THE CLASSIFICATION OF ELEMENTARY CATASTROPHES OF CODIMENSION* \( \leq 5 \).

by

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(Notes written and revised by David Trotman)

INTRODUCTION.

These lecture notes are an attempt to give a minimal complete proof of the classification theorem from first principles. All results which are not standard theorems of differential topology are proved. The theorem is stated in Chapter 1 in a form that is useful for applications [12].

The elementary catastrophes are certain singularities of smooth maps \( \mathbb{R}^r \to \mathbb{R}^r \). They arise generically from considering the stationary values of \( r \)-dimensional families of functions on a manifold, or from considering the fixed points of \( r \)-dimensional families of gradient dynamical systems on a manifold. Therefore they are of central importance in the bifurcation theory of ordinary differential equations. In particular the case \( r = 4 \) is important for applications parametrised by space-time.

The concept of elementary catastrophes, and the recognition of their importance, is due to René Thom [10]. He realized as early as about 1963 that they could be finitely classified for \( r \leq 4 \), by unfolding certain polynomial germs \((x^3, x^4, x^5, x^6, x^3y^2, x^2y^4)\). Thom's sources of inspiration were fourfold: firstly Whitney's paper [11] on stable-singularities for \( r = 2 \), secondly his own work extending these results to \( r > 2 \), thirdly light caustics, and fourthly biological morphogenesis.

*This paper, giving a complete proof of Thom's classification theorem, seems not to be readily available. In response to many requests from conference participants, Zeeman and his collaborator, David Trotman, agreed to make a revised version of the paper (July, 1975) available for the conference proceedings. I would like to express my appreciation to both Christopher Zeeman and David Trotman.

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However although Thom had conjectured the classification, it was some years before the conjecture could be proved, because several branches of mathematics had to be developed in order to provide the necessary tools. Indeed the greatest achievement of catastrophe theory to date is to have stimulated these developments in mathematics, notably in the areas of bifurcation, singularities, unfoldings and stratifications. In particular the heart of the proof lies in the concept of unfoldings, which is due to Thom. The key result is that two transversal unfoldings are isomorphic, and for this Thom needed a $C^\infty$ version of the Weierstrass preparation theorem. He persuaded Malgrange [3] to prove this around 1965. Since then several mathematicians, notably Mather, have contributed to giving simpler alternative proofs [4,5,7,8] and the proof we give in Chapter 5 is mainly taken from [1].

The preparation theorem is a way of synthesising the analysis into an algebraic tool; then with this algebraic tool it is possible to construct the geometric diffeomorphism required to prove two unfoldings equivalent. The first person to write down an explicit construction, and therefore a rigorous proof of the classification theorem, was John Mather, in about 1967. The essence of the proof is contained in his published papers [4,5] about more general singularities. However the particular theorem that we need is somewhat buried in these papers, and so in 1967 Mather wrote a delightful unpublished manuscript [6] giving an explicit minimal proof of the classification of the germs of functions that give rise to the elementary catastrophes. The basic idea is to localise functions to germs, and then by determinacy reduce germs to jets, thereby reducing the $\infty$-dimensional problem in analysis to a finite dimensional problem in algebraic geometry. Regrettably Mather's manuscript was never quite finished, although copies of it have circulated widely. We base Chapters 2, 3, 4, 6 primarily upon his exposition.

Mather's paper is confined to the local problem of classifying germs of functions. To put the theory in a usable form for applications three further steps are necessary. Firstly we need to globalise from germs back to
functions again, in order to obtain an open-dense set of functions, that can be used for modeling. For this we need the Thom transversality lemma, and Chapter 8 is based on Levine's exposition [2].

Secondly we have to relate the function germs, as classified by Mather, to the induced elementary catastrophes, which are needed for the applications. For instance the elliptic umbilic starts as an unstable germ \( \mathbb{R}^2 \to \mathbb{R} \), which then unfolds to a stable-germ \( \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^3 \), or equivalently to a germ \( f: \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R} \), and eventually induces the elementary catastrophe germ \( X_f: \mathbb{R}^3 \to \mathbb{R}^3 \). The relation between these is explained in Chapter 7.

Finally in Chapter 9 we verify the stability of the elementary catastrophes, in other words the stability of \( X_f \) under perturbations of \( f \). A word of warning here: although the elementary catastrophes are singularities, and are stable, they are different from the classical stable-singularities [1,2,4,5,11]. The unfolded germ is indeed a stable-singularity, but the induced catastrophe germ may not be. The difference can be explained as follows. Let \( M \) denote the space of all \( C^\infty \) maps \( \mathbb{R} \to \mathbb{R}^r \), and \( C \) the subspace of catastrophe maps. Then \( C \neq M \) because not all maps can be induced by a function. Therefore a stable-singularity, such as \( E_2 \), may appear in \( M \), but not in \( C \), and therefore will not occur as an elementary catastrophe. Conversely an elementary catastrophe, such as an umbilic, may appear in \( C \), and be stable in \( C \), but become unstable if perturbations in \( M \) are allowed, and therefore will not occur as a stable-singularity. For \( r = 2 \) the two concepts accidentally coincide, because Whitney [11] showed that the only two stable-singularities were the fold and cusp, and these are the two elementary catastrophes. However for \( r = 3 \) the concepts diverge, and for \( r = 4 \), for instance, there are 6 stable-singularities and 7 elementary catastrophes, as follows:
We are grateful to Mario De Oliveira and Peter Stefan for their helpful comments: these have led to several corrections in the text.*

*As editor, I would also like to express my gratitude to Sandra Smith for adapting the original manuscript to a form suitable for the Lecture Notes, and to Sarah Rosenberg for her skillful reproduction of the diagrams.

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Chapter 1. Stating Thom's Theorem
Chapter 2. Determinacy
Chapter 3. Codimension
Chapter 4. Classification
Chapter 5. The Preparation Theorem
Chapter 6. Unfoldings
Chapter 7. Catastrophe Germs
Chapter 8. Globalisation
Chapter 9. Stability
CHAPTER 1. STATING THOM'S THEOREM.

Let \( f: \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R} \) be a smooth function. Define \( M_f \subset \mathbb{R}^{n+r} \) to be given by
\[
\left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}, \frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_r} \right) = \text{grad}_x f = 0,
\]
where \( x_1, \ldots, x_n \) are coordinates for \( \mathbb{R}^n \), and \( y_1, \ldots, y_r \) are coordinates for \( \mathbb{R}^r \). Generically \( M_f \) is an \( r \)-manifold because it is codimension \( n \), given by \( n \) equations. Let \( X_f: M_f \to \mathbb{R}^r \) be the map induced by the projection \( \mathbb{R}^{n+r} \to \mathbb{R}^r \). We call \( X_f \) the catastrophe map of \( f \).

Let \( F \) denote the space of \( C^\infty \)-functions on \( \mathbb{R}^{n+r} \), with the Whitney \( C^\infty \)-topology. We can now state Thom's theorem.

**Theorem.** If \( r \leq 5 \), there is an open dense set \( F_\ast \subset F \) which we call generic functions. If \( f \) is generic, then

1. \( M_f \) is an \( r \)-manifold.
2. Any singularity of \( X_f \) is equivalent to one of a finite number of types called elementary catastrophes.
3. \( X_f \) is locally stable at all points of \( M_f \) with respect to small perturbations of \( f \).

The number of elementary catastrophes depends only upon \( r \), as follows:

<table>
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<tr>
<th>( r )</th>
<th>1</th>
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<th>3</th>
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<tr>
<td>elem. cats.</td>
<td>1</td>
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<td>11</td>
<td>( \infty )</td>
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Here equivalence means the following: two maps \( X: M \to N \) and \( X': M' \to N' \) are equivalent if \( \exists \) diffeomorphisms \( h, k \) such that the following diagram commutes:

\[
\begin{array}{ccc}
M & \xrightarrow{X} & N \\
\downarrow{h} & & \downarrow{k} \\
M' & \xrightarrow{X'} & N'
\end{array}
\]

Now suppose the maps \( X, X' \) have singularities at \( x, x' \) respectively. Then the singularities are equivalent if the above definition holds locally, with \( hx = x' \).
Remarks. The reason for keeping \( r \leq 5 \) is that for \( r > 5 \) the classification becomes infinite, because there are equivalence classes of singularities depending upon a continuous parameter. One can obtain a finite classification under topological equivalence, but for applications the smooth classification in low dimensions is more important. The theorem remains true when \( \mathbb{R}^{n+r} \) is replaced by a bundle over an arbitrary \( r \)-manifold, with fibre an arbitrary \( n \)-manifold.

The theorem stated above is a classification theorem: we classify the types of singularity that 'most' \( X_f \) can have. We find that if \( X_f \) has a singularity at \((x,y) \in \mathbb{R}^{n+r} \cap M_f\), and if \( \eta \) is the germ at \((x,y)\) of \( f|\mathbb{R}^n x y \), then the equivalence class of \( X_f \) at \((x,y)\) depends only upon the (right) equivalence class of \( \eta \) (Theorem 7.8). This result is hard and requires an application of the Malgrange Preparation Theorem, itself a consequence of the Division Theorem (Chapter 5), and study of the category of unfoldings of a germ \( \eta \) (Chapter 6).

To use it we have first to classify germs \( \eta \) of \( C^\infty \) functions \( \mathbb{R}^n, 0 \to \mathbb{R}, 0 \). We use two related integer invariants, determinacy and codimension, and the jacobian ideal \( \Delta(\eta) \) (the ideal spanned by \( \frac{\partial \eta}{\partial x_1}, \ldots, \frac{\partial \eta}{\partial x_n} \) in the local ring \( E \) of germs at 0 of \( C^\infty \) functions \( \mathbb{R}^n + \mathbb{R} \)). The determinacy of a germ \( \eta \) is the least integer \( k \) such that if any germ \( \xi \) has the same \( k \)-jet as \( \eta \) then \( \xi \) is right equivalent to \( \eta \). Theorem 2.9 gives necessary and sufficient conditions for \( k \)-determinacy in terms of \( \Delta \). Defining the codimension of \( \eta \) as the dimension of \( m/\hat{c} \), where \( m \) is the unique maximal ideal of \( E \), we use this theorem to show that \( \text{det} \eta - 2 \leq \text{cod} \eta \) in Lemma 3.1. If \( r \leq 5 \) and \( f \in F_\delta \) then if \( \eta = f|\mathbb{R}^n x y \), for any \( y \in \mathbb{R}^r \), we have \( \text{cod} \eta \leq r \). Hence since we can restrict to \( \text{cod} \eta \leq 5 \) we need only look at 7-determined germs in the vector space \( J^7 \) of 7-jets. We must restrict to \( r \leq 5 \), for if \( \text{cod} \eta \geq 7 \) there are equivalence classes depending upon a continuous parameter, and the definition of \( F_\delta \) ensures that if \( r = 6 \) then each of these equivalence classes contains an \( f|\mathbb{R}^n x y \) for some \( y \in \mathbb{R}^r \) and \( f \in F_\delta \).
The $7$-jets of codimension $\geq 6$ form a closed algebraic variety $\Sigma$ in $J^7$, and the partition by codimension of $J^7 - \Sigma$ forms a regular stratification (Chapters 3 and 8). We in fact use a condition implied by $a$-regularity (Definition 8.2). This is necessary to show that $F_*$ is open in $F$. That it is dense follows from Thom's transversality lemma, and transversality gives that $M_f$ is an $r$-manifold for $f \in F_*$ (Chapter 8).

The classification of germs of codimension $\leq 5$ is completed in Chapter 4 and in Chapter 7 the connection is made with catastrophe germs. Finally in Chapter 9 we show the local stability of $X_f$.

CHAPTER 2. DETERMINACY.

Definition. Suppose $C^\omega(M,Q)$ is the space of $C^\omega$ maps $M \to Q$, where $M$ and $Q$ are $C^\omega$ manifolds. If $x \in M$ and $f$ and $g \in C^\omega(M,Q)$ let $f \sim g$ if $\exists$ a neighborhood $N$ of $x$ such that $f|N = g|N$. The equivalence class $[f]$ is called a germ, the germ of $f$ at $x$.

Let $E_n$ be the set of germs at $0$ of $C^\omega$ functions $\mathbb{R}^n \to \mathbb{R}$. It is a real vector space of infinite dimension, and a ring with a 1, the 1 being the germ at $0$ of the constant function taking the value $1 \in \mathbb{R}$.

Addition, multiplication, and scalar multiplication are induced pointwise from the structure in $\mathbb{R}$.

Definition. A local ring is a commutative ring with a 1 with a unique maximal ideal.

We shall show that $E_n$ is a local ring with maximal ideal $m_n$ being the set of germs at $0$ of $C^\omega$ functions vanishing at $0$ (written as functions $\mathbb{R}^n, 0 \to \mathbb{R}, 0$).

Lemma 2.1. $m_n$ is a maximal ideal of $E_n$.

Proof. Suppose $\eta \in E_n$ and $\eta \not\in m_n$. We claim that the ideal generated by $m_n$ and $\eta$, $(m_n, \eta)_E$, is equal to $E_n$. 
Let the function \( e \in \mathfrak{n} \), i.e., \( \mathfrak{n} \) is the germ at 0 of \( e \), and choose a neighborhood \( U \) of 0 in \( \mathbb{R}^n \) such that \( e \neq 0 \) on \( U \). Then \( 1/e \) exists on \( U \). Let \( \xi \) be the germ \([1/e] \), then \( \xi e = [1/e] \cdot [e] = [1/e \cdot e] = [1] = 1 \).

Also \( \xi e \in (m, \mathfrak{n})_E \). Thus \( (m, \mathfrak{n})_E E = E \).

**Lemma 2.2.** \( m \) is the unique maximal ideal of \( E \).

**Proof.** Given \( I \subset E \), we claim \( I \subset m \). If not \( \exists \eta \in I - m \), and then as in Lemma 2.1 an inverse exists in \( E \), \( 1 = 1/\eta \cdot \eta \in I \), and so \( I = E \).

Lemma 2.1 and Lemma 2.2 show that \( E \) is a local ring.

Let \( G \subset E \) be the set of germs at 0 of \( C^\infty \) diffeomorphisms \( \mathbb{R}^n, 0 \rightarrow \mathbb{R}^n, 0 \).

\( G \) is a group with multiplication induced by composition. We shall drop suffices and use \( E, m \) and \( G \), when referring to \( E, m \) rather than \( E_s \) when \( n \neq s \), etc. Given \( \alpha_1, \ldots, \alpha_r \in E \), we let \( (\alpha_1, \ldots, \alpha_r)_E \) be the ideal generated by \( \{ \alpha_1, \ldots, \alpha_r \} \), and drop the suffix if there is no risk of confusion. Choose coordinates \( x_1, \ldots, x_n \) in \( \mathbb{R}^n \) (linear or curvilinear).

The symbol \( 'x_i' \) will be used ambiguously as:

(i) coordinate of \( x = (x_1, \ldots, x_n), x_i \in \mathbb{R} \).

(ii) function \( x_i : \mathbb{R}^n, 0 \rightarrow \mathbb{R}, 0 \).

(iii) the germ at 0 of this function in \( m \subset E \).

(iv) the k-jet of that germ (see below).

**Lemma 2.3.** \( m = (x_1, \ldots, x_n)_E \)

= ideal of \( E \) generated by the germs \( x_i \).

**Proof.** Given \( \eta \in m \), represent \( \eta \) by \( e : \mathbb{R}^n, 0 \rightarrow \mathbb{R}, 0 \). \( \forall x \in \mathbb{R}^n \),

\[
e(x) = \int_0^1 \frac{\partial e}{\partial t}(tx)dt = \int_0^1 \sum_{i=1}^n \frac{\partial e}{\partial x_i}(tx)(x)dt = \sum_{i=1}^n e_i(x)x_i(x).
\]

\( e = \sum_{i=1}^n e_i x_i \) as functions and so \( \eta = \sum_{i=1}^n e_i x_i \) as germs. Thus \( m \subset (x_1, \ldots, x_n) \).

\( (x_1, \ldots, x_n) \subset m \) because each \( x_i \in m \).
Corollary 2.4. \( m^k \) is the ideal generated by all monomials in \( x_i \) of degree \( k \).

Corollary 2.5. \( m^k \) is a finitely generated \( E \)-module.

We let \( j^k \) be the quotient \( E/m^k+1 \), and let \( j^k \) be \( m/m^k+1 \). \( j^k \) denotes the canonical projection \( E \rightarrow j^k \).

Lemma 2.6. \( j^k \) is 1) a local ring with maximal ideal \( j^k \),

2) a finite-dimensional real vector space (generated by monomials in \( \{x_i\} \), of degree \( \leq k \)).

Proof. 1) \( j^k \) is a quotient ring of \( E \) and thus is a commutative ring with a 1. There is a 1-1 correspondence between ideals:

\[
\begin{array}{cccc}
E & E/m^k+1 = j^k \\
U & U \\
I & I/m^k+1 \\
U & m^k+1 \\
\end{array}
\]

So \( j^k \) is a local ring.

2) \( j^k \) is a quotient vector space of \( E \) and is finite-dimensional.

For given \( n \in E \), the Taylor expansion at 0 is,

\[ n = n_0 + n_1 + \ldots + n_k + \rho_{k+1}, \]

where \( n_j \) is a homogeneous polynomial in \( \{x_i\} \) of degree \( j \), with coefficients the corresponding partial derivatives at 0, and \( \rho_{k+1} \in m^k+1 \).

Definition. The k-jet of \( n = j^k n = n_0 + \ldots + n_k \) = Taylor series cut off at k.

\( j^k \) and \( j^k \) are spaces of k-jets, or jet spaces.

Definition. If \( n, \xi \in E \) we say they are right equivalent (\( \sim \)) if they belong to the same \( G \)-orbit. \( n \sim \xi = \exists \gamma \in G \) such that \( n = \xi \gamma \).
Definition. If $n, \xi \in E$ we say they are $k$-equivalent ($\sim_k$) if they have the same $k$-jet. $n \sim_k \xi \iff j^k n = j^k \xi.$

Definition. $n \in E$ is $k$-determinate if $\forall \xi \in E, n \sim_k \xi \Rightarrow n \sim \xi.$ Clearly $n$ $k$-determinate $\Rightarrow n$ $i$-determinate $\forall i \geq k$. The determinacy of $n$ is the least $k$ such that $n$ is $k$-determinate. We write $\det n$.

Lemma 2.7. If $n$ is $k$-determinate then
1) $n \sim_k \xi \Rightarrow \xi \sim n$ $k$-determinate,
2) $n \sim \xi \Rightarrow \xi \sim n$ $k$-determinate.

Proof. 1) follows at once from 2), which we shall prove. Assume $n \sim \xi$, i.e. $n = \xi \gamma_1$, some $\gamma_1 \in G$. Suppose $\xi \sim \nu$, i.e. $j^k \xi = j^k \nu$, i.e. $j^k (\eta \gamma_1^{-1}) = j^k \nu.$

Then $j^k n = j^k (n \gamma_1^{-1}) = j^k (n \gamma_1^{-1}) \cdot j^k (\gamma_1) = j^k \nu \cdot j^k \gamma_1 = j^k (\nu \gamma_1).$ So $n \sim \nu \gamma_1$, which $\Rightarrow n \sim \nu \gamma_1$, i.e. $n = \nu \gamma_1 \gamma_2$ some $\gamma_2 \in G$. Then $\xi \gamma_1 = \nu \gamma_1 \gamma_2$, and $\xi = \nu \gamma_1 \gamma_2 \gamma_1^{-1}$, i.e. $\xi \sim \nu$. So 2) is proved.

Definition. If $n \in E$, choose coordinates $(x_i^1)$ for $E^m$, and let

$\Delta = \Delta(n) = (\frac{\partial n}{\partial x_1^1}, \ldots, \frac{\partial n}{\partial x_n})$. $\Delta$ is independent of the choice of coordinates. For

if $x = (\frac{\partial n}{\partial x_1^1})$ and $y = (\frac{\partial n}{\partial y_j})$, $\frac{\partial n}{\partial y_j} = \frac{\partial n}{\partial x_i} \frac{\partial x_i}{\partial y_j} \in \Delta_x$ and so $\Delta_y \subseteq \Delta_x$.

Similarly $\Delta_x \subseteq \Delta_y$, so $\Delta = \Delta_y$.

Lemma 2.8. If $n \in E - m$, and $n' = n - n(0) \in m$, then $\Delta(n) = \Delta(n')$, and $n$ is $k$-determinate $\Rightarrow n'$ is $k$-determinate.

Proof. $\Delta(n) = \Delta(n')$ is trivial. $n \sim_k \xi \Rightarrow \{n \sim_k \xi', \text{ trivially}.

Also $n = \xi \gamma \Rightarrow \{n' = \xi' \gamma, \gamma \in G
\{n(0) = \xi(0).

Thus $n \sim \xi \Rightarrow \{n' \sim \xi',
\{n(0) = \xi(0).

So from now on we shall suppose $n \in m.$
Theorem 2.9. If \( n \in M \) and \( \Delta = \Delta(n) \), then
\[
\mathfrak{m}^{k+1} \subset \mathfrak{m}^2 = \eta \text{ is } k\text{-determinate } = \mathfrak{m}^{k+1} \subset \mathfrak{m} \Delta.
\]

Proof. We shall use the following form of Nakayama's Lemma:

Lemma 2.10. If \( A \) is a local ring, \( \mathfrak{a} \) its maximal ideal, and \( M, N \) are \( A \)-modules (contained in some larger \( A \)-module) with \( M \) finitely generated over \( A \), then \( M \subset N + \mathfrak{a} M = M \subset N \).

Sublemma \( \lambda \in A \), \( \lambda \notin \mathfrak{a} = \lambda^{-1} \notin A \).

Proof. \( \lambda A \) is an ideal \( \notin \mathfrak{a} \). So \( \lambda A = A \ni 1 \), \( \exists \mu \) such that \( \lambda \mu = 1 \).

Proof of Lemma 2.10. We shall first prove the special case of \( N = 0 \), i.e.,
\( M \subset \mathfrak{a} M = M = 0 \). Let \( v_1, \ldots, v_r \) generate \( M \). \( v_i \in \mathfrak{a} M \) by hypothesis,
so
\[
v_i = \sum_{j=1}^{r} \lambda_{ij} v_j (\lambda_{ij} \in A)
\]
or
\[
\sum_{j=1}^{r} (\delta_{ij} - \lambda_{ij}) v_j = 0, \text{ i.e. } (I - \Lambda) v = 0, \text{ where } \Lambda \text{ is an } (r \times r) \text{-matrix}
\]
and \( \Lambda = (\lambda_{ij}) \). The determinant \( |I - \Lambda| = 1 + \lambda \), some \( \lambda \in A \). Now
\[
1 + \lambda \notin \mathfrak{a}, \text{ else } 1 \notin \mathfrak{a} \text{ and } A = A.
\]
So \( (1+\lambda)^{-1} \) exists by the sublemma. Then \( (I-\Lambda)^{-1} \) exists, giving \( v = 0 \) and \( M = 0 \).

