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The absolute simplicial approximation theorem, which dates back to Alexander (1), states that there is a simplicial approximation g to any given continuous map f between two finite simplicial complexes (see for instance (2), p. 37 or (3), p. 86). The relative theorem given here permits us to leave f unchanged on any subcomplex, on which f happens to be already simplicial.

The only previous mention of this modification that I have seen* in the literature is a remark ((3), Remark I, p. 87) to the effect that the relative case is an immediate generalization of the absolute case. In fact a strict generalization of the absolute case is not true, as is shown by the counter-example at the end of the paper. It is evidently necessary to give special treatment to the neighbourhood of the subcomplex to be kept fixed.

Presumably the relative theorem has been somewhat neglected, because simplicial approximation has only been used in the context of *algebraic* topology. If L denotes the subcomplex, then the absolute approximation theorem ensures that $gL = fL$, and that $g|L$ is an approximation to $f|L\colon L \to fL$, which is sufficient for homological applications. However, for recent applications in *geometric* topology the stronger result $g|L = f|L$ is necessary (as, for example, in the proof of (4), Lemma 2·7).

Notation. Let $K, L, \ldots$ denote finite simplicial complexes, and let $|K|$ denote the polyhedron underlying K. We shall assume simplexes to be *closed*. Any point $x \in |K|$ lies in the interior of a unique simplex in K, called the *carrier* of x.

If A is a simplex of K let $\mathrm{st}(A, K)$ denote the *star* of A in K, which is the open subset of $|K|$ consisting of the union of the interiors of all simplexes having A as a face. The stars of all the vertices of K form an open covering of $|K|$, called the *star covering* of K. The *double star* of a simplex is defined

$$\mathrm{st}^2(A, K) = \bigcup_{v \in A} \mathrm{st}(v, K),$$

where the union is taken over all vertices of A.

If L is a subcomplex of K, let $N(L, K)$ denote the *simplicial neighbourhood* of L in K, which is the subcomplex of K consisting of all simplexes meeting L, together with their faces. The *double* neighbourhood is defined

$$N^2(L, K) = N(N(L, K), K).$$

In particular the neighbourhood of a simplex $|N(A, K)| = \mathrm{cl}(\mathrm{st}^2(A, K))$.

If $L \subset K$, let $(K \bmod L)'$ denote the *barycentric derived complex of K modulo L*, which is obtained from K by subdividing barycentrically all simplexes of K-L in some order of

* I am indebted to the referee for drawing my attention to it.

decreasing dimension. In particular $(K \bmod L)'$ contains L as a subcomplex. Therefore we can define inductively

$$K_0 = K,$$
$$K_r = (K_{r-1} \bmod L)'.$$

THEOREM. *Let K, M be finite simplicial complexes, and L a subcomplex of K. Let $f\colon |K| \to |M|$ be a continuous map such that the restriction $f|L$ is a simplicial map from L to M. Then there exists an integer r, and a simplicial map $g\colon K_r \to M$ such that $g|L = f|L$ and g is homotopic to f keeping L fixed.*

If $x \in |K|$, let $\Delta(fx)$ denote the carrier of fx; that is the unique simplex of M whose interior contains fx.

ADDENDUM TO THE THEOREM. *Given an arbitrary neighbourhood U of $|L|$ in $|K|$, then we can choose g and the homotopy such that*

(i) *if $x \notin U$, the homotopy of x is the straight interval in $\Delta(fx)$ from fx to gx;*
(ii) *if $x \in |L|$, the homotopy leaves x fixed at fx;*
(iii) *if $x \in U - |L|$, the homotopy of x is contained in $N^2(\Delta(fx), M)$.*

Remark. Condition (iii) reveals why the relative theorem is not a strict generalization of the absolute theorem. In the absolute theorem all the points satisfy condition (i), but the example at the end of the paper shows that if we insist on condition (ii), then it is impossible to satisfy (i) for all points in the neighbourhood of $|L|$.

COROLLARY. *Given a continuous map $f\colon X \to Z$ between two polyhedra, such that the restriction of f to a subpolyhedron Y, $\subset X$, is piecewise linear, then there is an arbitrarily close piecewise linear map $g\colon X \to Z$ such that $g|Y = f|Y$, and an arbitrarily small homotopy from f to g keeping Y fixed.*

The corollary is obtained from the theorem and the addendum by triangulating Z sufficiently finely.

LEMMA. *If $r \geqslant 1$, then L is full in K_r; that is to say*

(i) *no simplex of $K_r - L$ has all its vertices in L, and*
(ii) *every simplex of $K_r - L$ meets L in a face or the empty set.*

For the proof of the lemma see (5), Lemma 4.

Proof of the theorem. We are given a continuous map $f\colon |K| \to |M|$ such that $f|L$ is simplicial. The customary proof of the absolute simplicial approximation theorem breaks down because, in general, for no r does the star covering of K_r refine f^{-1} (the star covering of M). The difficulty is illustrated by the counter-example at the end. Therefore we first perform a homotopy of $|K|$ on itself that retracts a neighbourhood of L onto L.

