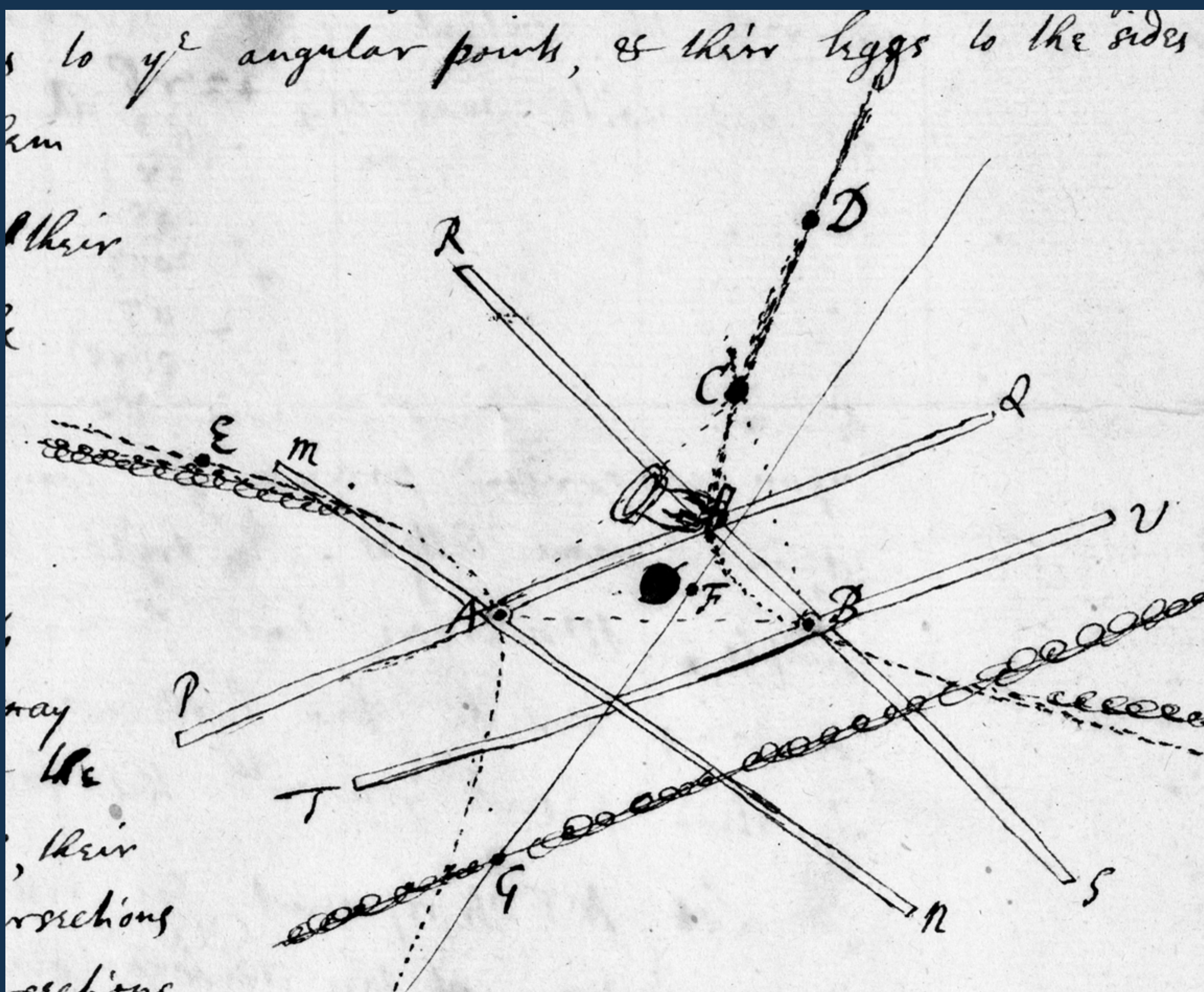
LONDON
MATHEMATICAL
SOCIETY
EST. 1865

NEWSLETTER

Issue: 477 - July 2018



TRENDS IN DIOPHANTINE APPROXIMATION

NEWTON AS A MATHEMATICIAN

MINIMISERS FOR NONLOCAL ENERGIES

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Charity registration number: 252660

COVER IMAGE

Image courtesy of Cambridge University Library.
See page 17 for its mathematical details.

ISSN: 2516-3841 (Print)
ISSN: 2516-385X (Online)
DOI: 10.1112/NLMS

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Invited Lecture Series: 9-13 July, Warwick (tinyurl.com/ybz5oq7s)

Society Meeting at ICM: 7 August, Rio de Janeiro (tinyurl.com/y8yxplq6)

Noether Celebration: 11 September, London (tinyurl.com/ycj98buu)

Popular Lectures: 19 September, Birmingham (tinyurl.com/hu58wjk)

Bond Review: Knowledge Exchange in the Mathematical Sciences

A review was recently undertaken by Philip Bond on Knowledge Exchange in the Mathematical Sciences. His report, entitled *The Era of Mathematics* and launched on 26 April this year, contains some major recommendations.

The report outlines in considerable detail the importance of mathematics and its impact across a very wide range of disciplines, technologies and industries. It emphasizes the need for improved infrastructure to facilitate effective communication and interaction between mathematicians and potential users. It recognises the need for top-level mathematicians within academia who are keen to engage with government, industry and wider research challenges. It also highlights the current shortage of such individuals and makes a strong case for increased investment in the discipline, with emphasis on impact but at the same time assuring commitment to fundamental research.

In addition, the report makes an extremely strong case for a very significant increase in funding for the Mathematical Sciences, and outlines various mechanisms for the development of suitably skilled mathematicians and for strengthening the national infrastructure for knowledge exchange and engagement between mathematicians and the discipline's many users. The full report can be found at tinyurl.com/yc8ykx4m.

Shortly after the launch, The Council for the Mathematical Sciences (CMS) met with mathematicians from the review board to discuss the report and its recommendations. Broadly welcoming the report, CMS has released a statement in response which can be found at tinyurl.com/yby8865c.

In particular, there was strong support for the establishment of an Academy for the Mathematical Sciences, while at the same time being mindful of the need for very careful consideration of the membership and remit of such a body and its relationship to existing learned societies. The meeting also considered how the other recommendations of the report might be implemented. Further discussions will be ongoing in the coming months.

Members are encouraged to examine the report.

Caroline Series
LMS President

LMS Prizes 2018

The winners of the LMS Prizes in 2018 were announced at the Society Meeting on Friday 29 June 2018. The Society extends its congratulations to these winners and thanks to all the nominators, referees and members of the Prizes Committee for their contributions to the Committee's work this year.

PROFESSOR KAREN VOGTMANN of WARWICK UNIVERSITY is awarded the PÓLYA PRIZE for her profound and pioneering work in Geometric Group Theory, particularly the study of automorphism groups of free groups.

PROFESSOR FRANCESCO MEZZADRI of the UNIVERSITY OF BRISTOL is awarded a FRÖHLICH PRIZE for his profound and wide-ranging contributions to random matrix theory and its applications.

PROFESSOR DR MARC LEVINE of THE UNIVERSITY OF DUISBURG-ESSEN is awarded a SENIOR BERWICK PRIZE in recognition of his paper: "A comparison of motivic and classical stable homotopy theories", *J. Topol.* 7 (2014), no.2, 327–362.