To prove the general case consider the quotient by \( N, (M+N)/N \subset N/N + (\mathfrak{a}M+N)/N \). We claim the R.H.S. = \( \mathfrak{a}(M+N)/N \). (*) Then by the special case,
\( (M+N)/N = 0 \), giving \( M \subset N \). Q.E.D.

The \( A \)-module structure on \( (M+N)/N \) is induced by that on \( M+N \) by \( \lambda(v+N) = \lambda v + N \).

\[
\mathfrak{a}(M+N)/N = \{\lambda(v+N) : \lambda \in A, v \in N\}
\]
\[
= \{\lambda v+N : \lambda \in A, v \in M\}
\]
\[
= (\mathfrak{a}M+N)/N, \text{ proving (*)}.\]

Continuing the proof of Theorem 2.9, we assume \( \mathfrak{m}^{k+1} \subset \mathfrak{m}^2 \), and must show that
\( n_k \sim \xi \Rightarrow n_k \sim \xi \). The idea of the proof is to change \( n \) into \( \xi \) continuously
with the assumption \( n_k \sim \xi \). Let \( \phi \) denote the germ at \( 0 \times \mathbb{R} \) of a function
\( \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R} \) given by \( \phi(x, t) = (1-t)n(x) + t\xi(x), x \in \mathbb{R}^n, t \in \mathbb{R} \). Let
\[ \phi^t(x) = \phi(x,t) = \begin{cases} \eta(x) & t = 0 \\ \xi(x) & t = 1. \end{cases} \]

**Lemma 1.** Fixing \( t_0, 0 \leq t_0 \leq 1 \), \( \exists \) a family \( \Gamma^t \in G \) defined for \( t \) in a neighborhood of \( t_0 \) in \( \mathbb{R} \) such that 1) \( \Gamma^t 0 = \text{identity} \)

2) \( \phi^1 = \phi^{t_0} \).

**Lemma 1** will give \( \eta \sim \xi \): Using compactness and connectedness of \( [0,1] \), cover by a finite number of neighborhoods as in Lemma 1, then pick \( \{ t_i \} \) in the overlaps, and construct \( \gamma \) satisfying \( \eta = \xi \gamma \) by a finite composition of \( \{ \Gamma^t \} \), i.e. \( \eta = \phi^0 \sim \ldots \sim \phi^1 = \xi \).

**Lemma 2.** For \( 0 \leq t_0 \leq 1 \), \( \exists \) a germ \( \Gamma \) at \( (p,t_0) \) of \( \mathcal{C}^\infty \) maps \( \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n \) satisfying

(a) \( \Gamma(x,t_0) = x \),

(b) \( \Gamma(0,t) = 0 \),

(c) \( \phi(\Gamma(x,t),t) = \phi(x,t_0) \),

for all \( (x,t) \) in some neighborhood of \( (0,t_0) \).

**Lemma 2** will give **Lemma 1**: Define \( \Gamma^t(x) = \Gamma(x,t) \) from a neighborhood of \( 0 \) in \( \mathbb{R}^n \) to \( \mathbb{R}^n \); \( \Gamma^t \) is a germ of \( \mathcal{C}^\infty \) maps \( \mathbb{R}^n,0 \rightarrow \mathbb{R}^n,0 \) by (b); \( \Gamma^{t_0} \) is the identity by (a). \( \mathcal{C}^\infty \) diffeomorphisms are open in the space of \( \mathcal{C}^\infty \) maps \( \mathbb{R}^n,0 \rightarrow \mathbb{R}^n,0 \) (because they correspond to maps with Jacobian of maximal rank, i.e. to the non-vanishing of a certain determinant), and so \( \exists \) a neighborhood of \( t_0 \) such that \( \Gamma^t \) is a germ of diffeomorphisms for \( t \) in that neighborhood, i.e. \( \Gamma^t \in G \).

**Lemma 3.** (c) in Lemma 2 is equivalent to,

\[ \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}(\Gamma(x,t),t) \frac{\partial \Gamma}{\partial t}(x,t) + \frac{\partial \phi}{\partial t}(\Gamma(x,t),t) = 0. \]

(c) \( \Rightarrow \) (c') by differentiation with respect to \( t \).

(c') \( \Rightarrow \) (c):

\[ 0 = \int_{t_0}^{t} \frac{d}{dt}(\phi(\Gamma(x,t),t)) = \phi(\Gamma(x,t),t) = \phi(\Gamma(x,t_0),t_0) \]

\[ = \phi(\Gamma(x,t),t) - \phi(x,t_0) \] by (a) in Lemma 2.

Thus we have (c).

**Lemma 4.** For \( 0 \leq t_0 \leq 1 \), \( \exists \) a germ \( \Gamma \) at \( (0,t_0) \) of a \( \mathcal{C}^\infty \) map \( \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n \)
satisfying (d) \( \psi(0,t) = 0 \),

\[
(a) \quad \sum_{i=1}^{\tilde{n}} \frac{\partial \Phi}{\partial x_i} (x,t) \psi_i(x,t) + \frac{\partial \tilde{\psi}}{\partial t} (x,t) = 0,
\]

for all \((x,t)\) in some neighborhood of \((0,t_0)\).

**Lemma 4 = Lemmas 3 and 2:** The existence theorem for ordinary differential equations gives a solution \( \Gamma(x,t) \) of \( \frac{\partial \Gamma}{\partial t} = \psi(\Gamma,t) \), with initial condition \( \Gamma(x,t_0) = x \) (i.e. (a) of Lemma 2). In (e) put \( x = \Gamma(x,t) \) to give (c'). (d) \( \Gamma = 0 \) is a solution, i.e. \( \Gamma(0,t) = 0 \) for all \( t \) in some neighborhood of \( t_0 \), which is (b).

Let \( A \) denote the ring of germs at \((0,t_0)\) of \( C^\infty \) functions \( \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \). Projection \( \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) induces an embedding \( \mathcal{E} \subset A \) by composition.

Let \( \Omega = \left( \frac{\partial \Phi}{\partial x_1}, \ldots, \frac{\partial \Phi}{\partial x_n} \right)_A \).

**Lemma 5.** \( m^{k+1} \subset m^2 \Delta = m^{k+1} \subset m^2 \mathcal{E} \).

**Lemma 5 = Lemma 4 as follows:**

\[
\frac{\partial \tilde{\psi}}{\partial t} = \tilde{\xi} - \eta \in m^{k+1} \quad (\eta \sim \xi) \quad c m^2 \mathcal{E}.
\]

Thus \( \frac{\partial \tilde{\psi}}{\partial t} = \sum_{j} \omega_j \mu_j \), \( \omega_j \in m^2 \), \( \mu_j \in \Omega \). (finite sum)

\[
= \sum_{j} \omega_j a_{ij} \frac{\partial \Phi}{\partial x_i}, \text{ where } \omega_j = \sum_{i} a_{ij} \frac{\partial \Phi}{\partial x_i}, a_{ij} \in A.
\]

\[
= -\tilde{\psi} \frac{\partial \Phi}{\partial x_i}, \text{ setting } \tilde{\psi} = \sum_{a \geq j} a_{ij} \in A.
\]

This gives (e).

Now \( \mu_j = \mu_j(x) \) and \( a_{ij} = a_{ij}(x,t) \). \( \psi = \{ \psi \} \) is a germ at \((0,t_0)\) of a map \( \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \), and \( \psi(0,t) = 0 \) as each \( \mu_j(0) = 0 \), so (d) holds for \( \psi \).

**Proof of Lemma 5.** (and hence the completion of the proof of a sufficient condition for \( k \)-determinacy)

\[
\frac{\partial \Phi}{\partial x_i} = \frac{3n}{3x_i} + t \frac{3}{3x_i} (\xi - \eta) \quad (t \in A, \xi - \eta \in m^{k+1})
\]

i.e.

\[
\frac{\partial \tilde{\psi}}{\partial x_i} \in \frac{\partial \Phi}{\partial x_i} + A m^k c \Omega + A m^k.
\]
So $\Delta \subset \Omega + Am^k$.

Denote the maximal ideal of $A$ by $a$, i.e. those germs vanishing at $(0,t_0)$. Then $m \subset a$. Now $Am^{k+1} \subset Am^2$ (hypothesis)

\[ c Am^2 (\Omega + Am^k) = m^2 \Omega + Am^{k+2} \subset m^2 \Omega + aAm^{k+1}. \]

Now apply Nakayama's Lemma 2.10 for $A$, $a$, $M$, $N$ where $M = Am^{k+1}$ is finitely generated by monomials in $(x_1)$ of degree $k + 1$ by Corollary 2.4, and 

$N = m^2 \Omega$. This gives $Am^{k+1} \subset m^2 \Omega$. In particular $m^{k+1} \subset m^2 \Omega$, completing

Lemma 5.

Now we prove that $m^{k+1} \subset mA$ is a necessary condition of $k$-determinacy. 

\[ \exists \text{ a natural map } \quad m \xrightarrow{\pi} j^{k+1} \xrightarrow{j^k} \pi = j^{k+1}/m. \]

\[ \eta \xrightarrow{j^{k+1} \eta} j^k \eta \]

Let $P = \{ \xi \in m: \eta \sim \xi \}$, and $Q = \{ \xi \in m: \eta \sim \xi \} = \text{orbit } \eta G$.

Assuming that $\eta$ is $k$-determinate then $P \subset Q$, so that $\pi P \subset \pi Q$. (\textbf{*})

$P = n + m^{k+1}$, so $\pi P = z + m^{k+1}/m^{k+2} = z + m^{k+1}$. (Letting $z = j^{k+1} \eta$). The 

tangent plane to $\pi P$ at $z$, $T_z(\pi P) = \pi m^{k+1}$.

Let $G^k$ denote the $k$-jets of germs belonging to $G$; $G^k$ is a finite-

dimensional Lie group. Now $j^{k+1}(\eta \gamma) = j^{k+1}(\eta)j^{k+1}(\gamma)$ for $\gamma \in G$, i.e. $\pi$ is 
equivariant with respect to $G$, $G^{k+1}$. So $\pi Q = \pi (\eta G) = zG^{k+1}$, an orbit under

\[ \text{a Lie group, and hence is a manifold. In particular } T_z(\pi Q) \text{ exists.} \]

\textbf{Lemma 2.11.} $T_z(\pi Q) = \pi (mA)$.

Now (\textbf{*}) gives $T_z(\pi P) \subset T_z(\pi Q)$. Then Lemma 2.11 gives $\pi m^{k+1} \subset \pi(mA)$, 

i.e. $m^{k+1} \subset mA + m^{k+2}$. Apply Nakayama's Lemma 2.10 with $A = E$, $a = m$, 

$M = m^{k+1}$, $N = mA$, using Lemmas 2.1, 2.2 and Corollary 2.4, to yield $m^{k+1} \subset mA$.

\textbf{Proof (of 2.11).} Suppose $\gamma \in G$. As $\mathbb{R}^n$ is additive we can write $\gamma = 1 + \delta$, 

where $1$ is the germ of the identity map, and $\delta$ is the germ at $0$ of a $C^\infty$ 
map $\mathbb{R}^n, 0 \to \mathbb{R}^n, 0$. Join $1$ to $\gamma$ by a continuous path of map-germs, $\gamma^t = 1 + t\delta$, 


0 ≤ t ≤ 1. When t = 0 or 1, γ^t is a diffeomorphism-germ. Diffeomorphisms are open in the space of C maps, and so ∃ t_o > 0 such that γ^t ∈ G, o ≤ t ≤ t_o.

Then {γ^t} is a path in G starting at l,
{ηγ^t} is a path in Q starting at n,
{πηγ^t} is a path in πQ starting at z.

The tangent to the path at t = 0 is given by

\[ \frac{d}{dt} (πηγ^t)_{t=0} = π \left[ \frac{d}{dt} η(1+tδ) \right]_{t=0} \]

Now δ = (δ_1, ..., δ_n) where δ_i is a germ of a C^∞ function R^n, 0 → R, 0.

(Remember m is a ring and a vector space so we can define differentiation).

So \[ \frac{d}{dt} (πηγ^t)_{t=0} = π \left[ \sum_{i=1}^{n} \frac{δn}{δx^i_1} (1+tδ) \cdot δ_1 \right]_{t=0} \]

= π \left[ \sum_{i=1}^{n} \frac{δn}{δx^i_1} \cdot δ_1 \right]

∈ π(mΔ). (δ_1 ∈ m, \frac{δn}{δx^i_1} ∈ Δ).

This tangent is in T_z(πQ); moreover any tangent in T_z(πQ) arises from a path in πQ, so from a path in G^{k+1}, so from a path in G starting at l.

Allowing δ to vary in G gives all such paths. Hence T_z(πQ) ⊂ π(mΔ).

Given ξ ∈ mΔ, we can write ξ = \[ \sum_{i=1}^{n} \frac{δn}{δx^i_1} δ_i \], δ_i ∈ m. The δ_i assemble into δ determining a path in G.

Hence π(mΔ) ⊂ T_z(πQ), and we have T_z(πQ) = π(mΔ).

Corollary 2.12. n is finitely determinate ⇒ m^k ⊂ Δ, some k.

Proof. '⇒' follows as n, k-determinate ⇒ m^{k+1} ⊂ mΔ ⊂ Δ.

'⇐'. m^k ⊂ Δ, so m^{k+2} ⊂ m^2Δ, and n is (k+1)-determinate.

Corollary 2.13. n ∈ m - m^2 = n is 1-determinate.

Proof. n'(0) ≠ 0, i.e. some \[ \frac{δn}{δx^i_1} \] m, so Δ = E.

m^2Δ = m^2 and then n is 1-determinate by Theorem 2.9.

So we may effectively assume n ∈ m^2 from now on.
Definition. With chosen coordinates \( \{x_i\} \), the essence of \( \eta \) (with respect to this coordinate system) is the least \( k \) for which \( j^k \eta \) contains all the \( x_i \). We write \( \text{ess } \eta \).

Corollary 2.14. \( \det \eta \geq \text{ess } \eta \) (with respect to any coordinate system).

Proof. \( k < \text{ess } \eta = j^k \eta \) does not contain \( x_i \); some \( i \). Let \( \xi = j^k \eta \) as a germ. So \( \Delta(\xi) \) any power of \( x_i \),
\[ \not\exists \, m^k, \forall k. \]

Thus \( \xi \) is not finitely determinate (Corollary 2.12). But \( n^k \sim \xi \), so if \( \eta \) were \( k \)-determinate, Lemma 2.7 would give a contradiction, i.e. \( k < \det \eta \).

Counterexample 1. Let \( \eta = x^{k+1}, n = 1. \) Then \( \Delta = (x^k) = m^k \), and \( m \Delta = m^{k+1} \).

Det \( \eta \geq \text{ess } \eta = k + 1 \), (Corollary 2.14). \( \eta \) is not \( k \)-determinate and so the implication \( n \Rightarrow \text{determinate} \Rightarrow m^{k+1} \subset m \Delta \) in Theorem 2.9 is not reversible.

Counterexample 2. D. Siersma found \( \eta = x^\frac{3}{3} + xy^2, n = 2. \) Here \( \Delta = (x^2+xy^3, xy^2) \).
\[ m^2 = (x,y,xy^2), \text{ so } m^2 \Delta = x^4 + x^2y, x^3y, x^5, x^2y^2 + y^2, x^3y^2, x^2y^3, xy^4, \]
\[ \gg x^2y^5, x^3y^3 + y^6, x^2y^4, x^3y^2, x^4y, x^5y, x^6, \]
\[ \gg m^6 \text{ (5-determinate)} \]
\[ \not\exists \, y^5, \not\exists m^5. \]

Det \( \eta \geq \text{ess } \eta = 4. \) By computation it is 4-determinate, and so the implication \( m^{k+1} \subset m^2 \Delta = \eta \) \( k \)-determinate is not reversible.

CHAPTER 3. CODIMENSION.

Remember that we work in \( m^2 \) using Corollary 2.13.

Definition. The codimension of \( \eta = \dim_R m/\Delta(\eta) \). We write \( \text{cod } \eta \). The definition makes sense because if \( \eta \in m^2 \), each \( \frac{\partial \eta}{\partial x_i} \in m \) and so \( \Delta(\eta) \subset m \).

If \( \eta \) were in \( m - m^2 \), \( \Delta(\eta) = E \) and by convention \( \text{cod } \eta = 0 \).

Lemma 3.1. Either both \( \text{cod } \eta \) and \( \det \eta \) are infinite, or both are finite and \( \det \eta - 2 \leq \text{cod } \eta \).
Proof. \(\eta \in m^2 = \Delta \subset m.\) \((\Delta = \Delta(\eta))\)

We have a descending sequence of vector subspaces of \(m,\)

\[ m = m + \Delta \supset m^2 + \Delta \supset m^3 + \Delta \supset ... \supset m^{k+1} + \Delta \supset ... \quad (3.2) \]

Either (i) \(\exists k \text{ such that } m^{k-1} + \Delta = m^k + \Delta,\) and \(k\) is the least such, or

(ii) \(\not\exists\) such a \(k.\)

Case (i): \(m^{k-1} \subset m^k + \Delta,\) and we may apply Nakayama's Lemma 2.10 yielding

\(m^{k-1} \subset \Delta,\) so \(m^{k+1} \subset m^2 \Delta.\) By Theorem 2.9 \(\eta\) is \(k\)-determinate, so \(\det \eta \leq k,\)

i.e. \(\det \eta\) is finite. Now \(\text{cod } \eta = \dim m/\Delta \leq \dim m/m^{k-1},\) and \(m/m^{k-1}\) is

finitely generated, by monomials in \(\{x_i\}\) of degree \(\geq 1\) and \(\neq k - 1.\) So

\(\text{cod } \eta\) is finite. Now \(m^{k-1} + \Delta = \Delta,\) and so the above sequence (3.2) descends

strictly to the \(m^{k-1} + \Delta\) term, and we have,

\[
m/\Delta \supset (m^2 + \Delta)/\Delta \supset ... \supset (m^{k-1} + \Delta)/\Delta = 0
\]

\[\longrightarrow \text{k-2 steps} \quad \longrightarrow\]

Hence \(\text{cod } \eta = \dim m/\Delta \geq k - 2 \geq \det \eta - 2,\) as required.

Case (ii): If \(\det \eta\) is finite, then \(m^k \subset \Delta\) for some \(k\) (Corollary 2.12).

Then \(m^k + \Delta = m^{k+1} + \Delta,\) and we are in Case (i). So \(\det \eta\) is infinite.

\(m/\Delta \supset (m^2 + \Delta)/\Delta \supset ...\) is a strictly decreasing sequence and so \(\text{cod } \eta\)

\((= \dim m/\Delta)\) is infinity.

Let \(\Gamma_c = \{\eta \in m^2 : \text{cod } \eta = c\}\) (a 'c-stratum' of \(m^2\)), and let

\(\Omega_c = \{\eta \in m^2 : \text{cod } \eta \leq c\}\) and \(\Sigma_c = \{\eta \in m^2 : \text{cod } \eta \geq c\},\) so that

\[m^2 = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_c \cup \ldots \cup \Gamma_\infty.\]

(disjoint union)

Let \(\Gamma^k_c, \Omega^k_c, \Sigma^k_c\) be the images of \(\Gamma_c, \Omega_c, \Sigma_c\) under the map \(\pi: m^2 \to m^k\)

\((\pi = j^k/m^2),\) where \(j^k\) is defined as \(m^2/m^{k+1}\) just as \(j^k\) is \(m^k/m^{k+1}\).

Theorem 3.3. If \(0 \leq c \leq k - 2,\) then \(I^k_c = \Omega^k_c \cup \Sigma^k_c+1\) (disjoint union), and

\(\Sigma^k_{c+1}\) is a (closed) real algebraic variety.

Remark. Both statements are false for \(c > k - 2.\)

Lemma 3.4. \(\dim E/m^{k+1} = \frac{(n+k)!}{n!k!},\) \(\forall n, k \geq 0.\)
Proof. If \( n = 0 \), \( E = \mathbb{R} \), \( m = 0 \); L.H.S. = 1 = R.H.S. \( \forall k \). If \( k = 0 \), \( E/m = \mathbb{R} \);
L.H.S. = 1 = R.H.S. \( \forall n \). Use induction on \( n + k \).

Then \( E/m^{k+1} = \text{polynomials of degree } \leq k \text{ in } x_1, \ldots, x_n \)
= (polynomials of degree \( k \) in \( x_1, \ldots, x_{n-1} \))
+ \( x_n \) (polynomials of degree \( k - 1 \) in \( x_1, \ldots, x_n \))

So \( \dim E/m^{k+1} = \frac{(n+k-1)!}{(n-k)!k!} + \frac{(n+k-1)!}{n!(k-1)!} \) (by induction)
= \( \frac{(n+k)!}{n!k!} \)

Proof of Theorem 3.3. We define an invariant \( \tau(z) \) for \( z \in \mathbb{I}^k = m^2/m^{k+1} \).
Choose \( \eta \in \mathbb{R}^{\frac{1}{2}} \). \( n \in m^2 \), so \( \Lambda(\eta) = \Lambda \subset m \). Define \( \tau(z) = \dim m/(\Lambda + m^k) \).

We claim that \( \tau(z) \) is independent of the choice of \( \eta \). Let \( \eta' \) be another choice, \( \Lambda(\eta') = \Lambda' \). Then \( n - n' \in m^{k+1} \), so \( \frac{\partial n}{\partial x_i} - \frac{\partial n'}{\partial x_i} \in m^k \), and \( \frac{\partial n}{\partial x_i} \in \Lambda' + m^k \). Hence \( \Lambda + m^k \subset \Lambda' + m^k \).

Hence \( \Lambda + m^k = \Lambda' + m^k \) and \( \tau(z) \) is well defined.

We claim that,

\[
\begin{align*}
(1) & \quad \tau(z) \leq c \Rightarrow \text{cod } \eta = \tau(z), \quad \text{so } z \in \mathbb{I}^k_c \\
(2) & \quad \tau(z) > c \Rightarrow \text{cod } \eta > c, \quad \text{so } z \in \mathbb{I}^k_{c+1}. \quad \text{(cod } \eta \text{ perhaps } \neq \tau(z))
\end{align*}
\]

Because (1) and (2) are disjoint, \( \mathbb{I}^k \) is the disjoint union of \( \mathbb{I}^k_c \) and \( \mathbb{I}^k_{c+1} \),
once we have shown (1) and (2) hold.

