. To obtain the isotopy $P \times I \to Q \times I$ in case (i), and the ambient isotopy $R \times I \to R \times I$ in case (ii), map the base of the cone by the identity, map the vertex $p \times 1$ to $q \times 1$, and join linearly.

**Kernels**

For the proofs of Theorems 2 and 3 we introduce the technical term 'kernel'. The idea is to construct a second-derived neighbourhood inside a given regular neighbourhood, with respect to a triangulation in which
the given neighbourhood collapses simplicially, as in the proof of ((11) Lemma 11).

More precisely, suppose that $X$ is link-collapseable on $Y$, and let $N_1$ be a given regular neighbourhood of $X$ mod $Y$ in $M$. A kernel $N$ of $N_1$ is constructed as follows. Choose a triangulation $J, K, L, G$ of $M, X, Y, N$ such that $G$ collapses simplicially onto $K_*$, by ((11) Theorem 7). Choose a second derived $J'$ of $J$, and define $N = |N(X - Y, J')|$. If, further, $M$ is bounded, and $X \cap \hat{M}$ is link-collapseable on $Y \cap \hat{M}$, and $N_1$ meets the boundary regularly, then we impose the additional restriction on $N$ that $(N \cap \hat{M}) - Y$ be also a kernel of $(N_1 \cap \hat{M}) - Y$; this can be done by subdividing $J$ if necessary (see ((11) Theorems 4 and 7)).

Remarks.

(i) The kernel $N$ is a regular neighbourhood of $X$ mod $Y$ in $M$ (by Theorem 1).

(ii) If $X \cap \hat{M}$ is link-collapseable on $Y \cap \hat{M}$, then $N$ meets the boundary regularly (by Theorem 1).

(iii) $N$ is smaller than $N_1$, because $N_1 \supseteq |N(X - Y, J)|$, which contains a neighbourhood of $N - Y$ in $M$.

(iv) If $N_1, N_2$ are two given regular neighbourhoods, we can choose $N$ to be a kernel of both (again by ((11) Theorems 4 and 6)).

**Lemma 6.** Suppose $X$ link-collapseable on $Y$ in $M$. Let $N_1$ be a regular neighbourhood of $X$ mod $Y$ in $M$, and let $N$ be a kernel of $N_1$. Then there exists an increasing sequence of regular neighbourhoods of $X$ mod $Y$ in $M$,

$$N = U_0 \subseteq U_1 \subseteq \ldots \subseteq U_{2r} = N_1,$$

such that, for each $i$,

$$U_i = U_{i-1} \cup B_i,$$

$$F_i = U_{i-1} \cap B_i,$$

where $B_i$ is an $m$-ball ($m = \dim M$), and $F_i$ an $(m - 1)$-ball facing $B_i$.

**Proof.** The proof is similar to that of ((11) Lemma 11). With the same notation that was used in defining the kernel, we have $G \setminus K$ simplicially. Let $A_1, A_2, \ldots, A_{2r}$ be the simplexes of $G - K_*$, arranged in such order that the simplicial expansion $K_* / G$ is obtained by first adding $A_1$ and its face $A_2$, then adding $A_3$ and its face $A_4$, and so on. Let $A_i$ be the point at which $A_i$ is starred to form $J'$, and let $B_i$ be the $m$-ball

$$B_i = |N(A_i, G')|.$$

Define, inductively, $U_i = U_{i-1} \cup B_i$, $U_0 = N$. Then the intersection $U_{i-1} \cap B_i = \hat{U}_{i-1} \cap \hat{B}_i$ is an $(m - 1)$-ball by the proof of ((11) Lemma 11).

It remains to show that each $U_i$ is a regular neighbourhood of $X$ mod $Y$. By induction $U_i$ is an $m$-manifold. The second condition for being a
regular neighbourhood is satisfied because $U_i$ lies between two other regular neighbourhoods, $N \subseteq U_i \subseteq N_i$. Finally $U_i \setminus X_i$ because

$$N_i = U_{2r} \setminus U_{2r-1} \setminus \ldots \setminus U_0 = N \setminus X_i.$$

The proof of Lemma 6 is complete.