PROFESSOR CAUCHER BIRKAR of CAMBRIDGE UNIVERSITY is awarded a WHITEHEAD PRIZE in recognition of his outstanding research in higher dimensional algebraic geometry, most prominently his recent groundbreaking finiteness results on Fano varieties and Mori fibre spaces.

DR ANA CARAIANI of IMPERIAL COLLEGE LONDON is awarded a WHITEHEAD PRIZE for her important contributions to the Langlands programme.

DR HEATHER HARRINGTON of the UNIVERSITY OF OXFORD is awarded a WHITEHEAD PRIZE for her outstanding contributions to mathematical biology which have generated new biological insights using novel applications of topological and algebraic techniques.

PROFESSOR VALERIO LUCARINI of the UNIVERSITY OF READING is awarded a WHITEHEAD PRIZE for his work applying the ideas and methods of statistical physics to the theory and modelling of climate dynamics, and for his leadership in the field of mathematics applied to climate science.

DR FILIP RINDLER of the UNIVERSITY OF WARWICK is awarded a WHITEHEAD PRIZE for his solutions to fundamental problems on the border between the theory of partial differential equations, calculus of variations and geometric measure theory.

DR PÉTER VARJÚ of the UNIVERSITY OF CAMBRIDGE is awarded a WHITEHEAD PRIZE for his deep and groundbreaking contributions to analysis and probability on algebraic structures.

PROFESSOR JEREMY GRAY of the OPEN UNIVERSITY and the UNIVERSITY OF WARWICK is awarded the HIRST PRIZE and will be invited to give the associated HIRST LECTURESHIP for his research and books on the history of mathematics, especially differential equations and geometry in and around the nineteenth century.

DR LOTTE HOLLANDS of HERIOT-WATT UNIVERSITY is awarded an ANNE BENNETT PRIZE in recognition of her outstanding research at the interface between quantum theory and geometry and of her leadership in mathematical outreach activities.

LMS Invited Lecture Series 2019

In May 2019, Professor Søren Asmussen (Aarhus University) will visit the International Centre for Mathematical Sciences, Edinburgh, and give a condensed course on advanced topics in life insurance mathematics.

The course is based on inhomogeneous Markov models, with special emphasis on connections to financial mathematics and stochastic control. The required background is essentially masters-level probability, including a basic knowledge of martingales and stochastic calculus. Some familiarity with elementary differential equation theory is also useful. The ten-hour course will be held from 20 to 24 May 2019. See lms.ac.uk/events/lectures/invited-lectures for more information about the Invited Lectures.

Success Stories Project Launched

What does it mean to be a successful mathematician, and what is involved in a successful mathematical career? These are the questions the new LMS *Success Stories* project asks.

Aimed at school-age students, the project launched in May 2018 aims to exhibit the rich and varied paths to which a degree in mathematics can lead. It emphasises and celebrates the diversity of mathematical careers, and the success of mathematicians of all kinds. The *Success Stories* are available at lms.ac.uk/success-stories, and printable posters will be made available for use in schools.

The webpages will be updated regularly with new stories — if you have an interesting story to tell, please contact success.stories@lms.ac.uk.

Global Survey of Mathematical, Computing and Natural Scientists

The 2018 *Global Survey of Mathematical, Computing and Natural Scientists* is now open. The survey, managed by the International Mathematical Union (IMU) and funded mainly by the International Council of Scientific Unions (ICSU), aims to explore social dynamics in scientific fields by asking a large number of scientists and practitioners to share their interests, experiences and challenges. Analysis of the survey responses will allow comparisons to be made across regions, countries, scientific disciplines, sector of employment, age and gender. The 2018 survey forms part of ICSU's Gender Gap in Science project and its results will be used to help inform interventions by ICSU and member unions to increase participation in STEM fields, especially for women.

The survey is inviting responses from anyone working in mathematical, computing and natural sciences, at all levels including students. It is open until 31 October 2018, and can be found at tinyurl.com/y7t6ztbf.

Mathematicians Honoured by the Royal Society

The Royal Society has announced the election of several mathematicians as Fellows. The Fellowship of the Royal Society is made up of eminent scientists, engineers and technologists from or living and working in the UK and the Commonwealth. Fifty new Fellows and ten Foreign Members were announced this year.

Newly elected mathematicians include Professor Alexander Dawid (University of Cambridge), Professor Nancy Reid (University of Toronto), Professor Geordie Williamson (University of Sydney) and Professor Daniel Wise (McGill University). Other elected Fellows included computer scientist Professor Peter O'Hearn (University College London) and cryptographer Professor Adi Shamir (The Weizmann Institute of Science) who was elected a Foreign Member.

In other Royal Society news, Professor Nicholas Higham FRS (University of Manchester) has been awarded a Royal Society Research Professorship, its premier research award. Professor Higham aims to develop a new generation of numerical linear algebra algorithms that exploit evolving computer architectures. Six Research Professorships were awarded this year.

The Learned Society of Wales

The Learned Society of Wales has elected as Fellows Professor Biagio Lucini (Swansea University), an LMS member whose main research field is particle physics, and Professor Idris Eckley (Lancaster University), whose research is in multiscale methods in statistics. Fields medallist Professor Sir Vaughan F.R. Jones (Vanderbilt University) was also elected as an Honorary Fellow.

The Learned Society of Wales was set up in 2010 with the aim of recognising and representing excellence, and promoting scholarship and research in Wales. The LMS congratulates the new Fellows.

Other News of Members

INGRID DAUBECHIES, former IMU President, was awarded the William Benter Prize in Applied Mathematics. She is the first female recipient of the prize.

PETER GIBLIN has been awarded an OBE in the Queen's Birthday Honours.

JAMES HIRSCHFELD was awarded an Euler Medal by The Institute of Combinatorics and its Applications. This award recognizes distinguished lifetime career contributions to combinatorial research.

MICHAEL RUZHANSKY has won the 2018 Ferran Sunyer i Balaguer Prize, jointly with Durvudkhan Suragan, for the monograph *Hardy inequalities on homogeneous groups (100 years of Hardy inequalities)*.

SIMON TAVARÉ FRS, the immediate past President of the LMS, was elected to the USA's National Academy of Sciences as a Foreign Associate.

POLICY DIGEST

Two New Maths Hubs to be Created

The Department for Education (DfE) announced in May that schools in the North of England will receive funds from a £6 million investment with the aim of improving mathematics teaching across the region. £1.75 million of this funding will be used to create two new "Hubs" in Central Lancashire and Cheshire. The DfE writes that Hubs will "help spread best teaching practice and improve local pupils' knowledge, understanding and enjoyment of mathematics". Details at tinyurl.com/yaxktu46.

New Executive Chair of EPSRC

Science Minister, Sam Gyimah, has announced that Professor Lynn Gladden, CBE, FRS, FREng has been selected to be the next Executive Chair of the Engineering and Physical Sciences Research Council (EPSRC). Professor Gladden will take up the role in October in succession to Professor Philip Nelson who will step down at the end of September. Professor Lynn Gladden is currently Shell Professor of Chemical Engineering at the University of Cambridge. Details at tinyurl.com/y6v6djf9.

Educational Disadvantage: how does England compare?