We have

(Lemma 3.4)

Note that \( \tau(z) \) is finite,
although \( \text{cod } \eta \) may be infinite.
Case (ii): \( \text{cod } n \geq \tau(z) \) (from the diagram) \( \implies \) Thus (ii) holds.
   \( > c \) (hypothesis of (ii))

Case (i): \( k - 2 \geq c \) (hypothesis of the theorem)
   \( \geq \tau(z) \) (hypothesis of (i))

We have a sequence,

\[
0 = m/m = m/\Delta + m \leftarrow m/\Delta + m^2 \leftarrow \ldots \leftarrow m/\Delta + m^k
\]

\( k - 2 \geq \tau(z) = \dim m/\Delta + m^k \), so one step must collapse, i.e. \( \Delta + m^{i-1} = \Delta + m^i \), for some \( i \leq k \), i.e. \( m^{i-1} \subset \Delta + m^i \). Nakayama's Lemma 2.10 = \( (m^k)_m \subset \Delta \).

Therefore \( \Delta + m^k = \Delta \), and so \( \kappa(z) = 0 \) where \( \kappa(z) = \dim (\Delta + m^k)/\Delta \) as in the diagram. We observe that \( \tau(z) = \text{cod } n \), and so (i) holds.

Now \( \sigma(z) = \frac{(n+k-1)!}{n!(k-1)!} - 1 - \tau(z) \), from the diagram. If \( \tau(z) > c \), then

\[
c(z) < \frac{(n+k-1)!}{n!(k-1)!} - 1 - c = K, \text{ say.}
\]

\( \Sigma_{c+1}^k = \{ z \in I^k : \text{cod } n > c \} \)

\( = \{ z \in I^k : \tau(z) > c \} \) \( (c \leq k - 2) \)

\( = \{ z \in I^k : \sigma(z) < K \} \), which we shall show is an algebraic variety (real).

If \( x_1, \ldots, x_n \) are coordinates for \( \mathbb{R}^n \), let the monomials of degree \( \leq k \) in \( \{ x_i \} \) be \( \{ x_j^\beta \} \) as below:

\[
X_1^1 X_2 X_3 \ldots X_{n+1}^1 X_{n+2} X_{n+3} \ldots X_\beta^1, \quad (\beta = \frac{(n+k)!}{n!k!})
\]

\[
l x_1 x_2 \ldots x_n x_1^2 x_1x_2^2 \ldots x_n^k
\]

Now \( J^k \) is the space of polynomials in \( \{ x_i \} \) of degree \( \leq k \) with coefficients in \( \mathbb{R} \) and no constant term. \( z \in I^k \) can be written \( z = \sum_{j=n+2}^\infty \alpha_j x_j \) (\( \alpha_j \in \mathbb{R} \)).

Because \( \frac{\partial z}{\partial x_i} \) is a polynomial of degree \( k - 1 \) with no constant term it belongs to \( J^{k-1} \), so \( \frac{\partial z}{\partial x_i} = \sum_{j=2}^\infty a_{ij} x_j \), \( ([z] = \frac{(n+k-1)!}{n!(k-1)!}) \), where each \( a_{ij} \) is an integer multiplied by some \( a_k \).

Just as \( \Delta \) is the ideal of \( E \) generated by \( \{ \frac{\partial z}{\partial x_1} \} \), so \( (\Delta + m^k)/m^k \) is the ideal of \( J^{k-1} \) generated by \( \{ \frac{\partial z}{\partial x_1} \} \). Now \( J^k \) as a vector space has a basis \( X_2, \ldots, X_\beta \). \( (\Delta + m^k)/m^k \) is now the vector subspace of \( J^{k-1} \) spanned by
\[
\left( \frac{\partial z_j}{\partial x_1} \right). \text{ Let each } \frac{\partial z_i}{\partial x_1} x_j = \sum_{k=2}^{\infty} a_{ij,k} x_k, \text{ where each } a_{ij,k} \text{ is some } a_{ilm}.
\]

We put \( M = \text{ the matrix } (a_{ij,k}) \)

\[
\text{the coordinates of vectors spanning } (\Lambda^+m)/m^k.
\]

Now \( \sigma(z) < K = \dim (\Lambda^+m^k)/m^k \)

\[
\text{rank of } M < K
\]

\[
\text{all } K \text{-minors of } M \text{ vanish.}
\]

And so \( \Sigma_{c+1}^k \) is given by polynomials in the \( \{a_{ij,k}\} \), k.e. by polynomials in the \( \{a_i\} \), each \( a_i \in \mathbb{R} \). Hence \( \Sigma_{c+1}^k \) is a real algebraic variety in the real vector space \( J^k \) of dimension \( \frac{(n+k)!}{n!k!} - 1 \), itself a subspace of \( J^k \)

which is \( \frac{(n+k)!}{n!k!} - 1 \)-dimensional.

**Corollary.** \( I^k \) is the disjoint union \( \bigcup_{0}^{k} I_1^k \bigcup \ldots \bigcup I_{k-2}^k \bigcup \Sigma_{c+1}^k \), and each \( I_i^k \) is the difference \( \Sigma_{c+1}^k - \Sigma_i^k \) between 2 algebraic varieties.

Recall that the map \( \eta: m^2 \to I^k \) is equivariant with respect to \( G, G^k \);

also the image of the orbit \( nG \) is \( zG^k \), a submanifold of \( I^k \), as in the proof of Theorem 2.9.

**Theorem 3.7.** Let \( \eta \in m^2 \) and \( \text{cod } \eta = c \) where \( 0 \leq c \leq k - 2 \). Then \( zG^k \) is a submanifold of \( I^k \) of codimension \( c \).

**Proof.** By Lemma 2.11, \( T_z (zG^k) = \pi(m \Delta) \). \( (\Delta = \Delta(\eta)) \)

By Lemma 3.1, \( \det n - 2 \leq \text{cod } \eta = c \leq k - 2 \), by the hypotheses. So \( \det n \leq k \), i.e. \( \eta \) is \( k \)-determinate. By Theorem 2.9, \( m^{k+1} \subset m \Delta \).

The codimension of \( zG^k \) in \( I^k = \dim I^k - \dim \pi(m \Delta) \)

\[
= \dim m^2/m^{k+1} - \dim m \Delta/m^{k+1}
\]

\[
= \dim m^2/m \Delta.
\]

Now \( m/m \Delta = m/m^2 + m^2/m \Delta \), so \( \dim m^2/m \Delta = \dim m/m \Delta - \dim m/m^2 \). So the codimension of \( zG^k \) in \( I^k = \dim m/m \Delta - \dim m/m^2 \)

\[
= \dim m/\Delta + \dim \Delta/m \Delta - \dim m/m^2
\]

\[
= c + n - n.
\]
using the following lemma.

Lemma 3.8. If \( \eta \in m^n \) and \( \text{cod } \eta < \infty \), then \( \dim \Delta/m\Delta = n \).

This completes the proof of the theorem.

Proof of Lemma 3.8. Since \( \Delta \) is the ideal of \( E \) generated by \( \left\{ \frac{\partial \eta}{\partial x_i} \right\} \), every \( \xi \in \Delta \) can be written as \( \xi = \sum_{i=1}^{n} a_i \frac{\partial \eta}{\partial x_i} \) where \( a_i \in E, a_i = a_i + \mu_i, \mu_i \in m \), \( a_i \in R \). Then \( \xi = \sum_{i=1}^{n} a_i \frac{\partial \eta}{\partial x_i} \mod m\Delta \). So \( \left\{ \frac{\partial \eta}{\partial x_i} \right\} \) span \( \Delta \) over \( R \mod m\Delta \), and \( \dim \Delta/m\Delta \leq n \). It remains to prove \( \dim \Delta/m\Delta \geq n \).

Suppose not, i.e. that \( \dim \Delta/m\Delta < n \). Then \( \left\{ \frac{\partial \eta}{\partial x_i} \right\} \) are linearly dependent \( \mod m\Delta \). \( \exists a_1, \ldots, a_n \in R, \) not all zero, such that

\[
\sum_{i=1}^{n} a_i \frac{\partial \eta}{\partial x_i} = \sum_{i=1}^{n} \mu_i \frac{\partial \eta}{\partial x_i} \in m\Delta, \text{ some } \{\mu_i\} \in m.
\]

Then \( x\eta = \sum_{i=1}^{n} (a_i - \mu_i) \frac{\partial \eta}{\partial x_i} = 0 \) where \( X = \sum_{i=1}^{n} (a_i - \mu_i) \frac{\partial}{\partial x_i} \) is a vector field on a neighborhood of \( 0 \) in \( R^n \). \( X \) is nonzero at \( 0 \) because \( \{\mu_i\} \in m \) and so vanish at \( 0 \) and \( \{a_i\} \) are not all zero.

Change local coordinates so that \( X = \frac{\partial}{\partial y_1} \) where \( \{y_i\} \) are the new coordinates. Then \( \frac{\partial \eta}{\partial y_1} = 0 \). So \( \eta = \eta(y_2, \ldots, y_n) \). \( \text{Ess } \eta = \infty \) with respect to \( \{y_i\} \). But \( \det \eta \geq \text{ess } \eta, \) by Corollary 2.14. By Lemma 3.1., \( \text{cod } \eta = \infty \).

We have shown that \( \dim \Delta/m\Delta = n \).

Theorem 3.7 justifies the notation \( \text{cod } \eta \), as an abbreviation for codimension.
CHAPTER 4. CLASSIFICATION

Key: \[\square\] means orbit is in \(\mathbb{R}^7_6\)

\(\bigcirc\) round a polynomial \(z\)

signifies its orbit \(zG^7\).

+ means + and - give 2

distinct orbits

\[Q_p = \{\eta: \eta \sim x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_n^2\}.\]

codim at top = codimension of

stratum in \(I^7\).

codim at bottom = codimension of

stratum in \(J^7\).

Orbits involving one essential

variable are called cuspoinds,

e.g. \(x^5\).

Orbits involving two

essential variables are

called umbilics,

e.g. \(x^3 + y^4\).


Diagram 4.1. Classification of \(I^7\) (and \(J^7\)).
This chapter will complete the above classification of $I^7$ as in Diagram 4.1. Supposing we already have our classification it follows that:

**Theorem 4.2.** In $I^7 - T^7$ there are exactly $16n - 7$ orbits under $G^7$.

**Proof.** We merely add the orbits in Diagram 4.1.

- **Stable singularities**: $n + 1$
- **Cuspsoids**: $7n$
- **Umbilics**: $8(n-1)$
- **Total**: $16n - 7$

**Corollary 4.3.** If $0 \leq c \leq 5$, $\Gamma^7_c$ is a submanifold of $I^7$ of codimension $c$.

**Proof.** $\Gamma^7_c$ is the union of a finite number of orbits by Theorem 4.2. By Theorem 3.7, each of these is a submanifold of codimension $c$.

We note that our classification also gives that $T^7_6$ is the union of a finite number of parts each of codimension $\geq 6$ in $I^7$. (See Diagram 4.1)

**Theorem 4.4.** $I^7 = \Gamma^7_0 \cup \Gamma^7_1 \cup \Gamma^7_2 \cup \Gamma^7_3 \cup \Gamma^7_4 \cup \Gamma^7_5 \cup \Gamma^7_6$ (disjoint) and each $\Gamma^7_c$ is of codimension $c$ in $I^7$ and $T^7_6$ is of codimension 6 in $I^7$.

We shall now proceed with the classification.

**Lemma 4.5.** Let $\pi : m + J^1 = m/m^2 \cong \mathbb{R}^n$, where $\pi = j^1|m$. Then $\pi^{-1}(J^1-0)$ is the orbit of regular germs.

**Proof.** Given $\eta \in m$, if $j^1(\eta) \neq 0$, then $\eta = \eta_1 + \text{higher terms}$, where $\eta_1$ is a nonzero linear term. Then $\Delta = E$, as in Lemma 2.2. $m^2\Delta = m^2$ and by Theorem 2.9, $\eta$ is 1-determinate. So $\eta \sim \eta_1 = \sum_{i=1}^{n} a_i x_i = y_j$ for some linear change of coordinates. Thus $\eta \sim x_1$ by the linear map sending $y_j$ to $x_1$, and $\eta \in$ orbit of $x_1$ which as a function is regular.

These regular germs are precisely those with no singularity, or rather are not singularities. We observe that $J^7 = J^1 \times J^7/J^1$. Lemma 4.5 tells us that $(J^1-0) \times J^7/J^1$ is the regular orbit. The remainder, $0 \times J^7/J^1 = m^2/m^8 = I^7$, are the irregular orbits which we must classify.

$\eta \in m^2 = \eta = q + \text{higher terms}$, where $q$ is a quadratic form in
\( \{x_i\}, \text{ say} \)

\[ q(x_1, \ldots, x_n) = \sum_{i,j} a_{ij} x_i x_j \quad (a_{ij} = a_{ji}) \]

Write \( A \) for the matrix \((a_{ij})\), which is symmetric, and define \( \text{rank } \eta \), the Hermitian rank of \( \eta \) or of \( j^k \eta \) \((k \geq 2)\) to be rank \( A \). Then \( 0 \leq \text{rank } \eta \leq n \).

**Lemma 4.6.** Let \( \text{rank } \eta = \rho \). By an elementary theorem of linear algebra there is a linear change of coordinates such that \( q = y_1^2 + y_2^2 + \ldots + y_\rho^2 - y_{\rho+1}^2 - \ldots - y_n^2 \).

**Corollary 4.7.** \( \eta \sim (x_1^2 + \ldots - x_\rho^2) + \text{higher terms, if } \text{rank } \eta = \rho \).

Let \( Q_\rho = \{ q : \text{Hermitian rank of } q = \rho \} \), in \( I^2 = \mathbb{M}^2/\mathbb{M}^3 \) which is diffeomorphic to \( \mathbb{R}^{m(n+1)} \) because it is the linear space of all quadratic forms with coordinates \( \{a_{ij}\}, i \leq j \). Then \( I^2 = Q_n \cup Q_{n-1} \cup \ldots \cup Q_0 \).

**Lemma 4.8.** \( Q_{n-\lambda} \) is a submanifold of \( I^2 \) of codimension \( \frac{n}{2} \lambda (\lambda+1) \).

**Proof.** Each \( Q_\rho \) is a submanifold because each component is an orbit under the action of the general linear group.

Choose \( q \in Q_\rho \). By Lemma 4.6. we may assume that \( q = x_1^2 + \ldots - x_\rho^2 \).

Then the associated matrix is \( \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} \), where \( E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). Suppose \( q' \)

\[ \begin{pmatrix} A & B \\ B' & C \end{pmatrix} \]

has matrix \( \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} \). \( \exists \) a neighborhood \( N \) of \( q \) in \( I^2 \) such that if \( q' \in N \), then \( |A| \neq 0 \). There rank \( q' = \text{rank } \begin{pmatrix} A & B \\ B' & C \end{pmatrix} \)

\[ = \text{rank } \begin{pmatrix} A^{-1} & 0 \\ -B'A^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ B' & C \end{pmatrix} \]

\[ = \text{rank } \begin{pmatrix} I & A^{-1}B \\ 0 & C-B'A^{-1}B \end{pmatrix} \]

Thus rank \( q' = \rho = C = B'A^{-1}B \), i.e. the entries of \( C \) are determined by the entries of \( A \) and \( B \). Then \( Q_\rho \cap N \) only has the freedom of the entries of \( A \) and \( B \).
So the codimension of \( Q_\rho = Q_{n-\lambda} \) is \( \frac{1}{2}(\lambda+1) \), which is the number of free entries in symmetric \( C \).

Now \( I^7 = I^2 \times m^3/m^8 = (Q_{n} \times m^3/m^8) \cup \{ (Q_{\rho} \times m^3/m^8) \}_{\rho < \lambda} \). \( Q_{n} \times m^3/m^8 \) is the union of orbits of stable singularities (studied in Morse theory) and by Lemma 4.8 is an open set in \( I^7 \). It is in fact \( I^7_0 \) (clear).

Suppose now that \( \text{rank } \eta = \rho \) and that as in Lemma 4.6, we have chosen coordinates \( x_1, \ldots, x_n \) so that \( \eta = x_1^2 + \ldots - x_\rho^2 + \text{higher terms} \). We call \( x_1, \ldots, x_\rho \) the dummy variables and \( x_{\rho+1}, \ldots, x_n \) the essential variables. The following lemma justifies these terms.

**Lemma 4.9.** (Reduction Lemma) Let \( \eta \in m^2 \) and \( j^2 \eta = q = x_1^2 + \ldots - x_\rho^2 \).

Then \( \forall k, \exists n' \in m^2 \) such that \( \eta \sim \eta' \), and \( j^k \eta' = q + p(x_{\rho+1}, \ldots, x_n) \) where \( p \) is a polynomial in only the essential variables with \( 3 \leq \deg \) of monomials of \( p \leq k \).

**Proof.** Use induction on \( k \). The lemma is true for \( k = 2 \). Suppose it is true for \( k - 1 \). In \( j^k, j^k \eta' = q + p(x_{\rho+1}, \ldots, x_n) + \eta_k(x_1, \ldots, x_n) \), where \( 3 \leq \deg \) of monomials of \( p \leq k - 1 \), and \( \eta_k \) is homogeneous of degree \( k \).

Write \( \eta_k = 2x_1p_1 \) (all terms containing \( x_1 \); \( p_1 \) a homogeneous polynomial in \( x_1, \ldots, x_n \) of degree \( k - 1 \))

\[ + 2x_2p_2 \] (all terms containing \( x_2 \), not \( x_1 \))

\[ + 2x_3p_3 \] (all terms containing \( x_3 \), not \( x_1, x_2 \))

\[ + \ldots - 2x_{\rho+1}p_{\rho+1} - \ldots - 2x_{\rho}p_{\rho} \]

\[ + p_1(x_{\rho+1}, \ldots, x_n) \] (all terms not containing dummy variables).

First incorporate the \( 2 ' \)'s and \( - ' \)'s into the \( (p_1) \). Then let \( y_1 = \begin{cases} x_1 + p_1 & i \leq \rho \\ x_1 & i > \rho \end{cases} \)

If \( i \leq \rho \), \( y_1^2 = (x_1 + p_1)^2 = x_1^2 + 2x_1p_1 \) because monomials of degree \( > k \) vanish in \( I^k \). So \( j^k \eta' = y_1^2 + \ldots - y_\rho^2 + p(y_{\rho+1}, \ldots, y_n) + p_1(y_{\rho+1}, \ldots, y_n) \), completing the lemma.

**Addendum 4.10.** The function \( \eta \mapsto p \) is well-defined because the construction is explicit.
Lemma 4.11. If \( \text{rank } \eta \geq n - 3 \), then \( \text{cod } \eta \geq 6 \).

Proof. Either \( \eta \) is not finitely determinate, in which case \( \text{cod } \eta = \infty \),

(Lemma 3.1), or \( \eta \) is \( k \)-determinate, some \( k \), i.e. \( \eta \sim j^k \eta \), and \( j^k \eta \sim q + p \) (essentials), by Lemma 4.9. Then \( \text{cod } \eta = \text{cod } (q+p) \). \( \Delta(q+p) = \)

\[
(2x_1, \ldots, -2x_1, \frac{2p}{3x_1}, \ldots, \frac{2p}{3x_n}).
\]

So \( \text{cod } \eta = \dim m/\Delta(q+p) \)

\[
\geq \dim m/(\Delta(q+p)+m^3)
\]

= number of the missing linear and quadratic terms in the essentials.

If \( n \)-rank \( \eta = \lambda \), all \( \lambda \) linear terms are missing, as too are at least all

but \( \lambda \) of the \( \frac{1}{2}\lambda(\lambda+1) \) quadratic terms, So \( \text{cod } \eta \geq \lambda + \frac{1}{2}\lambda(\lambda+1) - \lambda = \frac{1}{2}\lambda(\lambda+1) \).

If \( \text{rank } \eta \leq n - 3 \), then \( \lambda \geq 3 \) and \( \text{cod } \eta \geq 6 \).

We have that \( \bigcup_{p \leq m-3} \{Q_p \times \mathbb{m}^3/\mathbb{m}^8\} \) consists of \( \eta \) with \( \text{cod } \eta \geq 6 \).

By Lemma 4.8., this subspace has codimension 6 in \( I^7 \). It remains to

investigate \( Q_{n-1} \times \mathbb{m}^3/\mathbb{m}^8 \) and \( Q_{n-2} \times \mathbb{m}^3/\mathbb{m}^8 \).

Lemma 4.12. (Classifying cuspoids) If \( \text{rank } \eta = n - 1 \), then \( \eta \sim q + x_n^k \),

\( 3 \leq k \leq 7 \), or \( \text{cod } \eta \geq 6 \).

Proof. By the reduction lemma 4.9., \( \eta \sim \eta' \) where \( j^7 \eta' = q \) and \( p \) is a

polynomial \( p(x_n) \) with \( 3 \leq \) degree of monomials of \( p \leq 7 \). Let \( k \) be the

least degree appearing, so that \( p = a_k x_n^k + \ldots \). Then \( j^k \eta' \) is \( k \)-determinate,

because \( \Delta(j^k \eta') = (x_1, \ldots, x_{n-1}, x_n^{k-1}) \) and so \( m^2 \Delta \supset m+1 \), and we can use

Theorem 2.9. Thus \( \eta' \sim j^k \eta' = q + a_k x_n^k \)

\[
\Rightarrow q + y_n^k, \text{ changing coordinates so that } |a_k^{1/k} x_n^k = y_n^k.
\]

If \( k \) is odd changing coordinates \( y_n^k - y_n^{k} \) makes \( \eta' = q - y_n^k \sim q + y_n^k \).

Classify as \( q + p \) where \( p = x_n^3 + x_n^4 + x_n^5 \pm x_n^6 + x_n^7 \) and \( \text{cod}(q+p) = 1 \).

Lemma 4.13. The cuspoids \( \eta \) with \( \text{cod } \eta \geq 6 \) form a submanifold of \( I^7 \) of

codimension 6.

Proof. If \( \eta \) is a cuspoid, \( j^2 \eta = q = x_1^2 + \ldots - x_{n-1}^2 \).