**Proof of Theorem 2.** We are given two regular neighbourhoods $N_1, N_2$ of $X \mod Y$ in $M$, where $X$ is link-collapsible on $Y$. Choose a kernel $N$ of $N_1$ and $N_2$, and a kernel $N_3$ of $N$. We have to prove that there exists an isotopy in $M$ moving $N_1$ onto $N_2$ through a continuous family of regular neighbourhoods, keeping $N_3$ fixed.

Let $U_i$ be as in Lemma 6. For each $i$, $1 \leq i \leq 2r$, we shall construct an isotopy

$$f_i : U_{i-1} \times I \rightarrow U_i \times I$$

in $U_i$ moving $U_{i-1}$ onto $U_i$ through a continuous family of regular neighbourhoods (of $X \mod Y$ in $M$), and keeping $N_3$ fixed. The composition

$$N \times I \xrightarrow{f_r} U_1 \times I \xrightarrow{f_r} U_2 \times I \rightarrow \ldots \rightarrow U_{2r-1} \times I \xrightarrow{f_r} N_1 \times I \xrightarrow{f_r} M \times I$$

gives an isotopy in $M$ moving $N$ onto $N_1$ keeping $N_3$ fixed. Similarly there is an isotopy moving $N$ onto $N_2$, and the reverse of the former followed by the latter gives what we want.

![Fig. 5](image)

It remains to construct the isotopy $f_i$. Suppose therefore that $i$ is fixed, and consider the face $F_i$ of the ball $B_i$ of Lemma 6. We claim that $\hat{F}_i$ does not meet $N_3$. For since $N_3$ is smaller than $N$, $N$ contains a neighbourhood of $N_3 - Y$ in $M$. Therefore $\hat{F}_i \subseteq \hat{M} - \hat{N}$, does not meet $N_3 - Y$. Hence $F_i \cap N_3 = \hat{F}_i \cap Y$. Now $\hat{F}_i \subseteq \hat{N}_i$, and $\hat{N}_i$ does not meet $Y$ by the second regular-neighbourhood condition. Therefore $\hat{F}_i \cap N_3 = \hat{F}_i \cap Y \subseteq \hat{N}_i \cap Y = \emptyset$. Consequently $F_i \cap N_3 \subseteq \hat{F}_i$, and so $F_i$ is link-collapsible on $\hat{F}_i \cup N_3$, because a ball is link-collapsible on its boundary.

Let $C_i$ be a regular neighbourhood of $F_i \mod \hat{F}_i \cup N_3$ in $U_{i-1}$ (see Fig. 5). Since $C_i$ is a regular enlargement of $F_i$, it is an $m$-ball by Lemma 2, meeting $B_i$ in the common face $F_i$.
By Lemma 5 there is an isotopy in $B_i \cup C_i$ moving $C_i$ onto $B_i \cup C_i$, and keeping $\hat{\mathcal{C}}_i - \hat{\mathcal{F}}_i$ fixed. Extend this by the constant isotopy on $U_{i-1} - C_i$ to the required isotopy in $U_i$ moving $U_{i-1}$ onto $U_i$, and keeping $N_3$ fixed. At each stage of this isotopy, the image is a regular neighbourhood, because it is a manifold, lies between two other regular neighbourhoods, and collapses to $X_\delta$ by the image of the collapse $U_{i-1} \setminus X_\delta$. The proof of Theorem 2 is complete.

The proof of Theorem 3 will be a development of that of Theorem 2 in three stages: in Lemma 7 we make the isotopy ambient in the case when $M$ is unbounded; in Lemmas 8 and 9 the bounded case is dealt with; and finally we move the smaller neighbourhood that is to be kept fixed during the isotopy from thesis to hypothesis. Recall the statement of the theorem:

**Hypothesis of Theorem 3.** Suppose that $X$ is link-collapsible on $Y$, and $X \cap M$ link-collapsible on $Y \cap M$. Let $N_1, N_2, N_3$ be three regular neighbourhoods of $X$ mod $Y$ in $M$, meeting the boundary regularly, and such that $N_3$ is smaller than $N_1$ and $N_2$. Let $P$ be the closure of the complement of a second-derived neighbourhood of $N_1 \cup N_2$ mod $Y$ in $M$.