New analysis by the Education Policy Institute (EPI) and the UCL Institute of Education (IOE) examines how disadvantaged students in England compare with those in other countries. The key findings for mathematics show that for the performance of disadvantaged students in England versus other countries:

- The performance of disadvantaged students in England ranks in the lower half of developed countries — standing at 25 out of 44 nations. Under the new GCSE grades, the average mathematics grade of disadvantaged students in England is 3.8 (lower than the current pass grade of 4).
- England's disadvantaged students lag behind several other Western nations including Estonia, Canada, the Netherlands and Ireland — achieving around a third of a grade lower (on average). Disadvantaged students in Asian nations of Macao, Singapore, Hong Kong, Taiwan and Japan are even further ahead — with England performing around half a grade lower in mathematics.

- England is marked by a long tail of underperformance amongst its disadvantaged students. Just 1 in 10 disadvantaged students in England achieve a high score in GCSE maths of grade 7 to 9 (A–A* under the old grading system). Nearly twice as many disadvantaged pupils in Singapore achieve this grade.
- Similarly, as many as 4 in 10 disadvantaged students in England fail to reach the new GCSE "standard" pass mark of a grade 4 in mathematics.

The full report is available at tinyurl.com/ybu7kloz.

Review of Post-18 Education and Funding

The independent panel appointed to inform the government's Review of Post-18 Education and Funding launched a call for evidence in March. The panel sought views from all interested parties on the four areas it has been asked to consider:

- Choice: identifying ways to help people make more effective choices between the different options available after 18, so they can make more informed decisions about their futures.
- Value for money: looking at how students and graduates contribute to the cost of their studies, to ensure funding arrangements across post-18 education in the future are transparent and do not stop people from accessing higher education or training.
- Access: enabling people from all backgrounds to progress and succeed in post-18 education, while also examining how disadvantaged students receive additional financial support from the government, universities and colleges.
- Skills provision: making sure we have a post-18 education system that is providing the skills that employers need.

The call for evidence closed on 2 May 2018. More information is available at tinyurl.com/y9z6gvsn.

Note: items included in the Policy Digest are not necessarily endorsed by the Editorial Board or the LMS.

LMS Council Diary: A Personal View

After what hopefully was a very relaxing and fruitful Easter break, Council congregated again at De Morgan House on Friday 13 April. The meeting began with an update on the President's activities since the last Council meeting, and then moved swiftly onto a lively debate on the advantages and disadvantages of having a regional sounding title for a national Society. Although nothing was decided, it became clear that further and wider discussion would be necessary.

Gwyneth Stallard then reported from the Early Career Research Committee, providing an update on discussions regarding the Early Careers Fellowship Scheme (formerly Postdoctoral Mobility Grants), and it was decided that a broader discussion was needed to determine in which form such a scheme could best support researchers just finishing their doctorates.

We heard a report from the Education Committee which included a discussion on the Advisory Committee for Mathematics Education (ACME). It was agreed in principle to support the hosting of a part-time support position at De Morgan House for the four Contact Groups, should a funding bid be successful. We also heard that a consultation on the Teaching Excellence Framework (TEF) was due shortly, and that the Education Committee would prepare a response for discussion.

The President then continued with a report from the first meeting of the IT Resources Committee, and presented the Annual Report from Prizes Committee.

The Publications Secretary reported on the next stage of the Publications Strategic Review. This included plans for some changes to the editorial process for the *Bulletin, Journal, Proceedings* and *Transactions* which it is hoped will speed up the time it takes to reach decisions to accept or reject submitted papers.

The Treasurer presented the half-year financial review for 2017–18, as well as operational plans for 2018–19. It was agreed to purchase video conferencing equipment for the Hardy Room, which would also provide the Society with the ability to live-stream events.

After hearing further updates from the Women in Mathematics Committee, the Standing Orders Review Group, and the Nominating Committee, Council agreed 19 applications for membership to be proposed to the Society meeting to be held on 25 May 2018. We finished the meeting on a lighter note, discussing the possibility of having a celebration in 2019 marking the 21st anniversary of the Society's occupancy of De Morgan House. We agreed that this would be a good way of documenting how the Society had developed, and would present a good opportunity for thanking De Morgan House staff.

Brita Nucinkis

REPORTS OF THE LMS

Report: LMS Meeting in Honour of Maryam Mirzakhani

This ordinary meeting of the LMS, to honour Professor Maryam Mirzakhani (1977–2017), occurred at the University of Warwick on 22 March 2018. It formed part of a week-long workshop on Teichmüller Dynamics, which itself was part of the 2017–18 EPSRC/Warwick Symposium on Geometry, Topology and Dynamics in Low Dimensions.

Mirzakhani was a world leader in Teichmüller dynamics; her presence at our workshop was greatly missed. Unlike the workshop, which took place in the Zeeman Building (Mathematics Institute), the LMS meeting was held in a massive lecture theatre in a new building near the centre of campus. The meeting consisted of LMS business, a pair of mathematical talks, tea, and a wine reception. Several members (including myself) signed the Membership Book. There was an audience of just over 75 members and visitors, and included many who were not participants in the rest of the workshop.

Professor Caroline Series (President of the LMS) presided during the first part of the meeting. She discussed how to join the LMS, details of the previous meeting, and details of the next meeting, to occur at the University of Leicester on Monday 4 June 2018.

Dr Alex Wright (Stanford) spoke on *Mirzakhani's Computation of Weil-Petersson Volumes and Intersection Numbers*, and Professor Anton Zorich (Paris) spoke on *Mirzakhani's Count of Simple Closed Geodesics*. These two talks together gave a lovely and accessible introduction to the thesis work of Maryam Mirzakhani, a masterful combination of ideas from analysis, geometry, topology, and combinatorics applied to the study of the moduli space of surfaces.

Dr Wright's talk explained Mirzakhani's remarkable computation of the Weil-Petersson volume in the simplest example of the moduli space of genus one, once-punctured hyperbolic surfaces using the famous McShane identity. Wright then discussed how she generalized the techniques to determine the volume of all moduli spaces of Riemann surfaces. He also discussed her proof of Witten's Conjecture concerning intersection numbers of tautological classes on moduli spaces of Riemann surfaces, emphasising the connection she established to Weil-Petersson volumes.

Professor Zorich explained how Mirzakhani used McShane's identity to compute integrals over moduli space and in particular, how she counted the "average" number of simple closed geodesics of bounded length. He presented one of her particularly beautiful results: the asymptotic frequencies of simple closed geodesics of different topological types do not depend on the hyperbolic metric and are explicitly computable.

Saul Schleimer
University of Warwick

Report: LMS Education Day 2018

Colleagues from across the sector gathered at De Morgan House on 1 May for the LMS Education Day 2018. The theme of the event was the potential imminent disruption of mathematics curricula in HE by the introduction of measures such as the Teaching Excellence Framework (TEF). Mathematics departments across the UK are considering ways to scrutinise and improve the way courses are structured and delivered. The LMS Education Day provided an opportunity for attendees to discuss the opportunities and challenges that such changes may bring about.

The day's morning session featured talks from experts with experience working on curriculum transformation and innovation. Barrie Cooper (University of Exeter), Peter Rowlett (Sheffield Hallam University) and Jane White (University of Bath) gave presentations focusing on measures aimed at reforming and improving curricula. There was also an update from Mary McAlinden (University of Greenwich) on the recent consultation for subject-level TEF ratings.

The afternoon session began with the attendees forming groups to discuss four questions central to the day's theme, and discussions on each group's conclusions.