Write \( \mathbb{m}^3/\mathbb{m}^8 = R \times S \), where \( R \) is the set of polynomials involving
one of $x_1, \ldots, x_{n-1}$ and such that $3 \leq \text{degree of monomials in } r \in R \leq 7,$ and $S$ is the set of polynomials in $x_n$ only, so that $S \cong R^5.$ Then $j^7n = q + r + s, r \in R, s \in S.$ The reduction lemma 4.9. gave a (unique algebraic) map $\theta: R + S$ such that $n \sim n',$ and $j^7n' = q + 0 + (\theta r + s).$

\[ \text{cod } n \geq 6 \iff \text{cod } n' \geq 6 \]

\[ \theta r + s = 0 \]

\[ s = -\theta r \]

\[ (r, s) \in M_\theta, \text{ where } M_\theta \text{ is the graph of } -\theta, \text{ and is a submanifold of } R \times S \text{ of codimension 5. (} \theta \text{ is algebraic and so graph } \theta \text{ is source of } \theta \text{.) As } q \text{ varies through } Q_{n-1} \text{ we find that the required set of cuspoids } n \text{ with } \text{cod } n \geq 6 \text{ form a bundle over } Q_{n-1} \text{ (of codimension 1 in } m^2/m^3 \text{ by Lemma 4.8) with fibre } M_\theta \text{ which has codimension 5 in } m^2/m^3. \]

Thus the bundle has codimension 6 in $m^2/m^3 = I^7.$

Now we classify the umbilics, $Q_{n-2} \times m^3/m^8.$ Let $n \in m^2$ be such that $j^2n = q,$ and $q = x_1^2 + \ldots - x_{n-2}^2.$ By the reduction lemma 4.9., $n \sim n'$ where $j^3n' = q + p$ and $p$ is a homogeneous cubic in $x_{n-1}, x_n.$

In place of $x_{n-1}, x_n$ we shall use $x, y$ respectively, for clarity.

Note that Lemma 4.12., which classifies cuspoids, has been interpreted in this way in Diagram 4.1 with $x$ replacing $x_n.$

Let $(x, y) \in R^2.$ The space of cubic forms in $x, y$ is,$\{(a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3): a_1, a_2, a_3, a_4 \in R\} = R^4.$ The action of $GL(2, R)$ on $R^2$ induces an action on $R^4.$

**Lemma 4.14.** There are 5 $GL(2, R)$-orbits in $R^4,$ and so each $p \in R^4$ is equivalent to one of 5 forms:

<table>
<thead>
<tr>
<th>dimension</th>
<th>codimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) $x^3 + y^3$ hyperbolic umbilic</td>
<td>4</td>
</tr>
<tr>
<td>(2) $x^3 - xy^2$ elliptic umbilic</td>
<td>4</td>
</tr>
<tr>
<td>(3) $x^2y$ parabolic umbilic</td>
<td>3</td>
</tr>
<tr>
<td>(4) $x^3$ symbolic umbilic</td>
<td>2</td>
</tr>
<tr>
<td>(5) 0</td>
<td>0</td>
</tr>
</tbody>
</table>
Proof. Consider the roots $x, y$ of $p(x,y) = 0, p \in \mathbb{R}^4$.

There are 5 cases (1) 2 complex, 1 real

(2) 3 real distinct

(3) 3 real, 2 same

(4) 3 real equal

(5) 3 equal to zero

Case (4): $p = (a_1x+a_2y)^3 = u^3$ by changing coordinates, \begin{align*}
u &= a_1x + a_2y \\
\sim x^3 \\
v &= \text{independent}.
\end{align*}

Case (3): $p = u^2v$ where $u, v$ are independent linear forms in $x, y$.

\begin{align*}
\sim x^2y
\end{align*}

Case (2): $p = d_1d_2d_3$, product of 3 linear forms, $d_4 = a_1x + b_1y$. We have $k_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \neq 0$ because the root of $d_2 \neq$ the root of $d_3$. Let $k_1d_1 = u'$ \( (*) \). We claim this is a nonsingular coordinate change.

\begin{align*}
u' &= k_1d_1 = u' \\
u - v &= k_2d_2 = v'
\end{align*}

$u, v \mapsto u', v'$ has a change of basis matrix with determinant $= \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$.

$x, y \mapsto u', v'$ has a change of basis matrix with determinant $= k_1k_2 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = k_1k_2k_3 \neq 0$

Adding \( (*) \), $2u = k_1d_1 + k_2d_2$

\begin{align*}
&= (a_2b_3-a_3b_2)(a_1x+b_1y) + (a_3b_1-a_1b_3)(a_2x+b_2y) \\
&= x(a_2b_3-a_3b_2+a_3b_1-a_1b_3) + y(...) \\
&= -k_3(a_3x+b_3y) \\
&= -k_3d_3.
\end{align*}

So $u^3 - uv^2 \sim 2u(u^2-v^2) = -k_1k_2k_3d_1d_2d_3 \sim p$. Thus $p \sim x^3 - xy^2$.

Case (1): This is the same as Case (2) except that $a_2 = a_1, b_2 = b_1$ and $a_3, b_3$ are real. $d_2 = \bar{d}_1, k_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = \bar{k}_1 = \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} = -\bar{k}_2$. 

\begin{align*}
&= \bar{d}_1, k_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = \bar{k}_1 = \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} = -\bar{k}_2.
\end{align*}
\[ k_3 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1\bar{b}_1 - \bar{a}_1b_1 = it, \quad t \in \mathbb{R}. \] Change coordinates, \( iu + v = k_1d_1 \) \( \left( ^* \right) \). \( iu - v = k_2d_2 \)

We claim this is a real change. Adding, \( 2iu = k_3d_3 = itd_3 \) and \( td_3 \) is real.

Subtracting, \( 2v = k_1d_1 - k_2d_2 = k_1d_1 + \bar{k}_1\bar{d}_2 \), which is real. So both \( u \) and \( v \) are real. It is a non-singular change because \( \begin{vmatrix} 1 & 1 \\ i & -1 \end{vmatrix} = -2i \neq 0 \). The product of \( \left( ^* \right) \) is \( 2u(-u^2-v^2) = k_1k_2tp \sim p \). So \( p \sim 2(u^3+uv^2) \), absorbing into the \( u \)-coordinate. \( 2(u^3+uv^2) \sim 2(u^3+3uv^2) \) absorbing \( 3^k \) into \( v \).

\[ u^3 + v^3 \text{ with } u^3 + v^3 \]
\[ x^3 + y^3 \text{ and } x^3 - xy^2 \text{ are both } 3 \text{-determinate and both } \text{cod}(x^2+y^3) \text{ and } \text{cod}(x^3-xy^2) \text{ equal } 3. \] Thus the orbits corresponding to these are of codimension 3 in \( \mathcal{L}^7 \) by Theorem 3.7.

**Lemma 4.15.** If \( \eta = q + p, \quad q \in \mathcal{O}_{n-2}, \quad p = x^2y + \text{higher terms}, \) then either

1. \( \eta \sim q + (x^2y + t^4) \) and \( \text{cod } \eta = 4 \) (the parabolic umbilic)

or

2. \( \eta \sim q + (x^2y^5) \) and \( \text{cod } \eta = 5. \)

or

3. \( \eta \) belongs to \( \mathcal{L}^7. \)

**Proof.** If \( k \geq 4 \), then if \( p = x^2y + y^k, \) \( \text{cod } p = k = \det p. \)

**Lemma 4.16.** If \( k \geq 4 \) and \( j^{k-1}p = x^2y \) then \( p \sim x^2y + y^k \), or \( p \sim p' \) and \( j^{k-1}p' = x^2y. \)

**Lemma 4.16.** clearly gives Lemma 4.15.

**Proof of Lemma 4.16.** \( j^{k}p = x^2y + \text{a polynomial of degree } k \)

\[ = x^2y + ax^k + 2xyP + by^k, \]

where \( P \) is a homogeneous polynomial of degree \( k - 2 \geq 2. \)

\[ (x+P)^2(y+ax^{k-2}) = (x^2+2xP)(y+ax^{k-2}) = x^2y + 2xyP + ax^k \text{ in } \mathcal{L}^k. \]

Put \( u = (x+P) \) and \( v = y + ax^{k-2}, \) \( v^k = y^k \) in \( \mathcal{L}^k. \) So \( j^kP = u^2v + bv^k. \) There are two cases. \( b \neq 0: j^{k}p \sim u^2v + v^k \) absorbing \( |b|^{1/k} \) into \( v, \) and absorbing \( 1/|b|^{1/2k} \) into \( u. \) \( b = 0: j^{k}p = u^2v \sim x^2y. \)
Lemma 4.17. If $\eta = q + p$, $p \in \mathbb{Q}_{n-2}$ and $p = x^3 + \text{higher terms in } x, y$, then either (1) $\eta \sim q + x^3 + y^4$ and $\text{cod} \eta = 5$

or (2) $\eta \in \Sigma^7_6$.

Proof. Calculation shows that $x^3 + y^4 = p'$ is 4-determinate and $\text{cod} p' = 5$.

Then $j^4 p = x^3 + a_0 x^4 + a_1 x^2 y^2 + a_2 x y^3 + a_4 y^4$. $a_4 \neq 0$: Put $v = y + \frac{a_3 x}{4a_4}$.

Then $j^4 p = x^3 + 3x^2 v + a_4 v^4$, where $P$ is a homogeneous polynomial of degree 2 in $x, v$. In $j^4 p = (x + v)^3 + a_4 v^4$

$\sim u^3 + v^4$, putting $u = x + P$ and absorbing $|a_4|^k$ into $v$.

$a_4 = 0$: As above we find that $j^4 p \sim x^3 + xy^3$, which is 4-determinate as stated in Chapter 2. (This is Siersma's germ) In any case a short calculation gives $\text{cod} \eta = \text{cod}(x^3 + xy^3) = 6$, so $\eta \in \Sigma^7_6$.

Lemma 4.14 and a straightforward calculation produce the following facts. The symbolic umbilic (S) is a twisted cubic curve of dimension 1 in $\mathbb{R}^3$. The parabolic umbilic (P) is a quartic surface with a cusp edge along S. The elliptic umbilic (E) is inside the cusp. The hyperbolic umbilic (H) is outside the cusp. (4.18)

CHAPTER 5. THE PREPARATION THEOREM.

This chapter is self-contained and is devoted to proving a major result, the Preparation Theorem, which we need for Chapter 6.

The words "near 0" will always be understood to mean "in some neighborhood of 0."

Theorem 5.1. (Division Theorem) Let $D$ be a $C^\infty$ function defined near 0, from $\mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}$, such that $D(t,0) = d(t)t^k$ where $d(0) \neq 0$ and $d$ is
$C^\infty$ near 0 in $\mathbb{R}$. Then given any $C^\infty: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ defined near 0, $\exists C^\infty$

functions $q$ and $r$ such that: \(1\) $E = qD + r$ near 0 in $\mathbb{R} \times \mathbb{R}^n$,

where \(2\) $r(t,x) = \sum_{i=0}^{k-1} r_i(x) t^i$ for $(t,x) \in \mathbb{R} \times \mathbb{R}^n$ near 0.

**Notation.** Let $P_k: \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ be the polynomial $P_k(t,\lambda) = t^k + \sum_{i=0}^{k-1} \lambda_i t^i$.

**Theorem 5.2.** (Polynomial Division Theorem) Let $E(t,x)$ be a $E$-valued $C^\infty$

function defined near 0 in $\mathbb{R} \times \mathbb{R}^n$. Then $\exists E$-valued $C^\infty$ functions $q(t,x,\lambda)$ and $r(t,x,\lambda)$ defined near 0 in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k$ satisfying:

\(1\) $E(t,x) = q(t,x,\lambda) P_k(t,\lambda) + r(t,x,\lambda)$, and

\(2\) $r(t,x,\lambda) = \sum_{i=0}^{k-1} r_i(x,\lambda) t^i$,

where each $r_i$ is a $C^\infty$ function defined near 0 in $\mathbb{R}^n \times \mathbb{R}^k$. Moreover if $E$ is $\mathbb{R}$-valued, then $q$ and $r$ may be chosen $\mathbb{R}$-valued.

Note that if $E$ is $\mathbb{R}$-valued we merely equate real parts of (1) in Theorem 5.2 to give the last part.

**Proof of Theorem 5.1 using Theorem 5.2.** Given $D, E$ we can apply Theorem 5.2 to find $q_D, r_D, q_E, r_E$ such that $D = q_D P_k + r_D$ and $E = q_E P_k + r_E$; let now $r_D(t,x,\lambda) = \sum_{i=0}^{k-1} r^D_i(x,\lambda) t^i$ (*).

Now $t^k d(t) = D(t,0) = q_D(t,0) P_k(t,0) + r_D(t,0)$ ($\lambda = 0$)

$= q_D(t,0) t^k + \sum_{i=0}^{k-1} r^D_i(0) t^i$.

Comparing coefficients of powers of $t$, $r^D_i(0) = 0$ and $q_D(0) \neq 0$ ($d(0) \neq 0$).

Write $s_i(\lambda) = r^D_i(0,\lambda)$. We claim that $|s_i(0)\lambda_j| \neq 0$.

\[ t^k d(t) = D(t,0) = q_D(t,0,\lambda)(t^k + \sum_{i=0}^{k-1} s_i(\lambda) t^i). \]

Differentiating with respect to $\lambda_j$ and setting $\lambda = 0, 0 = \frac{\partial q_D}{\partial \lambda_j}(t,0) t^k + \sum_{i=0}^{k-1} \frac{\partial s_i}{\partial \lambda_j}(0) t^i$.

Thus $\frac{\partial s_i}{\partial \lambda_j}(0) = 0$ if $i < j$ and $\frac{\partial q_D}{\partial \lambda_j}(0) = -q_d(0)$. So $\left(\frac{\partial q_D}{\partial \lambda_j}(0)\right)$ is a lower triangular matrix, and as $q_D(0) \neq 0$, $\left|\frac{\partial q_D}{\partial \lambda_j}(0)\right| \neq 0$. 

By the implicit function theorem, \( \exists C^\infty \) functions \( \theta_i(x) (0 \leq i \leq k-1) \) such that (a) \( r_i^j(x, \theta) = 0 \), and (b) \( \theta(0) = 0 \) (recall \( r_i^j(0) = 0 \)). Let \( \tilde{q}(t, x) = q(t, x, \theta) \) and \( P(t, x) = \tilde{q}_k(t, \theta) \). Then \( D(t, x) = \tilde{q}(t, x)P(t, x) \) (as \( r_D(t, x, \theta) = 0 \) by (a.).) As \( \tilde{q}(0) = q(t, 0) \neq 0 \), \( P(t, x) = \frac{D(t, x)}{\tilde{q}(t, x)} \) near 0 in \( \mathbb{R} \times \mathbb{R}^n \).

By (*), \( E(t, x) = q_E(t, x, \theta)P_k(t, \theta) + r_E(t, x, \theta) = q(t, x)D(t, x) + r(t, x) \),
where \( q(t, x) = \frac{q_E(t, x, \theta)}{\tilde{q}(t, x)} \) and \( r(t, x) = r_E(t, x, \theta) = \sum_{i=0}^{k-1} r_i^E(x, \theta) t^i \). Finally let \( r_i^E(x, \theta) = r_i^E(x, \theta) \).

Suppose \( f: \mathbb{C} \to \mathbb{C} \), \( f = u + iv \) and \( u, v: \mathbb{C} \to \mathbb{R} \). If \( z = x + iy \),
then \( \frac{\partial u}{\partial z} = \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \) and \( \frac{\partial v}{\partial z} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \). A similar result for \( v \) gives us that
\[
\frac{\partial f}{\partial z} = \frac{1}{2i} \left( \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \right) + \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \tag{5.3}
\]

**Lemma 5.4.** Let \( f: \mathbb{C} \to \mathbb{C} \) be \( C \) as a function \( \mathbb{R}^2 \to \mathbb{R}^2 \). Let \( \gamma \) be a simple closed curve in \( C \) whose interior is \( U \). Then for \( w \in U \),
\[
f(w) = \frac{1}{2\pi i} \oint_{\gamma \setminus w} \frac{f(z)}{z-w} \, dz + \frac{1}{2\pi i} \oint_U \frac{\partial f}{\partial z} \, dz \wedge \overline{dz} \cdot
\]
(If \( f \) is holomorphic this reduces to the Cauchy Integral Formula since \( f \) is holomorphic \( \Rightarrow \frac{\partial f}{\partial z} = 0 \).)

**Proof.** Let \( w \in U \) and choose \( \epsilon < \min\{|w-z| : z \in \gamma\} \). Let \( U_{\epsilon} = U - \) (disc radius \( \epsilon \) about \( w \), and \( \gamma_{\epsilon} = \partial U_{\epsilon} \).

Recall Green's Theorem for \( \mathbb{R}^2 \). If \( M, N: U_{\epsilon} \to \mathbb{R} \) are \( C^\infty \) on \( \gamma_{\epsilon} \), then
\[
\int_{\gamma_{\epsilon}} (Mdx + Ndy) = \iint_{U_{\epsilon}} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx \wedge dy.
\]
Green's Theorem and (5.3) for \( f = u + iv \) give
\[
\int_{\gamma_{\epsilon}} f \, dz = \int_{\gamma_{\epsilon}} (u+iv)(dx+idy) = 2i \int_{U_{\epsilon}} \frac{\partial f}{\partial z} \, dx \wedge dy.
\]
\[2i \, dx \wedge dy = dz \wedge d\overline{z}, \text{ so } \int_{\gamma_{\epsilon}} f \, dz = -\iint_{U_{\epsilon}} \frac{\partial f}{\partial z} \, dz \wedge d\overline{z} \tag{\star}
\]
Apply (*) to $\frac{f(z)}{z-w}$, noting that $\frac{1}{z-w}$ is holomorphic on $U$.

$$-\int_{C_{\epsilon}} \frac{\partial f(z)}{z-w} \frac{dz \wedge d\bar{z}}{z-w} = \int_{\gamma_{\epsilon}} \frac{f(z)}{z-w} dz - \int_{C_{\epsilon}} \frac{f(z)}{z-w} dz,$$

(*)

where $C_{\epsilon}$ is the circle, radius $\epsilon$, centre $w$.

With polar coordinates at $w$, $\int_{C_{\epsilon}} \frac{f(z)}{z-w} dz = \int_{0}^{2\pi} \frac{f(w+\epsilon e^{i\theta})i\theta}{\epsilon} d\theta$. As $\epsilon \to 0$, R.H.S. of (*) $\to \int_{\gamma} \frac{f(z)}{z-w} dz - 2\pi i f(w)$, and L.H.S. of (*) $\to -\int_{U} \frac{\partial f(z)}{z-w} \frac{dz \wedge d\bar{z}}{z-w}$.

(The limit exists because $\frac{1}{\partial z}$ is bounded on $U$, and $\frac{1}{z-w}$ is integrable over $U$.)

**Proof of Theorem 5.2.** Let $\tilde{E}(z,x,\lambda)$ be a $C^\infty$ function defined near $0$ in $E \times \mathbb{R}^n \times \mathbb{C}^k$ such that $\tilde{E}(t,x,\lambda) = E(t,x) \forall t \in \mathbb{R}$, i.e. $\tilde{E}$ is an extension of $E$.

Then $\tilde{E}(w,x,\lambda) = \frac{1}{2\pi i} \int_{\gamma} \frac{\tilde{E}(z)}{z-w} dz + \frac{1}{2\pi i} \int_{U} \frac{\partial \tilde{E}(z)}{z-w} \frac{dz \wedge d\bar{z}}{z-w}$, by Lemma 5.4. Let $P_k(z,\lambda) - p_k(w,\lambda) = (z-w)P_{k-1}(z,\lambda)+i_{\lambda}P_{k-1}(z,\lambda)$, i.e. $\frac{P_k(z,\lambda)}{z-w} = \frac{P_k(w,\lambda)}{z-w} + \sum_{i=0}^{k-1} i_{\lambda} P_{k-1}(z,\lambda)$.

In the expression for $\tilde{E}(w,x,\lambda)$ multiply top and bottom inside the integrals by $P_k(z,\lambda)$ and expand $\frac{P_k(z,\lambda)}{z-w}$ giving $\tilde{E} = qP_k + r$ on $E \times \mathbb{R}^n \times \mathbb{C}^k$ where

$$q(w,x,\lambda) = \frac{1}{2\pi i} \int_{\gamma} \frac{\tilde{E}(z,x,\lambda)}{P_k(z,\lambda)} \cdot \frac{dz}{z-w} + \frac{1}{2\pi i} \int_{U} \frac{\partial \tilde{E}(z,x,\lambda)}{P_k(z,\lambda)} \cdot \frac{dz \wedge d\bar{z}}{z-w},$$

and $r(z,\lambda) = \frac{1}{2\pi i} \int_{\gamma} \frac{\tilde{E}(z,x,\lambda)}{P_k(z,\lambda)} \cdot i_{\lambda} P_{k-1}(z,\lambda) \cdot dz + \frac{1}{2\pi i} \int_{U} \frac{\partial \tilde{E}(z,x,\lambda)}{P_k(z,\lambda)} \cdot \frac{p_k(z,\lambda)}{P_k(z,\lambda)} \cdot dz \wedge d\bar{z},$

so long as these integrals are well defined and yield $C^\infty$ functions.

The first integral in the definition of both $q$ and $r$ is well-defined and $C^\infty$ as long as the zeros of $P_k(z,\lambda)$ do not occur on the curve $\gamma$ for $\lambda$ near $0$ in $\mathbb{C}^k$. Such a $\gamma$ is easily chosen.

But $U$ may contain zeros of $P_k$. So we need $\tilde{E}$ such that $\frac{\partial \tilde{E}}{\partial z}$ vanishes on zeros of $P_k$ and for real $z$ to ensure $q, r$ well-defined. As the integrands are bounded we need $C^\infty \tilde{E}$ such that $\frac{\partial \tilde{E}}{\partial z}$ vanishes to infinite order on zeros of $P_k$ and for real $z$ to ensure $q$ and $r$ $C^\infty$.

**Lemma 1.** (Nirenberg Extension Lemma) Let $E(t,x)$ be a $C^\infty$ $E$-valued function defined near $0$ in $\mathbb{R} \times \mathbb{R}^n$. Then $\exists$ a $C^\infty$ $E$-valued function $\hat{E}(z,x,\lambda)$ defined near $0$ in $\mathbb{C} \times \mathbb{R}^n \times \mathbb{C}^k$ such that,
(1) \( \hat{E}(t,x,\lambda) = E(t,x) \quad \forall \ t \in \mathbb{R}. \)

(2) \( \frac{\partial \hat{E}}{\partial z} \) vanishes to infinite order on \( \{ \text{Im} \ z = 0 \}. \)

(3) \( \frac{\partial \hat{E}}{\partial \overline{z}} \) vanishes to infinite order on \( \{ \text{Re} \ z \lambda = 0 \}. \)

**Lemma 2.** (E. Borel's Theorem) Let \( f_0, f_1, \ldots \) be a sequence of \( C^\infty \) functions on a given neighborhood \( N \) of 0 in \( \mathbb{R}^n \). Then \( \exists \) a \( C^\infty \) function \( F(t,x) \) on a neighborhood of 0 in \( \mathbb{R} \times \mathbb{R}^n \) such that \( \frac{\partial F}{\partial t^1}(0,x) = f_i(x) \quad \forall \ i. \)

**Proof.** Let \( \rho : \mathbb{R} \to \mathbb{R}^n \) be such that \( \rho(t) = \begin{cases} 1 & |t| \leq \frac{1}{2} \\ 0 & |t| \geq 1 \end{cases} \)

Let \( F(t,x) = \sum_{i=0}^{\infty} \frac{1}{i!} \rho(\mu_i t) f_i(x) \), where \( \{ \mu_i \} \) is a rapidly increasing sequence of real numbers tending to \( \infty \), so that \( F \) is \( C^\infty \) near 0.