**Thesis of Theorem 3.** There exists an ambient isotopy of $M$ moving $N_1$ onto $N_2$ and keeping $N_3 \cup P$ fixed.

For the next three lemmas, Lemmas 7, 8, and 9, we assume the hypothesis of Theorem 3, and make the following construction: let $N$ be a kernel of $N_1$ and $N_2$, and let $N_4$ be a kernel of $N$ and $N_3$.

**Lemma 7.** Suppose $M$ unbounded. Then there exists an ambient isotopy of $M$ moving $N$ onto $N_1$, keeping $N_4 \cup P$ fixed.

**Proof.** The proof is an addendum to the proof of Theorem 2. Using the same notation (with the proviso that for $N_3$ now read $N_4$), we show that the isotopy moving $U_{i-1}$ onto $U_i$ can now be realized by an ambient isotopy of $M$ keeping $N_4 \cup P$ fixed. The composition of these ambient isotopies for $i = 1, 2, \ldots, 2r$ will give the required isotopy for Lemma 7.

Continuing with the same notation, let

$$E_i = \text{cl} \left( \hat{U}_i - \hat{U}_{i-1} \right) = \hat{\mathcal{B}}_i - \hat{\mathcal{F}}_i,$$

which is an $(m-1)$-ball facing the $m$-ball $B_i$. We claim that $\hat{E}_i$ does not meet $P$. For $E_i \subseteq N_1$; therefore $E_i \cap Y \subseteq E_i \cap N_1 \cap Y = E_i \cap Y_\delta$, by the second regular-neighbourhood condition, and $E_i \cap Y_\delta \subseteq E_i \cap U_{i-1} = E_i$. Hence $\hat{E}_i \subseteq N_1 - Y$. But by hypothesis $M - P$ is the interior of a second-derived neighbourhood of $N_1 \cup N_2$ mod $Y$ in $M$, and therefore $\hat{E}_i \subseteq N_1 - Y \subseteq M - P$. We have verified the claim that $\hat{E}_i$ does not meet $P$. 
Therefore \( E_i \) is link-collapsible on \( E_i \cup P \), because a ball is link-collapsible on its boundary.

Let \( D_i \) be a regular neighbourhood of \( E_i \) mod \( E_i \cup P \) in \( \overline{M - U_i} \) (see Fig. 6). Then \( D_i \), being a regular enlargement of the ball \( E_i \), is an \( m \)-ball meeting \( B_i \) in the common face \( E_i \).

Let \( H_i = B_i \cup C_i \cup D_i \), which is an \( m \)-ball whose interior does not meet \( N_i \cup P \), by construction. By Lemma 5 there is an ambient isotopy of \( H_i \) moving \( C_i \) onto \( B_i \cup C_i \), and keeping \( H_i \) fixed. Extend this by the constant isotopy of \( M \setminus H_i \) to the required ambient isotopy of \( M \) moving \( U_{i-1} \) onto \( U_i \), and keeping \( N_i \cup P \) fixed. The proof of Lemma 7 is complete.

**Lemma 8.** Suppose that \( M \) is bounded. Then \( N_1 \) collapses admissibly to \( X_1 \), i.e.

\[
N_1 \setminus N_1 \cap M \cup X_1 \setminus X_2.
\]

**Proof.** Since \( N \) is a kernel of \( N_1 \), \( N_1 \setminus N \) by Lemma 6. Since \( N \) is a second-derived neighbourhood, \( N \setminus N \cap M \cup X_1 \setminus X_2 \) by Lemmas 2 and 4(iii). We shall produce a homeomorphism \( h \) of \( N_1 \) onto itself, throwing \( N \cap M \) onto \( N_1 \cap M \), and keeping \( X_1 \) fixed. Then the image under \( h \) of the collapse \( N_1 \setminus N \cap M \cup X_1 \setminus X_2 \) will give what we want.