The first group considered what a new mathematics curriculum should look like. The view was taken that better communication to applicants was needed in order to highlight what to expect from courses at different institutions. Core modules, with some degree of flexibility, should be used, with the aim of developing skills as well as teaching content.

Asked how a new curriculum should be delivered, the second group emphasised the importance of active learning in addition to lecturing. The third group, discussing assessment, argued for more flexibly-constructed exams to cater to more students' needs. The fourth group summarised a strategy for effectively transforming the curriculum: via setting outcomes, considering implications for departments, and deciding on the delivery of broad or deep subject coverage.

The Education Day concluded with a lively debate on whether mathematics degrees are already fit for purpose in the 21st century. Discussions focused on the importance of a coherent curriculum and the contribution to students' personal development, provided by general purpose mathematics education, versus the need to prepare for modern technologies and industries such as programming and artificial intelligence. Comments were invited from attendees at the end of the debate.

The Education Secretary, Kevin Houston, closed the Education Day by praising the strong ideas that were produced and discussed. The topic of the 2019 Education Day is likely to be the Teaching Excellence Framework measures currently under consultation.

George Ross
LMS Publications and Communications

Report: Harmonic Analysis and PDEs Workshop

Participants gathering outside Aston Webb Great Hall, University of Birmingham

This workshop was a one day meeting of the UK Harmonic Analysis and PDEs Research Network held on 19 April 2018 at the University of Birmingham. This network was funded by an LMS Scheme 3 grant, and received partial support from ERC. Currently, it includes Birmingham, Edinburgh, Madrid and Warwick as its nodes.

Around 20 people attended the workshop, including a good number of postgraduate students from the other nodes of the network. There were four talks, delivered by Michael Lacey (Georgia Institute of Technology), Giuseppe Negro (Instituto de Ciencias Matemáticas), Luz Roncal (Basque Center for Applied Mathematics), and Julien Sabin (Université Paris-Sud). The talks focused on exciting new developments in continuous and discrete harmonic analysis and the analysis of PDEs, including sharp Fourier restriction theory, extension problems, and sparse domination. They were typically followed by several questions and animated discussion.

This stimulating and exceptionally warm day ended with an informal dinner in the historic Jewellery Quarter district in Birmingham.

Susana Gutierrez, Alessio Martini,
Diogo Oliveira e Silva, Maria Carmen Reguera
University of Birmingham

Records and Proceedings at LMS meetings
Ordinary Meeting in Honour of Maryam Mirzakhani, 22 March 2018

The meeting was held at the University of Warwick as part of a workshop on *Teichmüller Dynamics* (held 19-23 March 2018). Over 50 members and visitors were present for all or part of the meeting.

The meeting began at 2.00 pm with The President, Professor Caroline Series FRS, in the Chair.

No members were elected to membership.

Three members signed the Members' Book and were admitted to the Society during the meeting, in addition the President extended an invitation to those present to look through the Members' Book during the tea break.

Professor John Smillie introduced the first lecture, given by Professor Alex Wright (Stanford) on *Mirzakhani's Computation of Weil-Petersson Volumes and Intersection Numbers*.

After tea, Professor John Smillie introduced the second lecture, given by Professor Anton Zorich (Paris) on *Mirzakhani's Count of Simple Closed Geodesics*.

The President thanked the speakers for their talks, and further extended her thanks to the local organisers for holding such an interesting and touching meeting in honour of Maryam Mirzakhani.

Professor Corinna Ulcigrai thanked the speakers, and invited those present to attend a wine reception and Society Dinner, which was held at the Zeeman Building.

Trends in Diophantine Approximation

DEMI ALLEN

What is Diophantine approximation? What type of problems does it tackle? We give an overview of this subject and some hints as to why the subject has a broad appeal both within the field of pure mathematics and beyond.

Foundations of Diophantine approximation

As early as the ancient Greeks and Chinese, rational approximations to π were being used in astronomy to accurately predict the position of planets and stars. These days, among other applications, rational numbers are still used to approximate irrationals in computer systems since computers cannot deal with the infinite. Diophantine approximation, named after Diophantus of Alexandria, in short, attempts to quantify how well real numbers may be approximated by rational numbers.

Life of Diophantus

While relatively little is known of the life of Diophantus, some of what we do know is embedded in the following 6th century riddle:

'Here lies Diophantus,' the wonder behold.
Through art algebraic, the stone tells how old:
'God gave him his boyhood one-sixth of his life,
One twelfth more as youth while whiskers grew rife;
And then yet one-seventh ere marriage begun;
In five years there came a bouncing new son.
Alas, the dear child of master and sage,
After attaining half the measure of his father's life chill fate took him.
After consoling his fate by the science of numbers for four years, he ended his life.'

For any real (rational or irrational) number, say x, we can always find a rational number which is as close as we like to x. When x is irrational, in order to get closer, the denominator of our rational approximation will inevitably increase. A similar phenomenon can be observed in higher dimensions, in which case we are interested in approximating vectors in $\mathbb{R}^n$ with ones which only contain rational entries. One of the main topics of interest in Diophantine approximation is studying the trade-off between the closeness (quality) of the rational approximations and the denominator required (i.e., the complexity of the approximation).

It is worth noting that when we refer to "rationals" here we merely mean fractions of integers which may or may not be in their lowest form, unless otherwise explicitly specified.

Approximating reals by rationals

Given any real number x and any natural number q, it is not too hard to see that we can always find a rational number with denominator q such that

$$\left|x - \frac{p}{q}\right| < \frac{1}{q}.$$

A more sophisticated theorem, due to Dirichlet from 1842, tells us the following.

Dirichlet's Theorem. *For any real number x there are infinitely many rationals $\frac{p}{q}$ such that*

$$\left|x - \frac{p}{q}\right| < \frac{1}{q^2}. \tag{1}$$

Pursuing statements of this type, the best result is a theorem of Hurwitz from 1891.

Hurwitz's Theorem. *For any real number x there are infinitely many rationals $\frac{p}{q}$ such that*

$$\left|x - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}. \tag{2}$$

As soon as one tries to replace $\frac{1}{\sqrt{5}}$ with something smaller, Hurwitz's Theorem becomes false. At least, it no longer holds for *all* real numbers x. This leads us naturally to the question: *what can be said if we do replace* $\frac{1}{\sqrt{5}}$ *with something smaller or, indeed, if we replace the right-hand side of* (2) *with a more general function of* q*?*

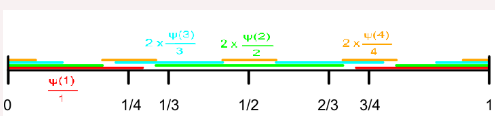
Well-approximable points

To address this question, suppose that we take some general non-negative real function, say ψ, which is defined on the natural numbers. A set which lies at the heart of modern-day Diophantine approximation is the set of real numbers x for which

$$\left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q}$$

has infinitely many rational solutions $\frac{p}{q}$. We denote this set of real numbers by $W(\psi)$ and say that the points in this set are ψ*-well approximable*. See below for a pictorial interpretation of the set $W(\psi)$.

Pictorial representation of $W(\psi)$



Part of the construction of the set $W(\psi)$.