If B is a simplex of L, then $fB \subset \mathrm{st}^2(fB, M)$, and so B is contained in the open set $f^{-1}(\mathrm{st}^2(fB, M))$. As $q \to \infty$, the compact sets

$$\mathrm{cl}(\mathrm{st}(B, K_q))$$

converge uniformly to the compact set B, and so, for sufficiently large q, are contained in $f^{-1}(\mathrm{st}^2(fB, M))$. Choose q, $\geqslant 1$, such that this is true for all simplexes of L. Therefore

$$f\{\mathrm{cl}(\mathrm{st}(B, K_q))\} \subset \mathrm{st}^2(fB, M) \quad \text{all } B \in L.$$

We now construct a subdivision K'_q of K_q, by subdividing barycentrically all simplexes of $K_q - L$ that meet L, in some order of decreasing dimension. Subdivisionwise, K'_q lies in between K_q and K_{q+1}. Let

$$N = N(L, K'_q) = N(L, K_{q+1}).$$

Next we construct a simplicial map h: $K'_q \to K_q$ as follows: if v is a vertex in $L \cup (K'_q - N)$ define $hv = v$; if v is a vertex in $N - L$ then v is the barycentre of some simplex $A \notin K_q$ meeting L, and we define hv to be a vertex of the face $A \cap L$ (it is a face by the lemma). Therefore

$$\operatorname{st}(v, K'_q) \subset \operatorname{st}(hv, K_q) \quad \text{all vertices } v \in K'_q.$$

By the absolute simplicial approximation theorem ((2), pp. 36–38), the vertex map h determines a simplicial map h, which is a simplicial approximation to the identity, and is homotopic to the identity by, say, a homotopy h_t: $|K| \to |K|$ $(0 \leqslant t \leqslant 1)$, where $h_0 = 1$ and $h_1 = h$. By our choice of h the homotopy leaves the subcomplex $L \cup \operatorname{cl}(K_q - N(L, K_q))$ of K'_q fixed. Any other point $x \in |K|$ lies in the interior of some simplex $A \in K_q - L$, that meets L, and the homotopy $h_t x$ of x is the straight interval in A from x to hx. By the lemma, the fullness of L in K_q ensures that $N = h^{-1}L$, and that locally

$$h(\operatorname{st}(y, N)) = \operatorname{st}(y, L), \quad \text{all vertices } y \in L.$$

Since $1 \simeq h$ it follows that $f \simeq fh$, and so we now need only approximate fh. Let

$$\beta = (fh)^{-1} (\text{the star covering of } M),$$

which is an open covering of $|K|$. Let V be the subcomplex of K_{q+1} complementary to N,

$$V = \operatorname{cl}(K_{q+1} - N).$$

If $r > q+1$ then the subdivision K_r of K_{q+1} induces subdivisions N_r, V_r, say, of N, V. In particular V_r is the $(r-q-1)$th barycentric derived complex of V, because V contains no simplexes of L. Let α_r be the star covering of K_r, and let α'_r be the subset

$$\alpha'_r = \{\operatorname{st}(v, K_r);\ v \in V_r\},$$

which is an open covering of a neighbourhood of $|V_r|$ in $|K|$. As $r \to \infty$, mesh $\alpha'_r \to 0$, because V_r is outside a neighbourhood of L, and so all the simplexes involved get smaller under subdivision, as opposed to those meeting L, which have to remain large. Choose r such that the mesh of α'_r is less than the Lebesgue number of β. Therefore α'_r refines β. We claim further than in fact α_r refines β, because, if v is a vertex of $N_r - V_r$, then

$$\operatorname{st}(v, K_r) = \operatorname{st}(v, N_r) \subset \operatorname{st}(y, N)$$

for some vertex $y \in L$. Therefore

$$\begin{aligned} fh(\operatorname{st}(v, K_r)) &\subset fh(\operatorname{st}(y, N)) \\ &= f(\operatorname{st}(y, L)) \\ &\subset \operatorname{st}(fy, M), \end{aligned}$$

since $f|L$ is simplicial. Since α_r refines β, we can choose a vertex map g: $K_r \to M$, such that

$$fh(\operatorname{st}(v, K_r)) \subset \operatorname{st}(gv, M), \quad \text{all vertices } v \in K_r.$$

By the absolute theorem, the vertex map g defines a simplicial map g, which is a simplicial approximation to fh, and is homotopic to fh by a homotopy that moves each point $x \in |K|$ along the straight interval in the carrier of fhx from fhx to gx. In particular, if y is a vertex of L, then

$$fy = fhy \in fh(\mathrm{st}(y, K_r)) \subset \mathrm{st}(gy, M),$$

and so $fy = gy$. Therefore $g|L = f|L$. Both homotopies $1 \simeq h$, $hf \simeq g$ keep L fixed, and so we have $f \simeq hf \simeq g$ keeping L fixed. The proof of the theorem is complete.