(Lemma 2 may be used to show that for any power series about 0 in \( \mathbb{R}^n \) \( \exists \) a \( C^\infty \) real-valued function with its Taylor series at 0 the given power series.)

**Lemma 3.** Let \( V, W \) be complementary subspaces of \( \mathbb{R}^n (= V+W) \). Let \( g, h \) be \( C^\infty \) functions near 0 in \( \mathbb{R}^n \), such that for all multi-indices \( \alpha \),

\[
\frac{\partial^{|\alpha|} g(x)}{\partial x^\alpha} = \frac{\partial^{|\alpha|} h(x)}{\partial x^\alpha} \quad \forall \ x \in V \cap W. \]

Then \( \exists \ C^\infty \) \( F \) near 0 in \( \mathbb{R}^n \), such that

\[
\forall \alpha, \quad \frac{\partial^{|\alpha|} F(x)}{\partial x^\alpha} = \begin{cases} \frac{\partial^{|\alpha|} g(x)}{\partial x^\alpha} & x \in V \\ \frac{\partial^{|\alpha|} h(x)}{\partial x^\alpha} & x \in W \end{cases} \] (A multi-index \( \alpha = (a_1, \ldots, a_n) \))

and \( |\alpha| = a_1 + \ldots + a_n \) so that

\[
\frac{\partial^{|\alpha|} g(x)}{\partial x^\alpha} = \frac{a_1 \ldots a_n}{\partial x_1 \ldots \partial x_n} g(x). \]

**Proof.** Without loss of generality \( h \equiv 0 \), for if \( F_1 \) is the required extension for \( (g-h) \) and 0, then \( F = F_1 + h \) is the required extension for \( g \) and \( h \).

Choose coordinates \( y_1, \ldots, y_n \) so that \( V = y_1 = \ldots = y_j = 0 \).
and \( W \equiv y_{j+1} = \ldots = y_k = 0 \). Let

\[
F(y) = \sum_{|\alpha| = 0}^{\infty} \frac{y^\alpha}{\alpha!} \beta(\alpha) (0, \ldots, 0, y_{j+1}, \ldots, y_n) \rho \left( u | y_{i=1}^{\alpha} y_1^{\alpha} \right),
\]

where \( \rho \) is as in

\[
a = (a_1, \ldots, a_j, 0, \ldots, 0)
\]

Lemma 2 and \( \{u_i\} \) increases to \( \infty \) rapidly enough so that \( F \) is \( C^\infty \) near 0.

If \( y \in W \), each term of \( \frac{3|\beta|}{\partial y^\beta} F(y) \) contains a factor \( \frac{3|\gamma|}{\partial y^\gamma} (0, \ldots, 0, y_{k+1}, \ldots, y_n) \).

Since \( (0, \ldots, 0, y_{k+1}, \ldots, y_n) \in V \cap W \), this factor = 0 (h=0). So \( \frac{3|\beta|}{\partial y^\beta} F(y) = 0 \).

If \( y \in V \), note that

\[
\frac{\partial |\gamma|}{\partial y^\gamma} \rho \left( u | y_{i=1}^{\alpha} y_1^{\alpha} \right) \bigg|_{y_1 = \ldots = y_j = 0} = \begin{cases} 1 & \gamma = 0, \\ 0 & \gamma \neq 0 \end{cases}
\]

and then \( \frac{3|\beta|}{\partial y^\beta} F(y) = \sum_{|\alpha| = 0}^{\infty} \frac{3|\beta|}{\partial y^\beta} \left[ \frac{\alpha}{\alpha!} \frac{\partial g(y)}{\partial y^\alpha} \right] \bigg|_{y_1 = \ldots = y_j = 0} \).

If \( b_i \neq a_i \) some \( i \leq j \), then this term is 0. In fact the only nonzero term is \( \frac{3|\beta|}{\partial y^\beta} g(y) \).

Lemma 4. Let \( f \) be a \( C^\infty \) \( \mathbb{C} \)-valued function near 0 in \( \mathbb{R}^n \) and let \( X \) be a vector field on \( \mathbb{R}^n \) with \( \mathbb{C} \) coefficients. Then \( \exists \ C^\infty \) \( \mathbb{C} \)-valued \( F \) near 0 in \( \mathbb{R} \times \mathbb{R}^n \) so that

(a) \( F(0, x) = f(x) \) \( \forall \ x \in \mathbb{R}^n \).

(b) \( \frac{\partial F}{\partial t} \) agrees to infinite order with \( XF \) at all \( (0, x) \in \mathbb{R} \times \mathbb{R}^n \).

Proof. Try \( \tilde{F}(t, x) = e^{tX} f = \sum_{k=0}^{\infty} \frac{t^k}{k!} X^k f \). Differentiating termwise at \( t = 0 \) gives (b). Clearly (a) holds. To ensure that \( \tilde{F} \) is \( C^\infty \) use Lemma 2 to choose \( C^\infty \) \( F \) such that \( F = \sum_{k=0}^{\infty} \frac{t^k}{k!} X^k \rho(u_{k, t}) \).

Proof of Lemma 1. We use induction on \( k \). If \( k = 0 \), \( F_k(z, \lambda) = 1 \), so we need \( C^\infty \) \( \tilde{E}(z, x) \) such that \( \tilde{E}(t, x) = E(t, x) \forall \ t \in \mathbb{R} \) and \( \frac{\partial \tilde{E}}{\partial t}(t, x) \) vanishes to infinite order \( \forall \ t \in \mathbb{R} \). Let \( z = s + it \), \( 2 = \frac{3z}{\partial s} + i \frac{3z}{\partial t} \). (Compare 5.3)

Then Lemma 4 with \( X = -i \frac{\partial}{\partial s} \) gives such an \( \tilde{E} \).

Suppose Lemma 1 is proved for \( k - 1 \). We show \( \exists \ C^\infty \mathbb{C} \-valued F(z, x, \lambda) \) and \( G(z, x, \lambda) \) such that
(1)' \( F \) and \( G \) agree to infinite order on \( \{ P_k(z, \lambda) = 0 \} \)

(2)' \( F \) is an extension of \( E \).

(3)' \( \frac{\partial F}{\partial z} \) vanishes to infinite order on \( \{ \text{Im} \ z = 0 \} \).

(4)' Let \( M = F \left[ \left( P_k(z, \lambda) = 0 \right) \right] \cdot \frac{\partial M}{\partial z} \) vanishes to infinite order on \( \left\{ \frac{\partial F}{\partial z} (z, \lambda) = 0 \right\} \).

(5)' \( \frac{\partial G}{\partial z} \) vanishes to infinite order on \( \{ P_k(z, \lambda) = 0 \} \).

Existence of \( F \) and \( G \) proves Lemma 1. Let \( u = P(z, \lambda) \in P_k(z, \lambda) \) and \( \lambda' = (\lambda_1', \ldots, \lambda_{k-1}') \). Consider \( (z, \lambda_0', \lambda') \rightarrow (z, u, \lambda') \) on \( E \times E \times E^{k-1} \). This is a valid coordinate change because \( \frac{\partial u}{\partial \lambda_0} = 1 \). In the new coordinates, \( \{ P_k(z, \lambda) = 0 \} \) is given by \( u = 0 \). By Lemma 3 \( \exists \tilde{E} \) agreeing to infinite order with \( G \) on \( u = 0 \) and to infinite order with \( F \) on \( \text{Im} \ z = 0 \). \( (u = 0 \) and \( \text{Im} \ z = 0 \) intersect transversally in \( \mathbb{R}^{2k+2} \). \( \text{(2)'}, \text{(3)'} \) and \( \text{(5)'} \) now imply \( \tilde{E} \) is the desired extension of \( E \).

Existence of \( F \) and \( G \). Suppose we have that \( F \) exists. In \( (z, u, \lambda') \)-coordinates, \( \frac{\partial}{\partial z} \) becomes \( \frac{\partial}{\partial z} + \frac{\partial P}{\partial z} \frac{\partial}{\partial u} \) and \( \frac{\partial}{\partial z} \) becomes \( \frac{\partial}{\partial z} + \frac{\partial P}{\partial z} \frac{\partial}{\partial u} \). So in these coordinates we need \( G(z, x, u, \lambda') \) such that

(a) \( F = G \) to infinite order on \( \{ u = 0 \} \), and

(b) \( \left( \frac{\partial}{\partial z} + \frac{\partial P}{\partial z} \frac{\partial}{\partial u} \right) G = 0 \) to infinite order on \( \{ u = 0 \} \).

Let \( X = -\frac{\partial P}{\partial z} \left( \frac{\partial}{\partial z} \right)^{-1} \frac{\partial}{\partial u} \). As in Lemma 4 we must find \( C^\infty G \) satisfying (a) and

\[
(b') \frac{\partial G}{\partial z} = XG \text{ to infinite order on } \{ u = 0 \}. \]

The formal solution is,

\[
G = \sum_{i=0}^{\infty} \left( \frac{\partial}{\partial z} \right)^i X^i M(z, x, \lambda') \rho(u_i |u|^2 ) \tag{*}
\]

As \( \frac{\partial M}{\partial z} = 0 \) to infinite order on \( \left\{ \frac{\partial F}{\partial z} (z, \lambda') = 0 \right\} \) by (4)', \( X^i M \) is \( C^\infty \) in \( (z, x, \lambda') \) \( \forall \ i \), so we can choose \( \{ u_i \} \) to increase quickly enough to make \( G \) \( C^\infty \).

We need only a \( C^\infty F \) so that in \( (z, x, u, \lambda') \)-coordinates,

(2)' \( F(t, x, u, \lambda') = E(t, x) \ \forall \ t \in \mathbb{R} \)

(3)' \( \frac{\partial F}{\partial z} = X F \) to infinite order on \( \{ \text{Im} \ z = 0 \} \).
(4)' If \( M = \mathcal{F}(u = 0) \), \( \frac{\partial M}{\partial z} = 0 \) to infinite order on \( \{ \frac{\partial P_k}{\partial z} = 0 \} \).

Consider \( u = 0 \) and the coordinate change \( \lambda' = (\lambda_1', \ldots, \lambda_{k-1}') \mapsto (\lambda_1, \ldots, \lambda_{k-1}') = \lambda'' \). The conditions are now that we find \( C^\infty M(z, x, \lambda'') \) such that,

(I) \( M(t, x, \lambda'') = E(t, x) \forall t \in \mathbb{R} \)

(II) \( \frac{\partial M}{\partial z} \) vanishes to infinite order on \( \{ \text{Im} \, z = 0 \} \), and

(III) \( \frac{\partial M}{\partial \lambda} \) vanishes to infinite order on \( \{ P_{k-1}(z, \lambda'') = 0 \} \).

The induction hypothesis gives such a \( C^\infty M(z, x, \lambda'') \), and we can view \( M \) as a \( C^\infty \) function of \( (z, x, \lambda') \).

Let \( F(z, x, u, \lambda') = \sum_{i=0}^{\infty} \mathcal{F}(u) u^i M(z, x, \lambda') \rho(\frac{u}{\bar{u}}^2) \). Compare (\#). By (III), \( X^1 M \) is \( C^\infty \) in \( z, x, \lambda' \), and so the \( \{ u_i \} \) may be chosen so that \( F \) is a \( C^\infty \) function satisfying (2)', (3)'. Also, on \( u = 0 \), \( F = M \) and (III) gives (4)'.

The complete the proof of Lemma 1.

The remarks before Lemma 1 state that this suffices to prove the (Polynomial Division Theorem) Theorem 5.2.

Let \( \pi \) be projection \( R^{n+s} \to R^s \). \( \pi \) induces \( \pi^*: E_s \to E_{n+s}^s \), where \( E_s \) is the set of germs at 0 of \( C^\infty \) functions \( R^s \to R \), as usual. Let \( M \) be an \( E_{n+s}^s \)-module, and let \( \overline{M} \) denote the same set regarded as an \( E_s \)-module with structure induced by \( \pi^* \).

Theorem 5.5. (Preparation Theorem) Suppose that

(1) \( M \) is a finitely generated \( E_{n+s}^s \)-module,

(2) \( M/(\pi^* M) \) is a finite-dimensional real vector space.

Then \( \overline{M} \) is finitely generated as an \( E_s \)-module.

Proof. There are 2 steps.

Step 1. Let \( \pi_1: R^s \times R \to R^s \) and \( t: R^s \times R \to R \) denote the projections. We prove the theorem for \( n = 1 \), \( s = \pi_1 \). Let \( v_1, \ldots, v_p \) be elements of \( M \) generating \( M \) as an \( E_{s+1} \)-module, whose images in \( M/(\pi^* M) \) span this vector space. Then any \( v \in M \) can be written \( v = \sum_{i=1}^{p} a_i v_i + \sum_{i=1}^{p} a_i v_i \) where \( a_i \in R \),
and \( a_{ij} \in (\pi \ast m_\ast)_{s+1} \). In particular \( \sum a_{ij} \in \mathbb{R}, \ a_{ij} \in (\pi \ast m_\ast)_{s+1} \ (1 \leq i, j \leq p) \), such that \( v_j = \sum_{i=1}^p (a_{ij} - a_{ij})v_j \). Let \( D \) be the determinant \( |t_{ij} - a_{ij} - a_{ij}| \); by Cramer's rule \( Dv_i = 0, i = 1, \ldots, p \). Expanding the determinant we see that \( D \) is regular of order \( k \), some \( k \leq p \), since \( D|_{0 \ast \mathbb{R} \ast} \) is a monic polynomial in \( t \) of order \( p \) \((a_{ij} = 0 \) on \( 0 \ast \mathbb{R} \)). Since \( D.M = 0, M \) is an \((E_{s+1} / D.E_{s+1})\)-module.

Now \( D \) is regular of order \( k \) \( (i.e., D(t,0) = d(t)t^k \), where \( d(0) \neq 0 \) and \( d \) is \( C^\infty \) near \( 0 \), and \( D \) is \( C^\infty \) defined near \( 0 \) in \( \mathbb{R}^s \times \mathbb{R} \)) and so using the Division Theorem 5.1., \( E_{s+1} / D.E_{s+1} \) is finitely generated as an \( E_s \)-module.

Since \( M \) is finitely generated as an \((E_{s+1} / D.E_{s+1})\)-module, it follows that \( M \) is finitely generated as an \( E_s \)-module.

Step 2. We complete the proof of the theorem. Factor \( \pi \) as follows:

\[
\mathbb{R}^s \times \mathbb{R}^n \xrightarrow{\pi_1} \cdots \xrightarrow{\pi_2} \mathbb{R}^s \xrightarrow{\pi_1} \mathbb{R}^s,
\]

where \( \pi_1 : \mathbb{R}^s \times \mathbb{R}^i \to \mathbb{R}^s \times \mathbb{R}^{i-1} \) is the germ of the projection,

\[
(y, a_1, \ldots, a_i) \longmapsto (y, a_1, \ldots, a_{i-1}).
\]

For each \( i, 0 \leq i \leq n + s \), we give \( M \) the \( E_{s+i} \)-module structure induced by \( (\pi_1 \circ \ldots \circ \pi_n)^* \). If \( i \) is 1 this is the \( E_s \)-module structure of \( M \) since \( \pi = \pi_0 \circ \ldots \circ \pi_n \).

Now we prove by decreasing induction on \( i \) that \( M \) is finitely generated as an \( E_{s+i} \)-module \( \forall \ i, 0 \leq i \leq n \). By hypothesis, it is true for \( i = n \), so it suffices to carry out the inductive step. Assume \( M \) is finitely generated as an \( E_{s+i+1} \)-module.

\[
(\pi \ast m_\ast)_s = (\pi_1 \ast \ldots \ast \pi_{i+1})^* (m_\ast)_s. \quad \text{(On the L.H.S. \( M \) is regarded as an \( E_{n+s} \)-module, and on the R.H.S. as an \( E_{s+i+1} \)-module.) So \( (\pi \ast m_\ast)_s \subset (\pi_{i+1} \ast m_{s+1})^* \).
\]

In particular \( M / (\pi_{i+1} \ast m_{s+1}) \) is finitely generated as a real vector space. In particular the hypotheses of the theorem are satisfied for \( \pi_{i+1} \) in place of \( \pi \). Thus we may apply Step 1 to see that \( M \) is finitely generated as an \( E_{s+1} \)-module.
This completes the inductive step and also the proof as \( i = 0 \) is the statement of the theorem.

**Definition.** Let \( \pi \) be projection \( \mathbb{R}^{n+s} \to \mathbb{R}^s \). A mixed homomorphism over \( \pi^* \) of finite type (a *mixture*) is a diagram:

\[
\begin{array}{cccccc}
 & & B & & \\
 & & \downarrow \beta & & \\
A & \xrightarrow{\alpha} & C & \xrightarrow{\pi^*} & E_{n+s} & \\
 & \leftarrow & \leftarrow & \leftarrow & \leftarrow & \\
B & \leftarrow & \leftarrow & \leftarrow & \leftarrow & \\
\end{array}
\]

where \( A \) is a finitely generated \( E_s \)-module, \( B \) is an \( E_{n+s} \)-module, \( C \) is a finitely generated \( E_{n+s} \)-module, \( \alpha \) is a module homomorphism over \( \pi^* \), i.e. \( \alpha(na) = (\pi^*n)(\alpha a) \), \( n \in E_s \) and \( a \in A \); \( \beta \) is an \( E_{n+s} \)-module homomorphism.

**Corollary 5.6.** \( C = \alpha A + \beta B + (\pi^*m)C = C = \alpha A + \beta B \).

**Proof.** Let \( C' = C/\beta B \) and \( \rho : C \to C' \) be the projection. As \( C \) is a finitely generated \( E_{n+s} \)-module so is \( C' \). \( (\pi^*m)C' = m_s C', \) so \( C'/(\pi^*m)C' = C'/m_s C' \).

Our hypothesis \( C' = \rho a A + (\pi^*m)C' = C' = \rho a A + m_s C' \) and this \( = C'/m_s C' \) is a finitely generated \( E_s \)-module. Choose now a finite base \( \{c_i\} \) for \( C' \mod m_s C' \) as an \( E_s \)-module. Any \( c \in C' \) can be written,

\[
c = \sum_{i=1}^{\infty} c_i \mod m_s C'.
\]

Now \( \eta_i = \eta_i(0) + \eta_i(0) \in \mathbb{R}, \eta_i(0) \in m_s \) in the notation of Lemma 2.8. So \( c = \sum_{i=1}^{\infty} c_i \mod m_s C' \). Because \( c \) was arbitrary we have shown that \( C'/m_s C' \) is a finite-dimensional vector space over \( \mathbb{R} \), and hence by (2) so is \( C'/(\pi^*m)C' \).

(1) and (4) for \( C' \) are the two hypotheses of the Preparation Theorem 5.5, and so \( C' \) is a finitely generated \( E_s \)-module. We can now apply Nakayama's Lemma 2.10 with \( A = E_s, a = m_s, M = C', N = \rho a A \) to (3).

Therefore \( C' = \rho a A \).

And so \( C' = \rho a A, \) i.e. \( C = \alpha A + \beta B \).
We define the category of unfoldings of \( \eta \), for fixed \( \eta \in \mathbb{R}^2 \). An object \((r,f)\) is a germ \( f: \mathbb{R}^n \times \mathbb{R}^r, 0 \to \mathbb{R}, 0 \) (shorthand for "is a germ \( f \) of a \( C^\infty \) function \( \mathbb{R}^n \times \mathbb{R}^r, 0 \to \mathbb{R}, 0 \)"), such that \( f|_{\mathbb{R}^n \times 0} = \eta \), i.e.

\[
\begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{n} & \mathbb{R} \\
1 \times 0 & \xrightarrow{f} & 1 \\
\mathbb{R}^{n+r} & \xrightarrow{\pi} & \mathbb{R}
\end{array}
\]

commutes.

A morphism \((\phi, \phi, \varepsilon): (s, g) \to (r, f)\) is a germ \( \phi: \mathbb{R}^{n+s}, 0 \to \mathbb{R}^{n+r}, 0 \), a germ \( \phi: \mathbb{R}^s, 0 \to \mathbb{R}^r, 0 \), a sheer germ \( \varepsilon: \mathbb{R}^s, 0 \to \mathbb{R}, 0 \), such that \( \phi|_{\mathbb{R}^n \times 0} = 1 \), and \( \pi: \mathbb{R}^{n+r} \to \mathbb{R}^r \) is projection, if \( \pi: \mathbb{R}^{n+s} \to \mathbb{R}^r \) and \( g = f\phi + \varepsilon\pi \).

**Definition.** \((r,f)\) is said to be **universal** if, \( \forall (s, g) \exists \) a morphism, \((s, g) \to (r, f)\).

**Definition.** \((\phi, \phi, \varepsilon)\) is an **isomorphism** if it has an inverse. Note that this requires \( r = s \), and \( \phi \) and \( \phi^{-1} = \phi \) are diffeomorphism-germs, so \((\phi^1, \phi^{-1}, -c\phi^{-1})\) will do.

**Prolongation of a germ.** Given \( \eta \in \mathbb{R}^2 \), let \( z = j^k_1 \eta \). Choose a representative function of \( \eta \), \( e: \mathbb{R}^n, 0 \to \mathbb{R}, 0 \). \( \mathbb{R}^n \) operates on \( e \) by translation as follows. Given \( w \in \mathbb{R}^n \), define \( w(e): \mathbb{R}^n, 0 \to \mathbb{R}, 0 \)

\[
x \mapsto e(w+x) - e(w).
\]

Graph \( w(e) = \text{graph } e \) with origin moved to \((w, e(w))\).