Take the real number line and imagine there is an interval of length $\frac{2\psi(1)}{1}$ centred at each integer. Next, suppose that there are intervals of length $\frac{2\psi(2)}{2}$ centred at each rational number with denominator 2 (that is, at all multiples of $\frac{1}{2}$). Continue in this way and suppose that there are intervals of length $\frac{2\psi(q)}{q}$ centred at all rational points with denominator q. For a fixed q, let us call the corresponding collection of intervals W_q. Then $W(\psi)$ is the set of points which lie in infinitely many of the sets W_q.

Broadly, in Diophantine approximation, we are interested in the "size" and structure of $W(\psi)$ and its various generalisations. In understanding the structure of these sets, we might be interested in whether certain numbers or classes of numbers are in $W(\psi)$ for a given function ψ. For example, in 1955, Klaus Roth showed that if $\psi(q) = q^{-(1+\varepsilon)}$ then there are no irrational algebraic numbers in $W(\psi)$ for any choice of $\varepsilon > 0$ (a real number is an *algebraic number* if it is the root of a polynomial with rational coefficients). This discovery played a large part in Roth being awarded a Fields medal in 1958 and, in fact, he is not the only mathematician to have received such significant recognition for work in Diophantine approximation.

For the remainder of this article, we will focus on discussing the branch of Diophantine approximation known as *metric* Diophantine approximation. This is concerned with studying the "size" of sets such as $W(\psi)$.

Khintchine's Theorem

The first notion of "size" by which we may attempt to measure $W(\psi)$ is *Lebesgue measure*. Lebesgue measure is essentially the usual measure of length, area, volume, etc. which we are familiar with. We write $\lambda(X)$ to denote the Lebesgue measure of a set X of real numbers. Loosely speaking, we say that a set of real numbers has FULL Lebesgue measure if a randomly chosen real number lies in that set with probability 1. Conversely, we say that a set of real numbers has ZERO Lebesgue measure if a randomly chosen real number lies in that set with probability 0. In 1924, Khintchine proved that if the function ψ is monotonic then the Lebesgue measure of $W(\psi)$ is determined via the following elegant criterion.

Khintchine's Theorem. *If* ψ *is monotonic, then*

$$\lambda(W(\psi)) = \begin{cases} ZERO & \text{if} \quad \sum_{q=1}^{\infty} \psi(q) < \infty\,, \\ FULL & \text{if} \quad \sum_{q=1}^{\infty} \psi(q) = \infty\,. \end{cases}$$

By now, this classical result has been extended in numerous directions and further extensions are still being explored. As well as extending Khintchine's Theorem to higher dimensions, one may ask questions such as what is the size of $W(\psi) \cap K$ for a general set K? For higher dimensional analogues of $W(\psi)$, the problem of understanding their intersection with a general set K has recently attracted a lot of attention in the case where the set K is a curve, a manifold or belongs to certain classes of fractals. In order to tackle such questions, and to better understand the size of such sets, one employs tools and concepts from various areas of mathematics including fractal geometry, dynamical systems, and ergodic theory.

Fractal geometry

Concepts from fractal geometry naturally come into play in Diophantine approximation since the sets we are typically interested in are somewhat fractal-like. As an example, compare the pictorial/geometric interpretation of $W(\psi)$ given above with the construction of the middle-third Cantor set, a canonical example of a fractal set.

The middle-third Cantor set

To construct the middle-third Cantor set, start with the unit interval and remove the middle third. Call the resulting collection of intervals C_1. Now, remove the middle third from all the intervals in C_1 and call the resulting set C_2. Continue in this manner forever, letting C_{n+1} be the collection of intervals obtained after removing the middle-third from all of the intervals in C_n. The set we are left with is the middle-third Cantor set.

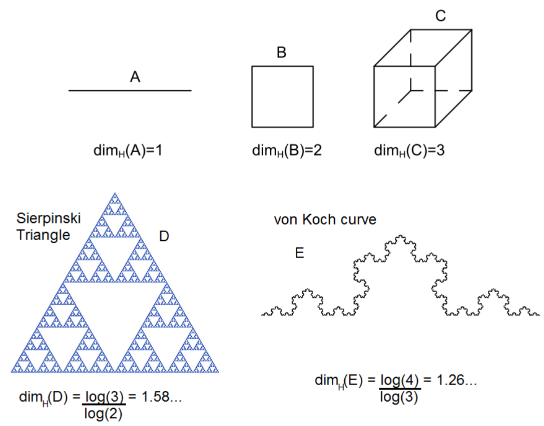
Construction of the middle-third Cantor set.

An equivalent, and perhaps presently more useful, definition is that the middle-third Cantor set is the set of points which lie in the intersection of all of the sets C_n. In other words, it is the set of points which lie in C_n for every natural number n.

Notice that while the constructions of the ψ-well approximable points and the middle-third Cantor set are similar, they do differ in the respect that a point is in the middle-third Cantor set if it lies in *all* of the sets C_n whereas for a point to be in $W(\psi)$ it only needs to lie in *infinitely many* of the sets W_q.

Since sets like $W(\psi)$ are fractal-like, another natural measure of "size" we might be interested in is *Hausdorff dimension*. Hausdorff dimension is a refinement of our intuitive notion of dimension but does not necessarily have to take integer values and so provides us with a means for assigning sensible values of dimension to, say, fractal sets. For example, the Hausdorff dimension of the middle-third Cantor set is $\frac{\log 2}{\log 3} \approx 0.63$. For a set X, we denote its Hausdorff dimension by $\dim_H X$.

We will not include a formal definition of Hausdorff dimension here, but one can be found in [3].



Hausdorff dimension of some sets.

Knowing the Hausdorff dimension of a set is often more enlightening than knowing its Lebesgue measure. In fact, if we have two functions, say ψ_1 and ψ_2, Hausdorff dimension can often provide a means of determining the relative sizes of $W(\psi_1)$ and $W(\psi_2)$ when Lebesgue measure (i.e., Khintchine's Theorem) fails to. As an example, consider the case when $\psi(q) = q^{-\tau}$. In this case, for $\tau > 0$, let us write $W(\tau)$ instead of $W(\psi)$. Notice that when $\tau > 1$ we have

$$\sum_{q=1}^{\infty} q^{-\tau} < \infty.$$

As a consequence of Khintchine's Theorem, we see that if $1 < \tau_1 < \tau_2$ then $\lambda(W(\tau_1)) = \lambda(W(\tau_2)) = 0$. Thus, Khintchine's Theorem does not allow us to differentiate between these sets. However, a theorem proved independently by Jarník (1929) and Besicovitch (1935), now known as the *Jarník–Besicovitch Theorem*, tells us that, for $\tau > 1$,

$$\dim_H(W(\tau)) = \frac{2}{\tau + 1}.$$

So, for these types of sets, Hausdorff dimension gives us a means of distinguishing their sizes while Lebesgue measure does not.

Badly approximable and very-well-approximable numbers

Earlier, we arrived at Hurwitz's Theorem by trying to improve upon Dirichlet's Theorem by decreasing the constant on the right-hand side of (1). For a given real number x, if we cannot improve this constant by making it arbitrarily small, then we say that x is *badly approximable*. That is, x is badly approximable if there exists a constant $c(x) > 0$ (depending on x) such that

$$\left|x - \frac{p}{q}\right| > \frac{c(x)}{q^2}$$

for all rationals $\frac{p}{q}$. We will denote the set of badly approximable numbers by $\mathcal{B}$.