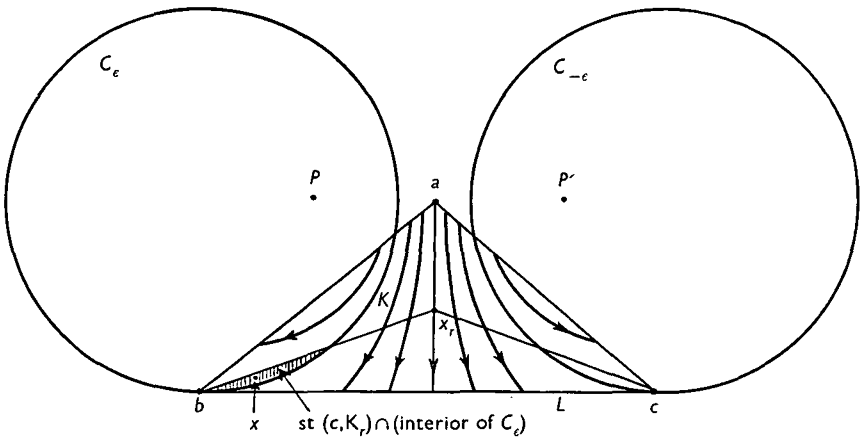

Fig. 1

Proof of the addendum. We are given a neighbourhood U of $|L|$ in $|K|$. Let

$$U_q = \bigcup_{y \in L} \mathrm{st}(y, K_q).$$

As $q \to \infty$, the sets U_q converge uniformly to $|L|$, and so for sufficiently large q are contained in U. In the above proof of the theorem we choose q so that in addition $U_q \subset U$. We proceed to verify the three cases.

(i) If $x \notin U$, then $x \notin U_q$, and so x is kept fixed under the homotopies $1 \simeq h$ and $f \simeq fh$, and under the homotopy $fh \simeq g$ moves along the straight interval in $\Delta(fx)$ from $fx = fhx$ to gx.

(ii) If $x \in |L|$, x is kept fixed.

(iii) If $x \in U - |L|$, then either $x \notin U_q$ or $x \in U_q - |L|$. In the first case condition (i) is satisfied, and so *a fortiori* condition (iii). In the second case x lies in the interior of some simplex $A \in K_q - L$ meeting L in the non-empty face B, say. The image of x under the homotopy h_t, $0 \leqslant t \leqslant 1$, is contained in A, and therefore

$$fh_t x \in fA \subset f\{\mathrm{cl}(\mathrm{st}(B, K_q))\} \subset \mathrm{st}^2(fB, M),$$

by our choice of q. In particular

$$fx = fh_0 x \in \mathrm{st}^2(fB, M).$$

But fx lies in the interior of $\Delta(fx)$. Therefore by the definition of double stars, $\Delta(fx)$ and fB have a vertex in common, and so $fB \in N(\Delta(fx), M)$. Therefore

$$fh_t x \in \mathrm{st}^2(fB, M) \subset |N(fB, M)| \subset |N^2(\Delta(fx), M)|,$$

as desired. The proof of the addendum is complete.

The counter-example. We conclude with an example of a continuous map $f\colon |K| \to |M|$ such that $f|L$ is simplicial, which illustrates the following two points:

(i) *For no r does the star covering of K_r refine f^{-1} (the star covering of M).*

(ii) *If we insist that $g|L = f|L$, then it is impossible to find a piecewise linear map $g\colon |K| \to |M|$ such that $gx \in \Delta(fx)$ for every point $x \in |K|$.*

Let $\{C_t\colon -1 < t < 1\}$ be a system of non-intersecting co-axial circles in the Euclidean plane, with limit points P, P'. Let a be the mid-point of PP'. Choose ϵ, $0 < \epsilon < 1$. The circles C_ϵ, $C_{-\epsilon}$ are of the same size and possess two common tangents parallel to PP'. Let b, c be the points of contact of the two circles with one of these common tangents. Let K be the 2-simplex abc, let L be the face bc, and let M be the boundary of K. Define the map $f\colon |K| \to |M|$ as follows: for each t, if the arc $K \cap C_t$ is non-empty, let f map the whole arc onto its end-point further than PP'. Then $f|L$ is the identity, and is therefore simplicial.

Proof of (i). Suppose the converse holds for K_r. Then $\mathrm{st}(c, K_r)$ is contained in f^{-1} (some star of M). In particular this star of M contains c, because $fc = c$, and so it is $\mathrm{st}(c, M)$. Hence $\mathrm{st}(c, K_r) \subset f^{-1}(\mathrm{st}(c, M))$. But $\mathrm{st}(c, K_r)$ contains the interior of the triangle bcx_r, where x_r is the point $(1/3)^r$ up the median through a, and so $\mathrm{st}(c, K_r)$ meets the interior of C_ϵ. But $f^{-1}(\mathrm{st}(c, M))$ is the intersection of K with the exterior of C_ϵ. We have a contradiction.

Proof of (ii). If $g\colon |K| \to |M|$ is piecewise linear, and $g|L = f|L$ ($=$ the identity), then g^{-1} (the interior of L) is an open simplicial neighbourhood of the interior of L in some subdivision of K. Any such neighbourhood meets the interior of C_ϵ, so choose a point x in the intersection. Then fx, gx lie respectively in the interiors of ab, bc. Therefore $\Delta(fx) = ab$, and so $gx \notin \Delta(fx)$.

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