Denote by \( j_1^k \eta \) the map obtained: \( \mathbb{R}^n, 0 \to \mathbb{R}, \eta \)

\[
w \mapsto \text{germ at } 0 \text{ of } w(e).
\]

Let \( j_1 \eta \) denote the germ at 0 of \( j_1^k \eta \) (we shall show this is unambiguous). \( j_1 \eta \) is called the **natural germ prolongation** of \( \eta \). \( j_1^k \eta = \pi_0 j_1 \eta \) is called the **natural k-jet prolongation** of \( \eta \), where \( \pi_0 \) is the usual projection \( m \to j^k \).
Lemma 6.2. (1) \( j_1^1 \eta \) and \( j_1^k \eta \) are uniquely determined by \( \eta \) (not by \( z \), necessarily), i.e. they are independent of the choice of \( e \).

(2) If \( \eta \) is \((k+1)\)-determinate, \( j_1^k \eta \) is the germ of an embedding \( \mathbb{R}^n, 0 \to J^k, z \).

(3) The tangent plane \( T_z(\text{Im } j_1^k \eta) \) lies in \( \pi \Delta(\Delta = \Delta(\eta)) \) transverse to \( \pi(m\Delta) \), and is spanned by \( \{ j^k \frac{\partial \eta}{\partial x_i} \} \).

Proof. If \( e \) and \( e' \) are 2 representatives of \( \eta \), then \( e = e' \) on \( N \), some neighborhood of \( 0 \) in \( \mathbb{R}^n \). \( \omega(e) = \omega(e') \) if \( w + x, w \in N \). So \( j_1^1 \eta \) is well defined (and clearly \( j_1^k \eta \) is too). (1) is proved. (2) follows using (3) and the definition of determinacy. (3). Clearly \( T_z(\text{Im } j_1^k \eta) \) is spanned by \( j^k \frac{\partial \eta}{\partial x_i} \) which are in \( \pi \Delta \). By the definition of \( \Delta \) (the ideal generated by \( \{ \frac{\partial \eta}{\partial x_i} \} \)), \( m\Delta \) and the space spanned by \( j^k \frac{\partial \eta}{\partial x_i} \) are transversal in \( \Delta \) (use Lemma 3.8). Quotient out by \( m^{k+1} \). Hence \( T_z(\text{Im } j_1^k \eta) \) is transverse to \( \pi(m\Delta) \) in \( \pi \Delta \).

We define the \( k \)-jet prolongation of an unfolding \((r,f)\) of a germ \( \eta \in m^2 \) in a similar way. Represent \( f \) by a function \( \tilde{f} : \mathbb{R}^{n+r}, 0 \to \mathbb{R}, 0 \). Let \( F \) be the germ at \( 0 \) of the map \( \mathbb{R}^{n+r}, 0 \to J^k, z \)

\[ (x', y') \to k \text{-jet at } 0 \text{ of the function } \]

\[ \mathbb{R}^n, 0 \to \mathbb{R}, 0 \]

\[ x \mapsto \tilde{f}(x + x, y') - \tilde{f}(x', y') \]

\( F \) is the \( k \)-jet prolongation of the unfolding \((r, f)\).
Definition. We say the unfolding \((r,f)\) is \(k\)-transversal if the germ \(F\) is transversal to the orbit \(zG_k^k\) in \(J^k\).

Let \(x_1, \ldots, x_n\) be coordinates for \(\mathbb{R}^n\) and \(y_1, \ldots, y_r\) be coordinates for \(\mathbb{R}^r\). Choose \(\tilde{f} \in f\), and then for each \(j = 1, \ldots, r\) we have a function \(\frac{\partial \tilde{f}}{\partial y_j}\) from \(\mathbb{R}^{n+r}\), 0 to \(\mathbb{R}\), \(\frac{\partial \tilde{f}}{\partial y_j}(0,0)\). Let \(\tilde{a}_j f\) be the germ at \(\tilde{a}_j f\) is in \(m\). \(v_f\) will denote the vector subspace of \(m\) spanned by \(\tilde{a}_1 f, \ldots, \tilde{a}_r f\). \(6.3\)

Lemma 6.4. An unfolding \((r,f)\) of a germ \(\eta\) is \(k\)-transversal
\(\Leftrightarrow m = \Delta + v_f + m^{k+1}\).

Proof. In \(J^k\), i.e. \(mod m^{k+1}\), the tangent to the orbit \(zG_k^k\) is \(m\Delta\) (Lemma 2.11), the tangent to the \(x\)-direction of \(F(=j^k_1\eta)\) is \(T_z(\text{im } j^k_1\eta)\), and these two are transverse in \(\Delta\) by Lemma 6.2 (3). The tangent to the \(y\)-direction of \(F\) is \(v_f\) (6.3). So \(F\) is transversal to \(zG_k = \Delta + v_f\) span \(m\) \(mod m^{k+1}\).

Corollary 6.5. Let \(\eta\) have finite determinacy and \(c = \text{cod } \eta\), then \(\exists\) an unfolding \((c,f)\), which is \(k\)-transversal \(\forall k > 0\).

Proof. Because \(\det \eta\) is finite, so is \(\text{cod } \eta = c\) finite by Lemma 3.1; by definition \(\text{cod } \eta = \dim m/\Delta (\Delta = \Delta(\eta))\). Choose \(u_1, \ldots, u_c \in m\) such that their images in \(m/\Delta\) form a basis for \(m/\Delta\). Define an unfolding \((c,f)\) by,

\(f: \mathbb{R}^n \times \mathbb{R}^c \to \mathbb{R}\)

Then \(\frac{\partial f}{\partial y_j}(x) = u_j(x)\), so \(\tilde{a}_j f = \frac{\partial \tilde{f}}{\partial y_j}(x) = \sum_{j=1}^{c} y_j u_j(x)\).

By the choice of \(\{u_j\}, \{\tilde{a}_j f\}\) span \(m/\Delta\). By 6.3 \(m = \Delta + v_f = \Delta + v_f + m^{k+1} \forall k > 0\). Now apply Lemma 6.4.
Lemma 6.6. Let $\eta$ have finite determinacy, with a universal unfolding $(r, f)$. Then $(r, f)$ is $k$-transversal $\forall k > 0$ and $r \geq \text{cod } \eta$.

Proof. Let $c = \text{cod } \eta$ and $(c, g)$ be the unfolding of Corollary 6.5, which is $k$-transversal $\forall k > 0$. By the definition of universality $\exists$ a morphism $(\phi, \tilde{\phi}, e): (c, g) \rightarrow (r, f)$. So $g(x, y) = f(\phi(0, y)) + e(y)$ where $(x, y) \in \mathbb{R}^n \times \mathbb{R}^c$,

by (6.1) $f(\phi(x, \tilde{\phi}y) + e(y)$ with $\phi(x) = \tau_y(\phi(x, y))$,

choosing $x_1', \ldots, x_n'$ and $y_1', \ldots, y_r'$ as coordinates for $\mathbb{R}^{n+r}$. Now we have

$$\frac{\partial g}{\partial y_j}(x, 0) = \sum_i \frac{\partial f}{\partial x_i}(\phi(0, x_i, \tilde{\phi}0)) \frac{\partial \phi}{\partial y_j}(x) + \sum_h \frac{\partial f}{\partial y_h}(\phi(0, \tilde{\phi}0)) \frac{\partial \phi}{\partial y_j}(0) + \frac{\partial e}{\partial y_j}(0).$$

$\phi_0 = \phi|\mathbb{R}^n \times 0 = 1$ and $\tilde{\phi}0 = 0$ by 6.1. Also $\frac{\partial \phi}{\partial y_j}(0) \in \mathbb{R}$. So the first sum is in $\Delta$, as $\frac{\partial f}{\partial x_1}(x, 0) = \frac{\partial n}{\partial x_1}(x)$, and the $h$th term in the second sum is $\frac{\partial f}{\partial y_h}(x, 0) \times \text{constant}$. Remember $\frac{\partial f}{\partial y_h}(x, 0) = \frac{\partial f}{\partial y_h}(0, 0) \in V_f^c$. So

$$V_g \subset \Delta + V_f.$$

Now $m = \Delta + V_g \forall k > 0$ by Lemma 6.4.

So $m \subset \Delta + V_f^c \forall k > 0$, i.e. $(r, f)$ is $k$-transversal $\forall k > 0$ by Lemma 6.4, $(\Delta, V_f^c \subset m)$. Also $r \geq \dim V_f \geq \dim m/\Delta = c$, follows at once.

Lemma 6.7. If $\eta$ is $k$-determinate and if $(r, f)$ and $(r, g)$ are $k$-transversal unfoldings of $\eta$, then they are isomorphic.

Proof. $(r, f)$ is $k$-transversal $\Rightarrow m = \Delta + V_f + m^{k+1}$ (Lemma 6.4) \begin{equation} (6.8) \end{equation}

Thus $m = \Delta + V_f^c$,

$$m \text{ is k-determinate} = m^{k+1} \subset m/\Delta \subset \Delta \text{ (Theorem 2.9)}$$

Let $\overline{\partial f}_j$ denote the image of $\partial f$ in $m/\Delta$. Then $(r, f)$ $k$-transversal means $\overline{\partial f}_j$ spans $m/\Delta$. $(r, f)$ and $(r, g)$ are isomorphic if $\exists$ a morphism $(\phi, \tilde{\phi}, e): (r, f) \rightarrow (r, g)$ where $\phi, \tilde{\phi}$ are diffeomorphisms. We write $f \cong g$.

Lemma 1. It suffices to prove Lemma 6.7 in the special case $\overline{\partial f}_j = \overline{\partial g}_j \forall j$.

Proof. We introduce a standard unfolding $(r, h)$ and show that $\exists h' \cong h$ such that $\overline{\partial f}_j = \overline{\partial g}_j$, $j = 1, \ldots, r$. By symmetry $\exists$ also $h'' \cong h$ such that $\overline{\partial f}_j = \overline{\partial g}_j$, $1 \leq j \leq r$. Assuming the special case of Lemma 6.7,

$$f \cong h' \cong h \cong h'' \cong g.$$
Choose \( u_1, \ldots, u_c \in \mathbb{R}^m \) such that \( \tilde{u}_1, \ldots, \tilde{u}_c \) form a base for \( \mathbb{R}^c / \mathcal{D} \), where \( c = \text{cod} \, \eta \), finite since \( \det \eta \) is finite. Define \( h: \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^{r-c} \times \mathbb{R} \)

\[
(x, v, w) \rightarrow \eta(x) + \sum_{j=1}^{r-c} v_{r-j} u_j(x) = \eta + v u,
\]

where \( v = (v_1 \ldots v_c) \), \( u = (u_1 \ldots u_c) \) are disconnected control coordinates, see below.

Now \( \overline{\partial}_j f = \sum_{h=1}^r a_{jh} \tilde{u}_h \), \( a_{jh} \in \mathbb{R} \). Denote the matrix \((a_{jh})\) by \( A \). \( A \) has rank \( c \) since \( \overline{\partial}_j f \) span \( \mathbb{R}^c / \mathcal{D} \). Choose a matrix \( B \) such that \( AB \) is nonsingular, where \( AB \) is,

\[
\begin{array}{cc}
\begin{array}{c}
c \\
r \end{array} & \begin{array}{c}
\mathbb{R}^c \\
r-c
\end{array} \\
A \\
B
\end{array}
\]

(not the matrix product)

Define \( \bar{\phi}: \mathbb{R}^r \rightarrow \mathbb{R}^c \times \mathbb{R}^{r-c} \), a linear isomorphism

\[
y \mapsto (yA, yB).
\]

This induces \( h': \mathbb{R}^{n+r} \xrightarrow{s = 1} \mathbb{R}^{n+r} \xrightarrow{h} \mathbb{R} \).

\((1 \times \bar{\phi}, 0): (r, h') + (r, h) = \langle x, y \rangle + (x, yA, yB) + \eta(x) + yAu \)

clearly an isomorphism, \( a_{jh} = \begin{cases} 
  u_j(x) & j \leq c, \\
  0 & j > c.
\end{cases} \)

So \( \overline{\partial}_h f = \sum_{h=1}^r a_{jh} \tilde{u}_h(x) = \overline{\partial}_j f \).

Lemma 2. \( m^E_{s n+s} \) = those germs in \( E_{n+s} \) vanishing on the \( \mathbb{R}^n \)-axis.

Proof. \( \leq: m_s \) is generated by \( \{y_j\} \) which vanish on the \( \mathbb{R}^n \)-axis, where \( x_1, \ldots, x_n \) are coordinates for \( \mathbb{R}^n \) and \( y_1, \ldots, y_s \) are coordinates for \( \mathbb{R}^s \).

\( \geq: \) Suppose the function \( \theta(x, y) \) vanishes on the \( \mathbb{R}^n \)-axis.

\[
\theta(x, y) = \left[ \theta(x, ty) \right]_{t=0}^{1} = \int_0^1 \frac{\partial \theta}{\partial t} (x, ty) dt = \int_0^1 \sum_{j} \frac{\partial \theta}{\partial y_j} (x, ty) y_j dt
\]

\[
= \sum_{j} \int_0^1 y_j \psi_j (x, y) \, dt,
\]

\( \psi_j \in E_{n+s} \).

The continuing proof of Lemma 6.7 now mimics the first half of

Theorem 2.9. Let \( E^t = (1-t)f + tg \). Then assuming \( \overline{\partial}_j f = \overline{\partial}_j g \),
\[ \frac{\partial E^t}{\partial j} = (1-t) \frac{\partial f}{\partial j} + t \frac{\partial g}{\partial j} = \frac{\partial f}{\partial j}. \] So \( E^t \) is \( k \)-transversal. For \( 0 \leq t \leq 1 \) we have a 1-parameter family of \( k \)-transversal unfoldings connecting \( f \) and \( g \).

Fix \( t_0 \), \( 0 \leq t_0 \leq 1 \).

**Lemma 3.** \( \exists \) an isomorphism \( (\psi^t, \varphi^t, \epsilon^t): (r, E^{t_0}) \rightarrow (r, E^t), \forall t \) in some neighborhood of \( t_0 \).

This implies Lemma 6.7 by the compactness and connectedness of \([0,1]\) (Cf. 2.9).

**Lemma 4.** \( \exists \) a germ \( \psi \) at \((0, t_0)\) of a map \( \mathbb{R}^{n+r} \times \mathbb{R}, 0 \times \mathbb{R} \rightarrow \mathbb{R}^{n+r}, 0 \).

1. \( \psi^{t_0} = 1 \) (so \( \frac{\partial \psi}{\partial t} = 1 \), and \( \epsilon^{t_0} = 0 \), and \( \forall t \) in a neighborhood of \( t_0 \),
2. \( \psi^t: \mathbb{R} \rightarrow \mathbb{R}, 0 \times \mathbb{R} \rightarrow \mathbb{R}^{n+r}, 0 \),
3. function \( \mathbb{R} \times \mathbb{R}, 0 \times \mathbb{R} \rightarrow \mathbb{R}^{n+r}, 0 \), such that

\( \psi^t \mathbb{R} \times \mathbb{R}, 0 \times \mathbb{R} \rightarrow \mathbb{R}^{n+r}, 0 \), and \( \epsilon^t \psi^t + \epsilon \mathbb{R} = \mathbb{R}^{n+r} \).

(i.e. \( E(x', y', t) + \epsilon(y, t) = E(x, y, t_0) \), where \( \psi^t(x, y) = (x', y') \).

**Lemma 4** = **Lemma 3** because the set of diffeomorphisms is open in the space of maps. (See proof of 2.9)

**Lemma 5.** We can replace (3) by

\[ \sum_{i=1}^{3n} \frac{\partial E^t}{\partial x_i} (x', y', t) \frac{\partial x_i}{\partial t} (x, y, t) + \sum_{j \neq 1}^{3n} \frac{\partial E^t}{\partial y_j} (x', y', t) \frac{\partial y_j}{\partial t} (y, t) + \frac{\partial E^t}{\partial y_1} (x', y', t) + \frac{\partial E^t}{\partial y_1} (y, t) = 0. \]

Differentiation of (3) with respect to \( t \) gives (4). Integration with respect to \( t \) from \( t_0 \) to \( t \) of (4) gives (3). (See 2.9)

**Lemma 6.** \( \exists \) a germ \( X \) at \((0, t_0)\) of a map \( \mathbb{R}^{n+r} \times \mathbb{R}, \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \), \( 0 \times \mathbb{R} \rightarrow \mathbb{R}^{n+r} \), \( 0 \),

1. \( \psi \mathbb{R} \times \mathbb{R}, 0 \times \mathbb{R} \rightarrow \mathbb{R}^{n+r}, 0 \),
2. \( \psi \mathbb{R} \times \mathbb{R}, 0 \times \mathbb{R} \rightarrow \mathbb{R}^{n+r}, 0 \),
3. function \( \mathbb{R} \times \mathbb{R}, 0 \times \mathbb{R} \rightarrow \mathbb{R}\), such that

\[ \sum_{i=1}^{3n} \frac{\partial E^t}{\partial x_i} (x, y, t) X_i (x, y, t) + \sum_{j \neq 1} \frac{\partial E^t}{\partial y_j} (x, y, t) Y_j (y, t) + \frac{\partial E^t}{\partial y_1} (x, y, t) + Z(y, t) = 0, \forall (x, y, t) \) in a neighborhood of \((0, t_0)\).
Proof that Lemma 6 = Lemma 5.

Let \((x', y') = \psi(x, y, t)\) be the unique solution of \[
\begin{align*}
\dot{x}' &= x(x', y', t), \quad x = x \text{ at } t = t_o \\
\dot{y}' &= y(y', t), \quad y = y \text{ at } t = t_o.
\end{align*}
\]

Let \(y' = \tilde{\psi}(y, t)\) for variables \(x, y, t\) in (5) and get (4).

\(\tilde{\psi}|_{R^n \times 0} = 1\) since \((x', y') = (x, 0)\) is a constant solution of

\[
\begin{align*}
X(R^n \times 0 \times R) &= 0 = \dot{x}', \\
Y(0 \times R) &= 0 = \dot{y}'.
\end{align*}
\]

We now choose a mixture. Let \(A\) be a free \(E_{r+1}\)-module on \((r + 1)\) variables (finitely generated), each \(a = (Y_1, \ldots, Y_r, Z)\), some \(Y_j, Z \in E_{r+1}\).

Let \(B\) be a free \(E_{n+r+1}\)-module on \(n\) variables, each \(b = x = (X_1, \ldots, X_n)\), some \(X_j \in E_{n+r+1}\). Let \(C\) be \(E_{n+r+1}\) (finitely generated).

\(\alpha: A \rightarrow C\) is given by \(\alpha a = \frac{3E}{3y}.Y + Z\); it is over \(*\) because it is linear in \(Y, Z\). (\(\pi\) is projection \(R^{n+r+1} \rightarrow R^{r+1}\))

\[
\begin{array}{c}
\begin{align*}
\alpha &\quad \Gamma \\
A &\quad \alpha \rightarrow C \\
E_{r+1} &\quad \Gamma \\
E_{n+r+1} &\quad \Gamma
\end{align*}
\end{array}
\]

\(\beta: B \rightarrow C\) is given by \(\beta x = \frac{3E}{3x}.X\). (Recall mixture of Chapter 5).

Lemma 7. \(C = \alpha A + \beta B + (\pi*_{n+r+1})C\).

Proof that Lemma 7 = Lemma 6. Apply Corollary 5.6 (to the Preparation Theorem) to give \(C = \alpha A + \beta B\). Then \(m_r C = \alpha(m_r A) + \beta(m_r B)\), where the \(E_r\)-module structures on \(C, A, B\) are induced by projection onto \(R^r\).

Now \(\frac{3E}{3t} = g - f\). And \(f|_{R^n \times 0} = n = g|_{R^n \times 0} (\forall t)\). So \(\frac{3E}{3t}\) vanishes on \(R^n \times 0 \times R\) in \(R^{n+r+1}\). By Lemma 2 \(\frac{3E}{3t} \in m_r C\), and so \(\frac{3E}{3t} \in \alpha(m_r A) + \beta(m_r B)\), i.e. \exists germs \(X \in m_r B, Y\) and \(Z \in m_r A\) such that

\(- \frac{3E}{3t} = \frac{3E}{3x}X + \frac{3E}{3y}Y + Z\), as germs. Lemma 6 follows applying Lemma 2 a few times.

Proof of Lemma 7. (And hence of Lemma 6.7) As \(E_r^x\) is \(k\)-transversal \(\forall t\), by (6.8) \(m_n = \Delta + V_{E_r^0}\). So \(E_n = \Delta + V_{E_r^0} + R\). Let \(\xi \in C\), and

\(\xi(x) = \xi(x, 0, t_0) \in E_{n+1}\). Then \(\xi(x) = \xi_{\xi} \frac{2n}{3x_1} X_1 + \xi_j \frac{2n}{3x_j} X_j + s\), where \(X_1 \in E_{n+1}, Y_j \in R\) and \(s \in R\).
Let \( \zeta(x,y,t) = \sum \frac{\partial E}{\partial y_i}(x,y,t)X_i(x,y,t) + \frac{\partial E}{\partial y_j}(x,y,t)Y_j(y,t) - \sum_j \frac{\partial E}{\partial y_j}(0,0,t)Y_j(0,t) + s. \) So \( \zeta = \frac{\partial E}{\partial x}X + \frac{\partial E}{\partial y}Y + Z \in \mathbb{R} \).

Now \( \zeta(x,0,t_0) = \xi(x) \) because \( E_{t_0}^{t_0} \mathbb{R}^n \times 0 = \eta \) and also

\[
\frac{\partial E}{\partial y_j} = \frac{E_{t_0}^{t_0}}{\partial y_j} \mathbb{R}^n \times 0 - \frac{\partial E}{\partial y_j} (0).
\]

So \( \xi - \zeta \) vanishes on the fibre \( \mathbb{R}^n \times 0 \times t_0 \).

By Lemma 2 \( \xi - \zeta \in (\#m_{rt+1})C \). Hence \( \xi \in \alpha \mathbb{A} + \beta B + (\#m_{rt+1})C \), proving Lemma 7.

Given an unfolding of \( \eta, (r,f), f: \mathbb{R}^{n+r}, 0 \rightarrow \mathbb{R}, 0 \), we introduce d disconnected controls as follows. Let \( g \) be the composition,

\[
\mathbb{R}^{n+r+d} = \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^d + \mathbb{R}^{n+r} \rightarrow \mathbb{R}
\]

\( (x,y,w) \mapsto (x,y) \mapsto f(x,y) = g(x,y,w) \).

We say \( (r+d,g) \) is \( (r,f) \) with d disconnected controls. Using the morphisms

\[
(1 \times \pi, \pi,0): (r+d,g) \rightarrow (r,f) \quad (1 \times 1,1,0): (r,f) \rightarrow (r+d,g),
\]

where \( \pi \) is the injection map, we see that \( (r,f) \) is universal \( \Rightarrow (r+d,g) \) is universal.