Another way in which we can attempt to improve upon Dirichlet's Theorem is by increasing the exponent in the denominator of (1). This leads us to two more sets of interest in Diophantine approximation.

If we can improve on the exponent in Dirichlet's Theorem for a given real number x, then we say that x is *very-well-approximable*. That is, a real number x is very-well-approximable if there exists some real number $\alpha > 0$ such that

$$\left|x - \frac{p}{q}\right| < \frac{1}{q^{2+\alpha}}$$

for infinitely many rationals $\frac{p}{q}$. If, for some irrational real number x, this statement holds for *any* $\alpha > 0$, then x is called a *Liouville number*. We denote the set of very-well-approximable numbers by $\mathcal{V}$ and the set of Liouville numbers by $\mathcal{L}$.

Regarding the sizes of these sets, it is known that

$$\lambda(\mathcal{B}) = \lambda(\mathcal{V}) = \lambda(\mathcal{L}) = 0.$$

Nevertheless, all three of these sets are infinite (in fact, they are all uncountable). We also know the Hausdorff dimension of each of these sets:

$$\dim_H(\mathcal{B}) = 1, \quad \dim_H(\mathcal{V}) = 1 \quad \text{and} \quad \dim_H(\mathcal{L}) = 0.$$

So, loosely speaking, all three of these sets are small when their Lebesgue measure is taken into consideration. However, if we look at Hausdorff dimension then $\mathcal{B}$ and $\mathcal{V}$ are seemingly large, indeed both of these sets have the same Hausdorff dimension as the real numbers. Moreover, the sets $\mathcal{B}$ and $\mathcal{V}$ are completely disjoint.

Two famous conjectures in Diophantine approximation

As Diophantine approximation keeps evolving and ever more applications are being discovered, there is no shortage of interesting problems for us to tackle. Here we will mention just two of these problems which are arguably the most famous unsolved problems in metric Diophantine approximation.

Duffin–Schaeffer Conjecture

Going back to our set $W(\psi)$, we see that to obtain the nice dichotomy for the Lebesgue measure given by Khintchine's Theorem, we are forced to assume that ψ is monotonic. However, not all functions are monotonic and so a question one might ask is whether this assumption is really necessary. In a paper of 1941 [1], Duffin and Schaeffer showed that the answer to this question is, unfortunately, yes. They did this by constructing a non-monotonic function θ with $\lambda(W(\theta)) = 0$ but where $\sum_{q=1}^{\infty} \theta(q) = \infty$.

In the same paper, Duffin and Schaeffer also conjectured a statement which should be true if we do not assume that ψ is monotonic. This is known as the *Duffin–Schaeffer Conjecture* and is one of the key unsolved problems in Diophantine approximation.

Duffin and Schaeffer considered a slight modification of the set $W(\psi)$. Given a non-negative real function ψ defined on the natural numbers, let $W'(\psi)$ be the set of real numbers x such that

$$\left|x - \frac{p}{q}\right| < \frac{\psi(q)}{q}$$

for infinitely many rationals $\frac{p}{q}$ with $\gcd(p, q) = 1$ (that is, rationals in reduced form). Finally, let φ be the Euler totient function; that is, for natural numbers q, $\varphi(q)$ is equal to the number of natural numbers p satisfying both $1 \leq p \leq q$ and $\gcd(p, q) = 1$. In 1941, Duffin and Schaeffer made the following conjecture:

Duffin–Schaeffer Conjecture. *Let ψ be* **any** *non-negative real-valued function defined on the natural numbers. Then,*

$$\lambda(W'(\psi)) = \begin{cases} ZERO & \text{if} \quad \sum_{q=1}^{\infty} \varphi(q)\frac{\psi(q)}{q} < \infty \,, \\[2ex] FULL & \text{if} \quad \sum_{q=1}^{\infty} \varphi(q)\frac{\psi(q)}{q} = \infty \,. \end{cases}$$

The ZERO measure part of the above statement is already known to be true so it is proving the FULL measure part which represents the significant remaining challenge.

There is also a higher dimensional version of the Duffin–Schaeffer Conjecture which was formulated by Sprindžuk. However, for this particular problem, it is curiously only the one dimensional case which still remains unproved. In dimensions two and higher the corresponding conjecture has been proved by Pollington and Vaughan [4].

Littlewood's Conjecture

Probably the single most famous unsolved problem in metric Diophantine approximation is Littlewood's Conjecture (c. 1930) on multiplicatively approximable points. Given a non-negative real function ψ, defined again on the natural numbers, the *multiplicatively approximable points* are the set of points $\mathbf{x} = (x, y)$ in the plane, $\mathbb{R}^2$, for which there are infinitely many pairs of rationals, $\frac{p_1}{q}$ and $\frac{p_2}{q}$, such that

$$\left|x - \frac{p_1}{q}\right|\left|y - \frac{p_2}{q}\right| < \frac{\psi(q)}{q^2}.$$

We write $W^{\times}(\psi)$ to denote the multiplicatively approximable points. Pictorially, this set corresponds to taking suitable hyperbolas around rational points (depending on their denominators). The multiplicatively approximable points will be the set of points lying in infinitely many of these hyperbolas.

In short, Littlewood's Conjecture says that, for any choice of $\varepsilon > 0$, if we take the function $\psi(q) = \frac{\varepsilon}{q}$, then the set $W^{\times}(\psi)$ of multiplicatively approximable points should be all of the points in $\mathbb{R}^2$. Thus, rewriting, Littlewood's Conjecture says the following:

Littlewood's Conjecture. *For any point $\mathbf{x} = (x, y)$ in $\mathbb{R}^2$ and for any $\varepsilon > 0$, there exist infinitely many pairs of rationals, $\frac{p_1}{q}$ and $\frac{p_2}{q}$, such that*

$$\left|x - \frac{p_1}{q}\right|\left|y - \frac{p_2}{q}\right| < \frac{\varepsilon}{q^3}.$$

Although this problem remains unsolved, various progress has been made towards its resolution since its formulation. Perhaps the most notable progress so far is that, in 2006, Einsiedler, Katok and Lindenstrauss [2] showed that if there are any counterexamples to Littlewood's Conjecture then the set of these counterexamples must be exceptionally small in the sense that it must have Hausdorff dimension zero. Indeed, this result contributed to Elon Lindenstrauss being awarded a Fields Medal in 2010.

Wider interest in Diophantine approximation

The ideas used so far to approach the Duffin–Schaeffer Conjecture and Littlewood's Conjecture, and indeed many other problems in Diophantine approximation, are drawn from a wide range of areas in mathematics. Conversely, so-called Diophantine properties often turn up in these other fields.

Recently, Diophantine approximation has additionally been found to have connections with more "exotic" topics. For example, it turns out that there is a reformulation of Littlewood's Conjecture in terms of mathematical quasicrystals which, in turn, are actual existing materials and so are of interest in materials science. The theory of Diophantine approximation has also been found to have numerous applications to wireless communication, so much so that we have recently been enjoying welcoming electronic engineers to Diophantine approximation conferences!

FURTHER READING

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Newton the Mathematician

NICCOLÒ GUICCIARDINI

Newton was one of the greatest mathematicians since Archimedes's times. But what was mathematics for Newton? What aim did he wish to pursue with his uniquely equipped mathematical mind?