Let \( M^* = N_1 \), \( X^* = X \cap M^* \), \( Y^* = Y \cap M^* \), \( N^* = (N_1 \cap M^*) - Y^* \) and \( N^* = (N \cap M) - Y \). Then, although \( X^* \neq X \cap M \) and \( Y^* \neq Y \cap M \) in general, it is nevertheless true that

\[
(X^*)_h = (X \cap M)_h, \quad (Y^*)_h = (Y \cap M)_h,
\]

because

\[
(X^*)_h = X^* - Y^* = (X - Y) \cap \overline{N_1} = (X - Y) \cap \overline{M} = (X \cap \overline{M}) - (Y \cap \overline{M}) = (X \cap M)_h,
\]

and

\[
(Y^*)_h = (X^*)_h \cap Y^* = (X \cap M)_h \cap (Y \cap N_1) = (X \cap M)_h \cap (Y \cap \overline{N_1}) = (X \cap M)_h \cap (Y \cap M) = (Y \cap M)_h.
\]
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Therefore by the hypothesis of Theorem 3 and the construction of $N$, it follows that $N_1^*$ is a regular neighbourhood of $X^* \mod Y^*$ in $M^*$, and $N^*$ is a kernel of $N_1^*$. Since $M^*$ is closed we can apply Lemma 7 to each component of $M^*$, and obtain an ambient isotopy $f^*$ of $M^*$ moving $N^*$ onto $N_1^*$, and keeping $X^*$ fixed. We wish to extend $f^*$ to an ambient isotopy of $N_1$, at the same time taking care not to move $X_3$.

Now $f^*$ is the composition of a finite number of isotopies $f_i^*$, $i = 1, 2, \ldots, 2r$, where $f_i^*$ is an ambient isotopy of $M^*$ keeping everything fixed except an $(m - 1)$-ball $H_i^*$, whose interior does not meet $X^*$ and hence does not meet $X_3$. Therefore $H_i^*$ is link-collapsible on $\bar{H}_i^* \cup X_3$ in $N_1$. Then $H_i$, being a regular enlargement of $H_i^*$, is an $m$-ball meeting $\bar{N}_i$ in the face $\bar{H}_i$. Hence $H_i$ is homeomorphic to a cone on $\bar{H}_i^*$ (the homeomorphism keeping $H_i^*$ fixed), and so the ambient isotopy $f_i^* \mid H_i^*$ of $H_i^*$ can be extended conewise to an ambient isotopy $f_i$ of $H_i$ keeping $\bar{H}_i - \bar{H}_i^*$ fixed. Extend $f_i$ to an ambient isotopy of $N_i$, fixed outside $H_i$. Let $f$ be the composition of the $f_i$, $i = 1, \ldots, 2r$. By our construction, $f$ is an ambient isotopy of $N_i$ moving $\bar{N}_i$ onto $N_i^*$ (since it is an extension of $f^*$), and keeping $X_3$ fixed.

Now $Y^* = Y_3 \subseteq X_3$, and so $f$ keeps $Y^*$ fixed. Therefore $f$ moves $N \cap M$ onto $N_1 \cap M$ because $N \cap M = N^* \cup Y^*$ and $N_1 \cap M = N_1^* \cup Y^*$. Consequently the end of the isotopy $f$ gives the homeomorphism $h$ that we want, throwing $N \cap M$ onto $N_1 \cap M$ and keeping $X_3$ fixed. The proof of Lemma 8 is complete.

**Lemma 9.** Suppose $M$ bounded. Then there exists an ambient isotopy of $M$ moving $N$ onto $N_4$ and keeping $N_4 \cup P$ fixed.

**Proof.** The lemma is the ‘bounded’ analogue of Lemma 7, and, as in the proof of Lemma 7, we construct for each $i$ an ambient isotopy of $M$ moving $U_{i-1}$ onto $U_i$, keeping $N_4 \cup P$ fixed. The boundary is a potential source of trouble, because if, for some $i$, it happened that $B_i \cap \bar{M} = \bar{B}_i - \bar{F}_i$, then an ambient isotopy would be impossible: we should have to push a bit of $M$ off the boundary, as it were. The trouble is precisely what does occur if a non-admissible collapse $N_1 \setminus X_3$ is used in Lemma 6 to construct the sequence of $U_i$'s, and it was to avoid this contingency that we proved the existence of an admissible collapse in Lemma 8.