Clearly also if \( (r,f) \) is k-transversal so is \( (r+d,g) \).

**Theorem 6.9.** If \( \eta \) has finite determinacy, and has \( (r,f) \) and \( (r,g) \) as universal unfoldings, then they are isomorphic.

**Proof.** By Lemma 6.6, \( (r,f) \) and \( (r,g) \) are both k-transversal, \( \forall k > 0 \).

Choose some \( k \) such that \( \eta \) is k-determinate. Then Lemma 6.7 provides an isomorphism.

**Theorem 6.10.** If \( \eta \) is k-determinate, then an unfolding \( (r,f) \) is universal \( \Rightarrow \) it is k-transversal.

**Proof.** By Lemma 6.6.

Given a k-transversal unfolding \( (r,f) \) we must show that for any unfolding \( (s,g) \) (also of \( \eta \)), \( \exists \) a morphism \( (s,g) \rightarrow (r,f) \). If \( c = \text{cod} \eta \), choose \( u_1, \ldots, u_c \) spanning \( m/\Delta \) as in Corollary 6.5. Let \( h \) be the map

\[
\mathbb{R}^n \times \mathbb{R}^{s+c} \rightarrow \mathbb{R}
\]

\( (x,y,v) \mapsto g(x,y) + \sum_{j=1}^{c} v_j u_j(x) \)

so that \( (s+c,h) \) is a k-transversal unfolding of \( \eta \) by Corollary 6.5.
Let \( s + c + d = r + d' \), i.e. choose such integers \( d, d' \) (one can be zero).

Let \((s+c,d,h')\) be \((s+c,h)\) with \(d\) disconnected controls, and \((r+d',f')\) be \((r,f)\) with \(d'\) disconnected controls. Both will be \(k\)-transversal (as noted above), and we can apply Lemma 6.7 to show the existence of an isomorphism

\[
(\phi, \phi', \varepsilon) \quad \text{with} 
\begin{align*}
(1 	imes j_1, j_1, 0, l) &\xrightarrow{(s+c,h)} (s+c+d, h') \xrightarrow{(1 	imes j_2, j_2, 0, l)} (s+c+d, h') \\
(r+d', f') &\xrightarrow{(r, f), \pi_l} (r, f), \text{with } j_1, j_2 \text{ obvious injections, } \pi_l \text{ a projection.}
\end{align*}
\]

This is the required morphism.

**Theorem 6.11.** If \( \eta \) has finite determinacy, it has a universal unfolding \((c,f)\) where \( c = \text{cod } \eta \), and moreover \( c \) is the minimum dimension of any universal unfolding of \( \eta \).

**Proof.** By Corollary 6.5 a \(k\)-transversal unfolding \((c, f)\) exists with \( k \geq \det \eta \). \((c,f)\) is universal by Theorem 6.10. Now use Lemma 6.6. for minimality.

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**CHAPTER 7. CATASTROPHE GERMS.**

Let \( \eta \in \mathbb{M}^2 \), and suppose \( \eta \) has an unfolding \( f: \mathbb{R}^{n+r}, 0 \to \mathbb{R}, 0 \).

Represent \( f \) by a function \( \tilde{f}: \mathbb{R}^{n+r}, 0 \to \mathbb{R}, 0 \) and define \( M_f \) to be the subset of \( \mathbb{R}^{n+r} \) on which \( \frac{\partial \tilde{f}}{\partial x_1} = \ldots = \frac{\partial \tilde{f}}{\partial x_n} = 0 \). Let the function \( \tilde{X}_f \) be the composition \( \tilde{X}_f \in \mathbb{R}^{n+r} \) \( \xrightarrow{g} \mathbb{R}^r \). Observe that \( 0 \in M_f \) because \( \eta \in \mathbb{M}^2 \). So we can define \( X_f \) to be the germ at 0 of \( \tilde{X}_f \). \( X_f \) is called the catastrophe germ of \( f \).

**Lemma 7.1.** Let \( \eta \in \mathbb{M}^3 \) and \( \text{cod } \eta = c \). Then \( \exists \) a universal unfolding \((c,f)\) such that \( M_f \) is diffeomorphic to \( \mathbb{R}^c \). Then \( X_f \) is a germ at 0 of a map \( \mathbb{R}^c, 0 \to \mathbb{R}^c, 0 \).

**Proof.** \( \eta \in \mathbb{M}^3 = \Delta \subset \mathbb{M}^2 \). And so when choosing a base \( u_1, \ldots, u_c \) for \( \mathbb{M}/\Delta \), we can demand that \( u_j(x) = \begin{cases} x_j \text{ if } j \leq n \\ \text{a monomial of degree } \geq 2, \text{ if } n < j \leq c \end{cases} \).
Let \( f(x,y) = n(x) + \frac{c}{j=1} y_j u_j(x) \); \((c,f)\) is \( k\)-transversal \( \forall k > 0 \), and so is universal using Theorem 6.10 with \( k \geq \det n \). 

\[
\frac{\partial f}{\partial x_1} = \frac{\partial n}{\partial x_1} + y_1 + \frac{c}{j=1} y_j \frac{\partial u_j}{\partial x_1} = 0 \equiv M_f, \text{ i.e. } M_f \text{ is the subset of } \mathbb{R}^{n+c} \text{ where } \forall i=1,\ldots,n \text{. So } \psi \text{ is a map } \mathbb{R}_x^n \times \mathbb{R}_y^n \rightarrow \mathbb{R}_y^n. \text{ The graph of such a polynomial map is diffeomorphic to its source, and } M_f = \text{ graph of } \psi \subset \mathbb{R}_x^n \times \mathbb{R}_y^c = \mathbb{R}_x^n \times \mathbb{R}_y^{c-n} \times \mathbb{R}_y^n, \text{ so } M_f \cong \mathbb{R}^c.
\]

We remark that \( M_f \) is not a manifold in general. E.g. \( n = x^5, f = \frac{x^5}{5} + \frac{ax^3}{3} \). \( \frac{\partial f}{\partial x} = x^4 + ax^2 \), and for \((x,a) \in \mathbb{R}^2, M_f \) looks like:

\[\text{Diagram: UNLATTED SEMI-CIRCULAR ARROWED CURVE WITH ARROWED POINTED UPWARDS AT} x \rightarrow x\]

**Lemma 7.2.** Suppose \( n \) has finite determinacy, and \( n = q + p \), where \( q = x_1^2 + \ldots + x_p^2 \) and \( p \) is a polynomial in \( x_{p+1}, \ldots, x_n \) only, consisting of monomials of degree \( \geq 3 \). Suppose \((r,f)\) is a universal unfolding of \( p \). Then if \( g = q + f \), \((r,g)\) is a universal unfolding of \( n \) and \( X_f = X_g \).

**Proof.** By Lemma 6.6 \((r,f)\) is \( k\)-transversal \( \forall k > 0 \), and in particular for \( k \geq \det n, \) Lemma 6.4 gives \( m_\lambda = \Delta(p) + V_f + m^{k+1} \) which, with \( m^{k+1} \subset \Delta(p) \) (Theorem 2.9) gives \( m_\lambda = \Delta(p) + V_f \). Here \( \lambda = n - p \), and \( m_\lambda \) is the ideal of \( E_\lambda \) generated by \( x_1, \ldots, x_p \). Similarly \( m_p \) is the ideal of \( E_\lambda \) generated by \( x_1, \ldots, x_p \). \( m \) and \( E \) denote \( m_n \) and \( E_n \). Then

\[
 m_p E + m_\lambda E = m_p E + \Delta(p) E + V_{f-g}.
\]

Now \( m = m_p E + m_\lambda E \) and \( V_{f-g} = V_g \). Also \( \Delta(n) = (x_1, \ldots, x_p, \frac{\partial f}{\partial x_{p+1}}, \ldots, \frac{\partial f}{\partial x_n}) = m_p E + \Delta(p)E\).

So \( m = \Delta(n) + V_{g} = \Delta(n) + V_{g} + m^{k+1} \) for \( k \geq \det n \) and so by Lemma 6.4 and Theorem 6.10, \((r,g)\) is universal.

\[
\begin{align*}
\text{If } i & < p, \quad \frac{\partial g}{\partial x_i} = 2x_i \quad (= 0 \text{ for } M_g) \\
\text{If } i & > p, \quad \frac{\partial g}{\partial x_i} = \frac{\partial f}{\partial x_i} \quad (= 0 \text{ for } M_g)
\end{align*}
\]

\( g \) = 0 \times M_f.
We have \( X_f : M_f \subset 0 \times \mathbb{R}^{r+\lambda} \xrightarrow{\pi} R_r \xrightarrow{y} R^r = X_f = X_g.\)

\( X_g : M_g \subset R^0 \times \mathbb{R}^{r+\lambda} \xrightarrow{\pi} R^r \)

**Lemma 7.3.** Suppose \((r,f)\) and \((s,g)\) are 2 unfoldings of \(\eta\), and \(\exists\) a morphism \((\phi, \psi, \epsilon) : (s,g) \rightarrow (r,f)\). Then \(M_g = \phi^{-1}M_f\), and \(X_g\) is the pullback of \(X_f\) under \(\phi, \overline{\phi}\).

**Proof.** We have \(R^n \xrightarrow{\phi^y} R^n \xrightarrow{\phi^{-y}}\)

\(\xrightarrow{\psi^y}\) commuting.

\(\xrightarrow{-\epsilon^y}\) sheer

\(\xrightarrow{\overline{\phi^y}}\)

Giving,

\( T_x(\phi^y) \)

\(\xrightarrow{\text{iso}}\)

\( T_{\phi^{-y}}(R^n \overline{\phi^y}) \)

\(\xrightarrow{T_x(\phi^y)}\)

\(\xrightarrow{T_{\phi^{-y}}(\overline{\phi^y})}\)

\(\xrightarrow{1}\)

\(\xrightarrow{R}\)

\(\phi^0 = 1\), so \(\phi^y\) is a diffeomorphism for small \(y\), and \(T_x(\phi^y)\) is an isomorphism for small \(y\). \((x,y) \in M_g \Rightarrow T_x(g^y) = 0\) (definition of \(M_g\))

\(\Rightarrow T_{\phi^{-y}}(\overline{\phi^y}) = 0\) (diagram commutes)

\(\Rightarrow (\phi^{-y}, \overline{\phi^y}) \in M_f\) (definition of \(M_f\))

\(\Rightarrow \phi(x,y) \in M_f\), i.e. \(M_g = \phi^{-1}M_f\).

We have that \(\phi^{-1}M_f \xrightarrow{\phi} M_f\)

\(\xrightarrow{X_g}\)

\(\xrightarrow{X_f}\)

\(\overline{\phi}\)

commutes, completing the lemma.

Recall that if \(\theta_i\) is a germ \(M_i, p_i \rightarrow M'_i, p'_i\) where \(M_i, M'_i\) are \(C^m\) manifolds, \(i = 1, 2\), then \(\theta_1 \sim \theta_2 \Rightarrow \exists\) diffeomorphism-germs \(\delta_1, \delta_2\) such that

\(\xrightarrow{\theta_1}\)

\(\delta_1\)

\(\xrightarrow{\theta_2}\)

\(\delta_2\)

commutes.
Corollary 7.4. If \((\phi, \overline{\phi}, \epsilon)\) is an isomorphism, \(X_g \sim X_f\).

Proof. \(\phi, \overline{\phi}\) will be diffeomorphism-germs; the requisite diagram is at the end of Lemma 7.3.

Lemma 7.5. If \((r, g)\) and \((r, f)\) are universal unfoldings of an \(\eta\), of finite determinacy, then \(X_f \sim X_g\).

Proof. This follows from Theorem 6.9 and Corollary 7.4.

Lemma 7.6. If \(\eta\) has finite determinacy and \((s, g)\), \((r, f)\) are universal unfoldings of \(\eta\) with \(s > r\), then \(X_g \sim X_f \times \mathbb{R}^{s-r}\).

Proof. Let \((s, f')\) be \((r, f)\) with \(s-r\) disconnected controls. Then \((s, f')\) is universal, so that \(X_{f'} \sim X_g\) by Lemma 7.5. Also \(M_{f'} = M_f \times \mathbb{R}^{s-r}\),

\[
\begin{array}{c}
X_f' \downarrow \quad X_f \\
\mathbb{R}^{s-r} \downarrow \quad \mathbb{R}^{s-r}
\end{array}
\]

i.e. \(X_{f'} = X_f \times \mathbb{R}^{s-r}\).

Lemma 7.7. If \(\eta\) has finite determinacy and is right equivalent to \(\eta'\), and if \((r, f)\) and \((r, f')\) are respective universal unfoldings, then \(X_f \sim X_{f'}\).

Proof. We have \(\eta' = \eta_\gamma\) where \(\gamma \in G\). Let \(g = f(\gamma x)\):

\[
\begin{array}{c}
\mathbb{R}^{n+r} \xrightarrow{\gamma \times 1} \mathbb{R}^{n+r} \xrightarrow{f} \mathbb{R} \\
\mathbb{R}^{n+r} \downarrow \mathbb{R}^{n+r} \downarrow \\
\mathbb{R}^r \xrightarrow{1} \mathbb{R}^r
\end{array}
\]

This induces \(M_g \xrightarrow{*} M_f\). \(*\) is a diffeomorphism because \(\gamma\) is.

\[
\begin{array}{c}
X_g \xrightarrow{*} X_f \\
\mathbb{R}^r \xrightarrow{1} \mathbb{R}^r
\end{array}
\]

And so \(X_f \sim X_g\).

Now \(g|_{\mathbb{R}^n} \times 0 = f_\gamma|_{\mathbb{R}^n} \times 0 = \eta_\gamma|_{\mathbb{R}^n} \times 0 = \eta'|_{\mathbb{R}^n} \times 0\). So \((r, g)\) unfolds \(\eta'\), and \((r, g)\) is a universal unfolding because \((r, f)\) is, clearly. By Lemma 7.5, \(X_g \sim X_{f'}\). Hence \(X_f \sim X_{f'}\).

Theorem 7.8. If \(\eta \in m^2\) of finite determinacy has a catastrophe germ \(X_f\), then the equivalence class of \(X_f\) depends only upon the equivalence class of \(\eta\). Moreover it is uniquely determined by the essential coordinates of \(\eta\).
Proof. Denote the equivalence class of $X_f$ by $[X_f]$. $[X_f]$ is independent of the choices of: $n$ by Lemma 7.2, universal unfolding $f$ by Lemma 7.5, $r$ by Lemma 7.6, and of $\eta$ by Lemma 7.7. Lemma 7.2 shows that $[X_f]$ is uniquely determined by the essential coordinates (of $\eta$).

Corollary 7.9. There are only 11 catastrophe germs if we restrict to those $\eta$ of codimension $\leq 5$.

Proof. If there are more than 2 essential coordinates of $\eta$, i.e. rank $n - 3$, then Lemma 4.11 shows $\operatorname{cod} \eta > 5$. So restrict to $n \leq 2 + \eta$ and $-\eta$ give the same $M_f$ and hence the same $X_f$. So the (distinct) essential coordinates giving distinct $[X_f]$'s are: $x^3, x^4, x^5, x^6, x^7, x^3 + xy^2, x^3 - xy^2, x^2y + y^4, x^3 + y^4, x^2y + y^5, x^2y - y^5$. These are the 11.

Definition. If $[X_f]$ is one of the 11 of Corollary 7.9 then $[X_f]$ is called an elementary catastrophe.

Corollary 7.10. If $\eta$ has finite determinacy and $(r,f)$ is a universal unfolding of $\eta$, where $r \leq 5$, then $[X_f]$ is an elementary catastrophe.

Proof. By Corollary 4.7 and the Reduction Lemma 4.9, $\eta \sim q + p$ and $p \in \mathbb{N}^3$. Also Lemma 6.6 tells us that $r \geq c = \operatorname{cod} \eta$, so that $c \leq 5$ and $p$ is one of the germs written out in the proof of Corollary 7.9 (cod $p \leq 5$ and consult Diagram 4.1). By Lemma 7.1 applied to $p \not\exists$ a standard universal unfolding $(c,g)$ of $p$ such that $X_g$ is a germ $\mathbb{R}^c, 0 \to \mathbb{R}^c, 0$. Now use Lemma 7.2 to provide a universal unfolding $(c,f')$ of $\eta$ such that $X_{f'} = X_g$. By Lemma 7.6 $X_f \sim X_{f'} \times 1^{r-c} = X_g \times 1^{r-c} : \mathbb{R}^r, 0 \to \mathbb{R}^r, 0$. Now $[X_f]$ is an elementary catastrophe by choice, and so in a certain obvious sense $[X_f]$ is an elementary catastrophe too. This is the same sense in which we said that "$[X_f]$ is independent of the choice of $r$" by Lemma 7.6" in Theorem 7.8.
CHAPTER 8. GLOBALISATION.

We shall first define the Whitney $C^\infty$ topology on the space of $C^\infty$ functions $\mathbb{R}^{n+r} \to \mathbb{R}$, denoted by $F$.

Given $f: \mathbb{R}^{n+r} \to \mathbb{R}$ define a map $f^k: \mathbb{R}^{n+r} \to \mathbb{R}$ (where, recall, $f^k_p = \frac{E}{n^m k^{k+1}}$) which sends $p \in \mathbb{R}^{n+r}$ to the $k$-jet at $0$ of the function $R^{n+r} \to \mathbb{R}$

$w \mapsto f(p+w)$.

Then given a function $\mu: \mathbb{R}^{n+r} \to \mathbb{R}_+$ we define a basic neighborhood of $0$ as $V^k_\mu = \{ f \in F : \forall p \in \mathbb{R}^{n+r}, |f^k_p| < \mu p \}$. For $f \in F$, $V^k_\mu(f)$ = \{g $\in F$ : $\forall p \in \mathbb{R}^{n+r}, |f^k_p - g^k_p| < \mu p \} is a basic open neighborhood of $f$. These form a base for a topology, called the Whitney $C^k$-topology. The topology with a base of all such $V^k_\mu(f), \forall k \geq 0$, is called the Whitney $C^\infty$ topology. $F$ will be assumed to have this topology.

**Theorem 8.1.** If $r \leq 5$, then $\exists$ an open dense set $F^*_r \subset F$ such that if $f \in F^*_r$, then $\tilde{X}_f$ has only elementary catastrophes as singularities (and these are already classified), and $M_f$ is an $r$-manifold.

We shall need several lemmas to prove the theorem.

Given $f \in F$, $\varepsilon > 0$, and $X \subset \mathbb{R}^{n+r}$, define an open set, $V^k_{\varepsilon, X}(f)$ as \{g $\in F$ : $\forall p \in X, |f^k_p - g^k_p| < \varepsilon \}, so that $\varepsilon$ controls all partial derivatives of order $\leq k$ on $X$. It is open because it is the union of all $V^k_\mu(f)$ for $\mu: \mathbb{R}^{n+r}, X \to \mathbb{R}_+(0, \varepsilon)$.

**Definition.** Let $J$ be a manifold. A stratification $Q$ of $J$ is a decomposition into a finite number of submanifolds $\{Q_i\}$ such that,

1. $\partial Q_j = \overline{Q}_i - Q_j$ is the union of $Q_j$ of lower dimension.
2. If $z \in Q_j \subset Q_i$ and a submanifold $S$ of $J$ is transverse to $Q_j$ at $z$, then $S$ is transverse to $Q_i$ in a neighborhood of $z$. (8.2)

Following the construction of the $k$-jet prolongation of an unfolding $(r,f)$ in Chapter 6, given $f \in F$ we let $F$ be the induced map
\[ \mathbb{R}^{n+r} \rightarrow J^k \]

\[ p = (x,y) \mapsto k\text{-jet at } 0 \text{ of the function } \mathbb{R}^n, 0 \rightarrow \mathbb{R}, 0 \]

\[ x' \mapsto f(x+x', y) - f(x, y). \]

Given \( X \subset \mathbb{R}^{n+r} \) we let \( F^X = \{ f \in F : \forall p \in X, F \text{ is transversal to } Q \text{ at } p \} \), where \( Q \) is either a submanifold or a stratification of \( J^k \).

Open Lemma 1. (OL1) If \( X \subset \mathbb{R}^{n+r} \) is compact and \( f \in F^X \), then \( \exists \) a neighborhood \( \bigcup_{\varepsilon, X}^{k+1}(f) \subset F^X \). (i.e. \( F^X \) is \( C^{k+1} \)-open.)

**Proof.** Given \( p \in X, F \) is transversal to \( Q \) at \( p \). By continuity and (8.2) (if appropriate), \( F \) is transversal to \( Q \) in a neighborhood of \( p \), in particular in a compact neighborhood \( N \) of \( p \). This remains true for all sufficiently small changes of \( F \) and \( T_F \) on \( N \), and so for all sufficiently small changes in \( f^{k+1} \) on \( N \). Because \( N \) is compact, \( \exists \varepsilon > 0 \) such that \( \bigcup_{\varepsilon, N}^{k+1}(f) \subset F^N \). Cover compact \( X \) by a finite number of such \( N_i \), and let \( \varepsilon = \min \varepsilon_i \). Then \( \bigcup_{\varepsilon, X}^{k+1}(f) \subset \bigcup_{\varepsilon, N_i}^{k+1}(f) \subset F^X \).

Open Lemma 2. (OL2) Let \( X = \bigcup X_i \), a countable union of disjoint compact \( X_i \) with neighborhoods \( Y_i \). Then \( F^X \) is \( C^{k+1} \)-open.

**Proof.** Choose a \( C^\omega \) bump function \( \beta_i : \mathbb{R}^{n+r} \rightarrow [0,1] \), which takes values \( 1 \) on \( X_i \) and \( 0 \) outside \( Y_i \), for each \( i \). Let \( \beta_0 = 1 - \sum_{i=1}^\infty \beta_i \). Given \( f \in F^X \), then \( f \in F^{X_i} \). So \( \exists \varepsilon_i > 0 \) such that \( \bigcup_{\varepsilon, X}^{k+1}(f) \subset F^{X_i} \).

Let \( \mu = \beta_0 + \sum_{i=1}^\infty \varepsilon_i \beta_i \). Then \( \bigcup_{\varepsilon, X}^{k+1}(f) \subset \bigcup_{i=1}^\infty \bigcup_{\varepsilon, X_i}^{k+1}(f) \subset F^X \).

Density Lemma 3. (DL3) \( \forall p \in \mathbb{R}^{n+r} \) and \( \forall f \in F, \exists \) a compact neighborhood \( N \) of \( p \) in \( \mathbb{R}^{n+r} \) and \( \exists \) a neighborhood \( V \) of \( f \in F \) such that \( F^N \) is \( C^\omega \)-dense in \( V \).