The achievements of a young mathematician

In 1661, an eighteen-year-old lad, after a journey on foot from Lincolnshire, entered Trinity College in Cambridge. Isaac Newton — this was his name — matriculated as a sub-sizar. The University, like England in general, was still unstable and in a state of flux. The Restoration of the Stuart monarchy had occurred only the previous year and Cambridge, a hotbed of Puritan sympathizers, was under pressure. It is easy to imagine the feeling of anxiety that the young Isaac must have experienced amidst such political turmoil. To Newton's generation the future looked uncertain, since no established and recognized authority that could validate truth and guarantee justice was easily discernible.

This instability also characterized the fields of natural philosophy and religion. Newton soon sought an answer to these concerns in a world of paper: in books that he could borrow or acquire, and notebooks he would fill in a minute yet legible handwriting. Mathematics captured Newton's attention with a strength that, I feel, was not due to any choice on his part but to the fact that his mind was extraordinarily equipped for mathematical inventiveness. His annotations after the winter of 1664 reveal the journey of an independent mind that took the existing literature on the most advanced mathematical topics as a springboard for discovering new concepts and methods. From 1663, he was helped by the first Lucasian Professor, Isaac Barrow, a mathematician who can be ranked amongst the co-discoverers of the calculus — even though, unfortunately, we do not know the details of this collaboration.

Indeed — so the story goes — in the 'marvellous years' 1665–6, Newton 'discovered calculus'. While this is broadly true, it remains a problematic statement: as historians of mathematics we should be aware of this. Most historians would agree that neither Newton nor Leibniz 'discovered' the calculus. Rather, the accepted view is that both Newton and Leibniz contributed, each in his own way, to a process that was begun by earlier generations of mathematicians and was concluded by posterity: the result of this long process is the calculus as we know it nowadays. But, in essence, what did Newton do in his marvellous years?

His first major discovery was the binomial series for fractional (positive and negative) exponents. Newton could expand $y = (1 - x^2)^{1/2}$ as a power series and integrate — or, as he put it, 'square' — term-wise, obtaining the area of the circle segment. Power series representations of the trigonometric functions immediately followed. Application of the binomial series to negative exponents led to interesting results. Most notably, Newton applied the binomial series to the hyperbola $y = (1 + x)^{-1}$, a result that he considered valid when x is small. He immediately applied this result, with an algorithmic dexterity that borders on insanity, to the calculation of logarithms up to 50 decimal places. What is bound to strike the modern reader about Newton's early discovery and application of the binomial series is the way he achieved it and the purpose he had in mind.

Newton obtained this fundamental result by guesswork rather than proof. He saw a pattern in what he called 'Oughtred's Analyticall Table', which tabulates the coefficients for positive integer exponents, and he extrapolated this to negative and fractional exponents. There is no rigorous proof here. Thus, Newton's mathematical practice was miles away from the rigour epitomized by Euclid's *Elements*. Newton's bold procedure belongs to the toolbox of the mathematical practitioners active as surveyors, astronomers, gunners, mariners, and the like. His aim was to obtain numerical results useful for table-making: logarithms and trigonometric functions.

Newton's second mathematical discovery was a notation and method to draw tangents to curves. Much work had already been done on this topic, and Newton never stressed the importance of the method of tangents as one of his most original discoveries. He was eager, even in his old age, to recognize Barrow's, Sluse's and Fermat's contributions. For example, he considered the cubic $x^3 - ax^2 + axy - y^3 = 0$. According to his rules the ratio of the 'fluxions' of the 'fluents' x and y is given by $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y + a\dot{y}x - 3\dot{y}y^2 = 0$ (where I use a dotted notation that Newton devised only in the 1690s). In Newton's terms, the 'fluents' are magnitudes varying continuously in time, the 'fluxions' are the instantaneous speeds. In Newton's example, the rules for the calculation of the fluxions of the sum $x + y$, the product xy, and the integer positive power x^n are simultaneously stated, respectively, as $\dot{x} + \dot{y}$, $x\dot{y} + y\dot{x}$, and $nx^{n-1}\dot{x}$. All this was well-known territory. Newton went beyond accepted knowledge on tangents to curves by explaining how to cope with non-polynomial equations. In ca. 1671 he wrote:

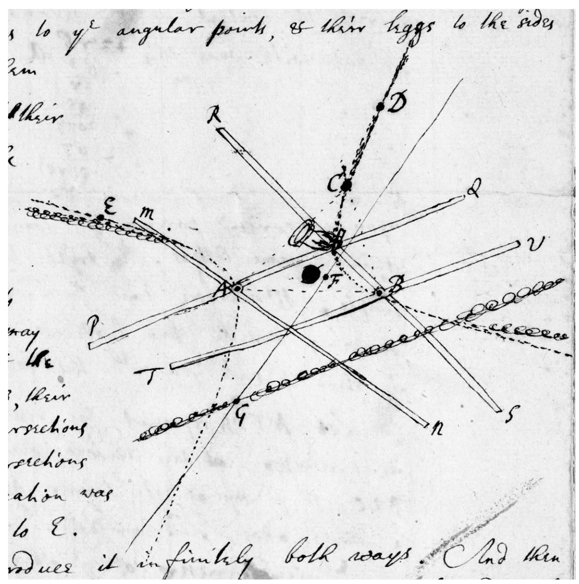
> Whenever complex fractions or surd quantities are present . . . , in place of each I put a corresponding letter and, supposing these to designate fluent quantities, I work as before. Then I suppress and exterminate the letters ascribed[1].

The problem with this method was of course the use of infinitely small magnitudes. Newton assumed that during a small interval of time o, a *moment*, as he called it, the generating motion is uniform, so that the ratio $\dot{y}o/\dot{x}o$ is the slope of the curve traced by a point $C(x, y)$. In his calculation of this ratio he cancelled higher-order infinitesimals. Once again, he was miles away from Euclid!

The third mathematical discovery was what we call the fundamental theorem of calculus. Newton certainly borrowed from Barrow, in this regard. This proved that, in a way, the operations for calculating the tangent to a curve and the calculation of the area of the surface subtended to a curve are one the inverse of the other. This fact was known to the cognoscenti. Newton, however, was the first to write integral tables based on the fundamental theorem. A few years later, Leibniz did the same, and with a better notation. Newton also dealt with the 'squaring of curves', such as $y = z^2/\sqrt{az - z^2}$, by 'comparing them to conic sections', that is to the squaring of a circle sector or of a conic area. This means that, by substitution of variables, Newton reduced these more difficult curves to a function that could be integrated in terms of trigonometric or logarithmic functions.

From this brief overview of Newton's three discoveries concerning calculus, the method of series and fluxions as he termed it, we get an initial picture of this young creative mathematician. He was daringly un-rigorous, he put his hands on what were super-difficult problems for his age, and his aims were practical. He searched for numbers, he loved tables, he mastered the new highly conjectural symbolism of algebra, he manipulated with no worries the infinite and infinitesimal. He proudly referred to his method as a 'new analysis', by which he meant a new method of discovery.

Newton was at home with mathematical practitioners. He shared, it seems, their aims and ethos. His first, and practically only, mathematical correspondent was John Collins, a Londoner, accountant and former 'mathematician on board' English ships in the service of the Republic of Venice. With him Newton spoke the same language. Via Collins, Newton got in touch with the 'gaugers' employed by the Excise Office: he applied the binomial theorem to calculate the volume of barrels. Newton never complained about meddling with practical people intent in solving down-to-earth challenging problems. He was a problem-solver.