More precisely, if the admissible collapse of Lemma 8 is used to define the ordering $A_1, A_2, \ldots, A_{2r}$ of the simplexes in the proof of Lemma 6, then there exists a $q$ such that

$$A_i \subseteq \bar{M}, \quad 1 \leq i \leq q,$$

$$\hat{A}_i \subseteq \hat{M}, \quad q < i \leq 2r.$$
If \( i > q \), then \( B_i \subseteq \widehat{M} \), because \( B_i \) is the closed star, in a second derived, of an interior vertex in a first derived, of a triangulation of \( M \). Therefore we can proceed to define the ambient isotopy as in Lemma 7. If \( i \leq q \), then \( B_t \) meets \( \widehat{M} \), and we proceed as follows.

For the rest of the proof of this lemma we use the superscript star to denote intersection with \( \widehat{M} \), \( U_i^* = U_i \cap \widehat{M} \), etc. Then \( B_t^* \) is an \((m - 1)\)-ball facing \( B_t \), and meeting \( F_t \) in the common \((m - 2)\)-face \( F_t^* \), and such that

\[
U_t^* = U_{t-1}^* \cup B_t^*, \quad F_t^* = U_{t-1}^* \cap B_t^*.
\]

Let \( J_t \) denote the \((m - 2)\)-ball \( J_t = \hat{F}_t - \hat{F}_t^* \), with boundary the \((m - 3)\)-sphere \( J_t^* \).

As in the proofs of Theorem 2 and Lemma 7, we can show that \( B_t \cap (N_4 \cup P) \subseteq J_t \), and so \( B_t \) is link-collapsible on \( J_t \cup N_4 \cup P \), and \( B_t^* \) is link-collapsible on \( J_t^* \cup N_4^* \cup P^* \). Let \( H_t \) be a regular neighbourhood of \( B_t \) mod \( J_t \cup N_4 \cup P \) in \( M \) that meets the boundary regularly (such exist by Theorem 1). Then \( H_t \) is an \( m \)-ball meeting \( \widehat{M} \) in the face \( H_t^* \), which is a regular neighbourhood of \( B_t^* \) mod \( J_t^* \cup N_4^* \cup P^* \) (by definition of meeting the boundary regularly). See Fig. 7. Therefore the pair \( H_t, B_t \) is homeomorphic to the cone on the pair \( H_t^*, B_t^* \), the homeomorphism keeping \( H_t^*, B_t^* \) fixed. By Lemma 5 there is an ambient isotopy of \( H_t^* \), fixed on the boundary, and moving \( U_{t-1} \cap H_t^* \) onto \( U_t \cap H_t^* \). We can extend this, first conewise to an ambient isotopy of \( H_t \), and then by the constant isotopy on the complement, to an ambient isotopy of \( M \), moving \( U_{t-1} \) onto \( U_t \), and fixed outside \( H_t \). The proof of Lemma 9 is complete.

Proof of Theorem 3. Lemmas 7 and 9, applied to both \( N_1 \) and \( N_2 \), give an ambient isotopy \( f \) of \( M \) moving \( N_1 \) onto \( N_2 \) and keeping \( N_3 \cup P \) fixed.

Let \( N_0 \) be a kernel of \( N_4 \), and let \( P' = M - (N_1 \cap N_2) \). Now \( N_4 \) is a kernel of \( N_0 \), and \( P' \) is contained in the closure of the complement of a second-derived neighbourhood of \( N_3 \) mod \( Y \) in \( M \), because \( N_3 \) is smaller than

\[
\text{FIG. 7}
\]
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Therefore we can appeal to Lemmas 7 and 9 to obtain an ambient isotopy of \( M \) moving \( N_1 \) to \( N_3 \), and keeping \( N_5 \cup P \) fixed. In particular, the end of this isotopy furnishes a homeomorphism \( h \) of \( M \) onto itself, throwing \( N_4 \) onto \( N_3 \), throwing \( N_1 \) onto itself and \( N_2 \) onto itself, and keeping \( P \) fixed. The image of \( f \) under \( h \), or, more precisely, the composition

\[
M \times I \xleftarrow{h \times 1} M \times I \xrightarrow{h \times 1} M \times I,
\]

gives the ambient isotopy of \( M \) that we want, moving \( N_1 \) onto \( N_2 \), and keeping \( N_3 \cup P \) fixed. The proof of Theorem 3 is complete.

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