**Proof.** Having chosen \( N \) and \( V \) we must show that \( \forall g \in V, \exists \) an arbitrarily
$C^\infty$-close $h \in F^N$. Now $F^N = \{ f : F \text{ is transversal to } Q \text{ in } N \}$, where $Q$ is (first) a submanifold of $J^k$. Given $f$ let $z = F(0,0)$ and w.l.o.g. $p = (0,0)$.

Case 1. $z \in Q$. This is hard.

Case 2. $z \in \overline{Q}-Q$. This does not occur if $Q$ is closed, but we need this case where $Q$ is one stratum of a stratification.

Case 3. $z \notin \overline{Q}$. This is trivial.

Case 1

Case 2

Case 3

Case 3. Pick $N$ such that $F^N \cap Q$, and $V$ such that $\forall g \in V$, $GN \cap Q$. Then $g \in F^N$, trivially. So $V \subset F^N$, and $h = g$ will do.

Case 1. Let $q$ be the codimension of $Q$ in $J^k$. Choose a product neighborhood $B$ of $z$ in $J^k$ and a projection $\theta : B \rightarrow \mathbb{R}^q$ such that $\theta^{-1}0 = B \cap Q$. Now $J^k$ is spanned by monomials in $x_1, \ldots, x_n$. Of these choose $u_1, \ldots, u_q$ spanning the $q$-plane transverse to $Q$ at $z$. Let $e_w$ be the function $\mathbb{R}^n \rightarrow \mathbb{R}$

$$x \mapsto \sum_{i=1}^q w_i u_i(x), \text{ where } w_i \in \mathbb{R} \text{ form } w \in \mathbb{R}^q,$$

and so $e : \mathbb{R}^q \times \mathbb{R}^n \rightarrow \mathbb{R}$. As usual $e$ induces $E : \mathbb{R}^q \times \mathbb{R}^n, 0 \rightarrow J^k, 0$

$$ (w,x) \mapsto k\text{-jet of the function } \mathbb{R}^n, 0 \rightarrow \mathbb{R}, 0. $$

Then $(F+E) : \mathbb{R}^q \times \mathbb{R}^{n+r}, 0 \rightarrow J^k, z$

$$x' \mapsto e_w(x+x') - e_w(x). \quad (w,x,y) \mapsto F(x,y) + E(w,x),$$

is convenient notation. Now choose a compact neighborhood $W \times N$ of $0$ in $\mathbb{R}^q \times \mathbb{R}^{n+r}$ such that $(F+E)(W \times N) \subset B$. 
Choose a neighborhood $V$ of $f$ in $F$ such that $\forall g \in V$, $(G+E)(W \times N) \subset B$. This is possible because $W \times N$ is compact and $B$ is open.

**Sublemma 1.** The matrix of partial derivatives with respect to $W$ at $0$ of the composite map $W \times N, 0 \xrightarrow{(F+E)} B, z \longrightarrow \mathbb{R}^q, 0$ is a nonsingular matrix.

**Proof.** $(F+E)(w,0,0) = F(0) + E(w,0) = z + E(w,0)$.

$E(w,0)$ is the $k$-jet at $0$ of $\mathbb{R}^n \rightarrow \mathbb{R}$

$$x' \rightarrow e_w(x') - e_w(0) = \sum_{i=1}^{q} w_i u_i(x').$$

So $(F+E)(w,0,0) = z + \sum_{i=1}^{q} w_i u_i$, which is in the $q$-plane transverse to $Q$ at $z$ by construction. Hence $\theta(F+E)$ is transversal to $0$ in $\mathbb{R}^q$.

**Corollary.** By choosing $W, N, V$ sufficiently small, the matrix of partial derivatives of the composition map $\phi: W \times N \xrightarrow{(G+E)} B \longrightarrow \mathbb{R}^q$ with respect to $W$ is nonsingular at $(w,p) \forall (w,p) \in W \times N, \forall g \in V$.

**Proof.** By continuity from Sublemma 1.

**Sublemma 2.** (Implicit Function Theorem) Given $W^q \times N^{n+r} \rightarrow \mathbb{R}^q$ with the matrix of partial derivatives of $\phi$ with respect to $W$ nonsingular $\forall (w,p) \in W \times N$, then $\exists$ a unique $C^m$ map $\psi: N^{n+r} \rightarrow W^q$ such that $\phi^{-1}0 = \text{graph} \psi$.

By Sard's Theorem choose a regular value $w^*$ of $\psi$, arbitrarily small. Let $\psi^*$ be the map: $N^{n+r} \rightarrow \mathbb{R}^q$

$$p \mapsto \psi(w^*, p).$$
If \( N = \text{east}, W = \text{north}, \)
\( R^q = \text{vertical}, \phi = \text{height} \)
above sea level, and \( \psi = \text{coastline,} \) then Sublemma 2
says that \( \exists \text{ a coastline.} \)

Sublemma 3. \( \phi^* \) is transversal to 0.

Proof. Suppose \( \phi^* p = 0. \) Let \( v = (w^*, p), \in \text{graph } \psi \subset W \times N \) as \( \psi v = 0. \)

Consider \( T_v(W, N) \xrightarrow{T_v \phi} T_0 R^q. \) \( T_v \phi \) is surjective by the Corollary to Sublemma 1.

\[ R^q \times R^{n+r} \xrightarrow{T_v \phi} R^q \]

Let \( K \) be the kernel of \( T_v \phi, K = (T_v \phi)^{-1} 0. \) \( \dim K = (q+n+r) - q = n+r, \) by surjectivity.

Because \( w^* \) is a regular value of \( \psi, \) the map \( K^{n+r} \subset R_w^q \times R^{n+r} \xrightarrow{T_0} R_w^q \) is
surjective. So \( K^{n+r} \) meets \( R^{n+r} \) transversely; \( \dim K^{n+r} \cap R^{n+r} = (n+r) + (n+r) - (n+r+q) = n + r - q. \)

Consider \( T_p(n^{n+r}) \xrightarrow{T_p \phi^*} T_0(R^q). \)

Kernel of \( T_p \phi^* = \text{kernel of } T_v \phi \cap R^{n+r} = K^{n+r} \cap R^{n+r}, \) and so is of dimension
\( n + r - q. \) Hence \( T_p \phi^* \) is surjective, and \( p \) is a regular point of \( \phi^*. \) Thus
\( \phi^* \) is transversal to 0.
We have chosen $N$ and $V$. Choose now a bump function 
$eta: \mathbb{R}^{n+r} \to [0,1]$ such that $\beta = 1$ on $N$, $\beta = 0$ outside a compact neighborhood of $N$. Given $g \in V$, choose $w^*$ (dependent upon $g$), a regular value of $\psi$, $w^*$ arbitrarily small. Define $h: \mathbb{R}^{n+r} \to \mathbb{R}$ by 
$h(x,y) = g(x,y) + \sum_{i=1}^{d} w^* u_i(x) \beta(x,y)$. Then by Sublemma $3$, $\partial H = \phi^*$ is transversal to $0$ on $N$. So $H$ is transversal to $Q$, and $h \in \mathbb{R}^N$. Given an arbitrary $C^\infty$-neighborhood $V^\delta_{\mu}(g)$, we can reduce the partial derivatives of $w^* u\beta$ of order $\leq \ell$, below $v$, on a compact neighborhood of $N$ by making $w^*$ sufficiently small. So $h \in V^\delta_{\mu}(g)$. $h$ is arbitrarily $C^\infty$-close to $g$.

This completes Case 1 of DL3.

Case 2. $z \in Q \subset 3Q' = \overline{Q'} - Q'$ where $\{Q\}$ form a stratification. Given $g \in V$ we must show $\exists h$ such that $H$ is transversal to both $Q$ and $Q'$ (and any other incident strata) at the same time, on $N$. Given $g$, find $h$ as in Case 1 arbitrarily $C^\infty$-close such that $H$ is transversal to $Q$ on $N$. Automatically by (8.2) $H$ is transversal to $Q'$ at all points in a compact neighborhood of $z$ in $B$.

Choose a product neighborhood $B'$ of $Q' \cap (B-L)$, and a map 
$\theta': B' \to \mathbb{R}^{q'}$ such that $\theta'^{-1}Q = Q' \cap (B-L)$, where $q'$ is the codimension of $Q'$ in $J^x$. Find now $h' \in \mathbb{R}^N$ arbitrarily $C^\infty$-close to $h$ so that

(a) $\partial H'$ remains transversal to all points of $\partial L$, and

(b) $\partial H'$ becomes transversal to $0$ in $\mathbb{R}^{q'}$ by Case 1 for $Q'$.

Then $H'$ is transversal to $Q$ and $Q'$ on $N$. By induction, $H'(s)$ is transversal to the stratification because there are only a finite number ($s+1$, say) of strata through $z$, by (8.2). Then $h(s) \in \mathbb{R}^N$ is arbitrarily $C^\infty$-close to $g \in V$.

Density Lemma 4. (DL4) If $X \subset \mathbb{R}^{n+r}$ is compact, then $F^X$ is $C^\infty$-dense in $F$.

Proof. Given $f \in F$, cover $X$ by a finite number of $N_i$ given by DL3. Let $V = \bigcap_{i=1}^{d} V_i$. Then $F_{N_i}^X$ is $C^\infty$-open by OLL (because $C^{k+1}$-open) and is $C^\infty$-dense (by DL3) in $V_i$. So $F_{N_i}^X$ is $C^\infty$ open dense in $V$. Now $F^X = \bigcap_{i=1}^{d} F_{N_i}^X$ is $C^\infty$-dense in $F$. 


open dense in \( V \). So \( F^X \) is dense in \( V \), i.e. \( \forall f \in F, \exists V \text{ such that } F^X \text{ is dense in } V \).

Therefore \( F^X \) is dense.

Density Lemma 5. (DL5) Let \( X = \bigcup X_i \) as in OL2, then \( F^X \) is \( C^w \)-dense.

Proof. Given \( f \in F \) and given a basic \( C^w \)-neighborhood \( V^\mu_\mu(f) \), we want \( g \in V^\mu_\mu(f) \cap F^X \). Let \( \{ \beta_i \} \) be as in OL2. For each \( i \) choose \( \varepsilon_i > 0 \) such that \( h \in V^\mu_{\varepsilon_i,Y_i} = \beta_i h \in V^\mu_\mu \). (This is possible by the boundedness of the derivatives of order \( \leq k \) of \( \beta_i \) on \( Y_i \)). By DL4, choose \( f_i \in V^\mu_{\varepsilon_i,Y_i}(f) \cap F^{X_i} \). Define

\[
g = \beta_0 (f + \sum_{i=1}^\infty \beta_i f_i). \quad \text{Then } g = f \text{ outside } \bigcup Y_i. \quad \text{On } Y_i, \quad g = (1-\beta_i)f + \beta_i f_i = f + \beta_i (f_i - f). \quad \text{Now } f_i - f \in V^\mu_{\varepsilon_i,Y_i} \text{ by choice, so } \beta_i (f_i - f) \in V^\mu_{\mu}. \quad \text{Meanwhile } g = f_i \text{ on } X_i. \quad \text{But } F_i \text{ is transversal to } Q \text{ on } X_i, \text{ and so } g \text{ is also transversal to } Q \text{ on } X_i. \quad \text{Therefore } g \in \cap F^{X_i} = F^X.
\]

So \( g \in V^\mu_\mu(f) \cap F^X \) as required.

The result of DL5 can also be proved by showing that \( F \) with the Whitney \( C^w \) topology is a Baire space, but the proof is longer.

Lemma 6. \( \mathbb{R}^{n+r} \) is \( C^{k+1} \)-open and \( C^w \)-dense in \( F \).

Proof. Choose \( X, X' \) each as in OL2 such that \( \mathbb{R}^{n+r} = X \cup X' \). Then \( F^{n+r} = F^X \cap F^{X'} \), each \( C^{k+1} \)-open and \( C^w \)-dense, by OL2 and DL5 respectively.

Proof of Theorem 8.1. We describe the stratification \( Q \) of \( J^7 \) resulting from the classification of orbits in \( I^7 \) in Chapter 4.

(a) the open subspace \( J^7 - I^7 \),

(b) \( n + 1 \) orbits of jets of stable germs in \( m^2 \) of codimension 0 in \( I^7 \),

(c) the orbits of jets of germs in \( m^2 \) of codimension 1, 2, 3, 4 and 5 in \( I^7 \),

(d) the strata of the algebraic variety of jets of germs in \( m^2 \) of codimension \( \geq 6 \) in \( I^7 \).

These come directly from Diagram 4.1.

Because \( \Sigma_6^7 \), i.e. (d), is of codimension \( n + 6 \) and we are not interested in its internal structure, we shall let \( Q \) be the stratification (a),
(b), (c) of \( J^7 - E_6^7 \). The strata are the \( \tau_c^7 \) for \( c = 0, 1, 2, 3, 4 \) and 5, together with \( J^1 - 0 \) (this last making \( J^7 - \Sigma_6^7 \), rather than \( I^7 - \Sigma_6^7 \)).

**Lemma 7.** \( Q \) satisfies (8.2) (and hence is a stratification).

Let \( F_o = \{ f \in F : F \text{ misses } \tau_6^7 \} \), i.e. where \( F \) is transversal to \( \tau_6^7 \) if \( r \leq 5 \) (\( F \) maps \( R^{n+1} \) into \( J^7 \)). By general position, \( F_o \) is \( C^\infty \)-open (and hence \( C^8 \)-open) and \( C^\infty \)-dense. Let \( F_\times = F_o \cap \tau_6^7 \), then \( F_\times \) \( \{ f \in F : F \text{ is transversal to } Q \text{ and } \tau_6^7 \} \), and is \( C^8 \)-open and \( C^\infty \)-dense, using Lemma 6.

Suppose \( f \in F_\times \). Then \( F \) is transversal to \( m^2 / m^8 = I^7 \), since \( I^7 \) is the union of strata of \( Q \) and \( \tau_6^7 \). So \( F^{-1}(I^7) \) is of codimension \( n \), and of dimension \( r \). (\( I^7 \) is of codimension \( n \) in \( J^7 \)). Now \( F^{-1}(I^7) \) is the set of points \( (x,y) \in R^{n+1} \) such that the 1-jet of \( x' \mapsto f(x+\epsilon x',y) - f(x,y) \) is zero, i.e. such that \( \frac{\partial f}{\partial x_1}(x,y) = ... = \frac{\partial f}{\partial x_n}(x,y) = 0 \). So \( F^{-1}(I^7) \) is precisely \( M_f \) and \( M_f \) is an \( r \)-manifold. Suppose that \( X_f : M_f \rightarrow R^7 \) has a singularity at \( (x,y) \). Let \( n \) be the germ at \( (x,y) \) of \( f|R^n \times y \). W.l.o.g. \( (x,y) = (0,0) \), so \( n \in m^2 \). The germ of \( f \) at \( (0,0) \) is a 7-transversal unfolding of \( n \), because \( f \in F_\times \) and so \( F \) is transversal to the orbit \( (j^7 n)G^7 \), contained in some stratum.

**Lemma 8.** If \( (r,f) \) is a 7-transversal unfolding of \( n \in m^2 \), and \( r \leq 5 \), then \( (r,f) \) is a universal unfolding.

**Proof.** By Lemma 6.4, \( m = \Delta + V_f + m^8 \). \( \Delta = \Delta(n) \). So \( \dim m/(\Delta + m^8) \leq \dim V_f \leq r \leq 5 \), using (6.3). In the notation of Theorem 3.3, \( \tau(j^7 n) \leq 5 \). But \( \text{cod } n = \tau(j^7 n) \leq 5 \), by (3.5), and so by Lemma 3.1, \( \text{det } n \leq 7 \), and we can apply Theorem 6.10 to show that \( (r,f) \) is universal.

By Corollary 7.10 we now know that if \( X_f \) is the germ at \( (0,0) \) of \( \tilde{X}_f \), then \( [X_f] \) is an elementary catastrophe.

So the only singularities of \( \tilde{X}_f \) are elementary catastrophes.

**Proof of Lemma 7.** (Which we have used to complete Theorem 8.1). \( Q \) has a finite number of strata, each of which is a submanifold by Corollary 4.3. (There are in fact 7 strata.) Condition (1) of (8.2) follows from Corollary 3.6 since each \( \Sigma_6^7 \) is closed (Theorem 3.3). Note that \( \tau_c^7 \) now refers to the closure
in $J^7 - \Sigma^7_0$.

Condition (2): Let $Q_1$, $Q_2$ be strata, $z_1 \in Q_1 \subset \partial Q_2$, and $S$ a submanifold of $J^7 - \Sigma^7_0$ transverse to $Q_1$ at $z_1$. Then $S$ is transverse to $z_1 G^7$ at $z_1$.

Write $\alpha$ for the $C^\infty$ map $J^7 \to C^\infty(G^7, J^7)$. $\alpha(z_1)$ is now transversal to $S$ in a neighborhood $U$ of the identity $e$. Spanning, and hence transversality, is an open property, so $\exists$ an open neighborhood $V$ of $\alpha(z_1)$ in $C(G^7, J^7)$ and a neighborhood $U_1$ of $e$ (perhaps smaller than $U$) so that $\beta \in V$ implies $\beta$ is transversal to $S$ in $U_1$. $\alpha^{-1}(V)$ is open and contains $z_1$, and if $z \in \alpha^{-1}(V)$, $\alpha(z)$ is transversal to $S$ in $U_1$; in particular $z G^7$ is transverse to $S$ at $z$. But $Q_2$ is the finite union of such orbits $z G^7$. Hence $S$ is transverse to $Q_2$ in $\alpha^{-1}(V)$, a neighborhood of $z_1$.

Thus condition (2) is satisfied, completing the proof of Lemma 7.

CHAPTER 9. STABILITY.

Given $f \in F_s$, let $X_f : M_f \to \mathbb{R}^r$ be induced by projection. (See Chapter 1) We have to show that $X_f$ is locally stable at all points of $M_f$.

Definition. $X_f$ is locally stable at $(x_0, y_0) \in M_f$ if given a neighborhood $N$ of $(x_0, y_0)$ in $\mathbb{R}^{n+r}$, $\exists$ a neighborhood $V$ of $f$ in $F_s$, such that given $g \in V$, $\exists (x_1, y_1)$ in $N \cap M_g$ such that $X_f$ at $(x_0, y_0)$ is locally equivalent to $X_g$ at $(x_1, y_1)$.

Let $\tilde{f}, X_{\tilde{f}}$ denote the germs of $f$, $X_f$ at $(x_0, y_0)$ and $\tilde{g}, X_{\tilde{g}}$ the germs of $g$, $X_g$ at $(x_1, y_1)$. Then $X_{\tilde{f}}, X_{\tilde{g}}$ agrees with the notation in Chapter 7, and we also have that

\[(9.1) \quad X_{\tilde{f}} \sim X_{\tilde{g}} \Rightarrow X_f \approx X_g \text{ at } (x_0, y_0) \text{ is locally equivalent to } X_g \text{ at } (x_1, y_1).

Theorem 9.2. If $r \leq 5$ and $f \in F_s$, then $X_f$ is locally stable at each point of $M_f$. 

Proof. $f$ induces $F: \mathbb{R}^{n+r} + J$ as at the beginning of Chapter 8. Let $(x_0, y_0)$ be in $M_f$, and $F(x_0, y_0) = z_0$. We suppose we are given a neighborhood $N$ of $(x_0, y_0)$. Since $f \in F_\ast$, $F$ is transversal to $z_0^7$ at $z_0$; hence we can choose a disc $D^q$ with centre $(x_0, y_0)$ contained in $N$, where $q$ is the codimension of $z_0^7$ in $J^7$, whose image under $F$ intersects $z_0^7$ transversely at $z_0$, and so that $F|_{D^q}$ is an embedding. $F(D^q)$ will then have intersection number $1$ with $z_0^7$. If $F$ is perturbed slightly to $G$, $G(D^q)$ will still be a $q$-disc whose intersection number with $z_0^7$ is still $1$. I.e. $\exists$ an open neighborhood $V_o$ of $f$ in $F$ with this property for $g \in V_o$. Write $V = V_o \cap F_\ast$. Given $g \in V$, $G$ is transversal to $z_0^7$ and we may choose $(x_1, y_1) \in D^q$ such that $G(x_1, y_1) = z_1 = G(D^q) \cap z_0^7$. Then $z_1$ and $z_0$ are in the same orbit and are right equivalent as germs $\mathbb{R}^n, 0 \rightarrow \mathbb{R}, 0$.

Let $f_0(x, y) = f(x+x_0, y+y_0) - f(x_0, y_0)$ and $g_1(x, y) = g(x+x_1, y_1) - g(x_1, y_1)$ define $f_0$ and $g_1: \mathbb{R}^{n+r}, 0 \rightarrow \mathbb{R}, 0$. Then $z_0 = j^7(f_0|_{\mathbb{R}^n \times 0})$ and $z_1 = j^7(g_1|_{\mathbb{R}^n \times 0})$. Note that $F(\mathbb{R}^{n+r})$ is the same point-set as $F_0(\mathbb{R}^{n+r})$ and so $F_0$ is transversal to $z_0^7$ and $(r, \hat{x}_0)$ is a $k$-transversal unfolding of the germ $z_0$; so we can apply Lemma 8 in Chapter 8 (similarly for $\hat{g}_1$). As $r \leq 5$ the proof of this lemma gives that $z_0$ (and so also $z_1$) is finitely determined as a germ. The result of the same lemma tells us that $\hat{z}_0$ and $\hat{z}_1$ are also universal unfoldings of germs $z_0$, $z_1$ respectively. Now apply Lemma 7.7 which says $X_{f_0} \sim X_{g_1}$ (germs at $(0, 0)$ of $X_{f_0}, X_{g_1}$).

Now $M_f$ is merely a translate of $M_{f_0}: M_f = M_{f_0} + (x_0, y_0)$.

And so

$$X_{f}(x, y) = X_{f_0}(x-x_0, y+y_0) + y_0.$$
Similarly $X_{\hat{g}} \sim X_{\hat{g}_1}$.

Hence $X_{\hat{f}} \sim X_{\hat{f}_0} \sim X_{\hat{g}} \sim X_{\hat{g}_1}$. This completes Theorem 9.2.

(Observe that $(x_0, y_0) \in M_{\hat{f}}$ and $M_{\hat{f}} = F^{-1}(1^7)$ so that $z_0$ and $z_0G^7 \in I^7$.

Then $z_1 \in I^7$ and $(x_1, y_1) \in M_{\hat{g}} = G^{-1}(1^7)$, i.e. $(x_1, y_1) \in N \cap M_{\hat{g}}$ as required.)

Remark. This is a result about local stability. It would be interesting and useful to have a similar global stability result.
REFERENCES