Organic construction of a conic through five given points. The conic, in this case a hyperbola, passes through the five given points *A, B, C, D, E*. Newton to John Collins (20 August 1672). Source: Add. 3977.10, f.1v. Reproduced by kind permission of Cambridge University Library.

A letter[2] to Collins in 1672 gives us a vivid impression of the practical orientation of Newton's early mathematics and also allows us to glimpse a mathematical field, other than algebra and calculus, that occupied the young Newton: geometry. Newton was interested in what was known as 'organic geometry', the art of tracing curves via instruments ('organa' in Greek). Briefly stated, Newton's organic description

[1][8], 3, p77
[2][7], 2, pp230–1

can be presented by noting that if two angles of given magnitudes turn about their respective vertices (the poles, which are fixed) in such a way that the point of intersection of one pair of arms always lies on a straight line (the describing line), the point of intersection of the other pair of arms will describe a conic. When reading this letter, one wonders if Newton had anticipated by a century and a half Jakob Steiner's projective generation of conics, formulated by the Swiss founder of the modern practice of synthetic geometry (the approach to geometry without the use of coordinates or formulas). Let us leave this question to be debated among historians.

What's interesting for me to note is that the drawing accompanying the letter to Collins is quite realistic and suggests that Newton actually made use of real instruments to trace curves. Indeed, in a manuscript Newton described his 'organon' in quite practical terms:

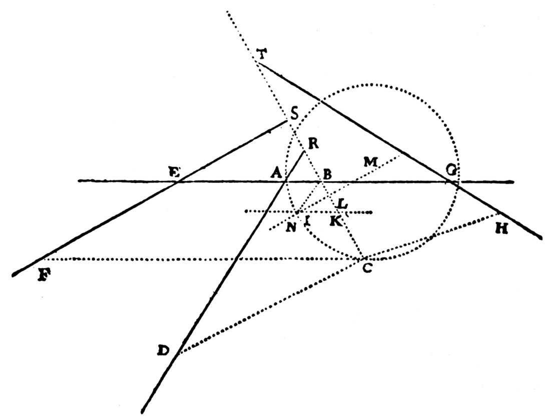
> Two rules . . . are to be manufactured so that their legs . . . can be inclined to each other, at will, in any given angle . . . . And at the junctures there should be a steel pin-point around which the rules may be rotated while the pin is fixed on some given point as its centre. To be sure, the steel nail by which the legs of the sector are joined might be finely sharpened at one end, and on the other threaded to take a nut more or less tightly (as the need arises) which will clamp the legs of the sector in the given angle [8], 2, p135.

Newton's mind was trained to do things. With his pen and ink (that he had manufactured by himself) he calculated decimals of logarithms, with pins and nuts he traced hyperbolas. His mind, at least to the historian of his early mathematics, does not seem intent on 'voyaging through strange seas of thought'.

A turn towards geometry

The image I have given of Newton the practitioner is, however, only a tiny part of the whole picture. Since his early studies, Newton showed himself to be a man of great ambitions, which far exceeded the intellectual horizons of Collins and the London gaugers. For him mathematics was the instrument to achieve certainty, or better to approach it as far as possible, in fields as diverse as optics, mechanics, and astronomy. Later in his life, he was to use mathematics for the purpose of injecting certainty into demandingly technical research devoted to the study of the Temple of Salomon and the determination of the chronologies of ancient kingdoms.

It is in the 1670s that Newton began to distance himself from his early mathematical research. It is the extrapolations he used in handling infinite series and the bold assumptions concerning infinitesimals in his methods of series and fluxions that began to worry him. He tried to formulate a theory of limits to establish his methods on firmer foundations . . . and turned to geometry [1]. The kind of geometry he was interested in was the solution of locus problems, such as the so-called Pappus problem for 3 or 4 lines. He tried to solve these problems not by using algebra, but via geometrical means that anticipate ideas in projective geometry developed much later by Steiner, Michel Chasles and Jean-Victor Poncelet.



The Pappus problem. Given four lines (solid) and four corresponding angles, find the locus of a point C such that the angled distances d_i (dotted) from the point to each line maintain the constant ratio $d_1 d_2 : d_3 d_4$. In this case the locus is a circle [3], p61.

The extent of Newton's interests in the geometry of Greek mathematicians, such as Apollonius, was revealed for the first time in the masterful edition of the mathematical papers edited by Tom Whiteside [8]. However, already in the *Principia* (1687) Newton had disclosed some geometrical gems, including the organic construction of conics we have just looked at (hardly a topic of interest for gravitation theory). As Nicole Bloye has shown in her thesis, Newton developed many techniques and achieved results that pertain to projective geometry, going so far as to anticipate the Cremona transformations. Indeed, Newton was aware that his organic rulers could create or resolve singularities, and he explicitly used them to do this. He stated that if the describing line is a conic (rather than a straight line), then the rulers will draw a cubic and showed how to use the organic rulers to construct a cubic through seven given points[3].

Now the question I would like to pose is: why this interest in geometry, which apparently set in in the middle of the 1670s and persisted throughout Newton's life? Has it anything to do with Newton's tardiness in publishing the calculus? Is this interest related to the geometrical character of many parts of his masterpiece, the *Principia*? As is always the case when dealing with the intentions an author might have had, these are difficult questions. In the rest of this article, I will suggest some tentative answers [4].

On the Ancients

It is interesting to note that in the period in which Newton began to turn his mathematical eye to the works of the Greek geometers, he also began to regard himself as a restorer of the pristine Wisdom of the Ancients. He saw himself both as a restorer of the Ancients' mathematical 'method of discovery', which he claimed was superior to the algebra of the moderns, and of the ancient knowledge in natural philosophy and religion, which in his opinion had been corrupted by pagan idolatry and by a Christianity imbued with metaphysical neo-Platonism. A harking-back to the past strongly marks Newton's view of religion. Of course, the reference to the 'Ancients' we find in his mathematical work has an entirely different meaning to the one we find in his religious manuscripts: referring to Euclid is not the same as referring to Moses! The former was admired for his mathematical method, the latter revered as a prophet. Nevertheless, there is a common thread linking these references despite their differences, which shows the mentality of the author, who sought to take a stance against the corruption of the moderns. There is a stylistic, rhetorical similarity, which brings us closer to the feelings of a natural philosopher, theologian and mathematician who, across very different disciplines, regarded his contemporaries with suspicion and was keen to style his work as a restoration of lost knowledge.

Deeply intertwined with this admiration for the Ancients is Newton's anti-Cartesianism. Descartes, and later Leibniz, were viewed by Newton as the moderns par excellence. Their philosophical 'romances' were dangerous for religion, since they led to a denial of God's providential action. Indeed, their natural philosophies depicted Nature as self-sufficient, its laws being subject to conservation principles, such as Leibniz's conservation of *vis viva* — we would say, conservation of energy — which made God's creation autonomous from its Creator.

After writing the *Principia* Newton could claim that his natural philosophy required the intervention of an 'intelligent Being'. In Newton's opinion, in impacts between the ultimate constituents of matter, perfectly hard particles, 'motion is lost'. Further, the mutual gravitational interaction of masses leads to a many-body problem that, he claimed, cannot result in the order we observe in the planetary system. The 'wonderful Uniformity in the Planetary System' can only be the 'Effect of Choice'. In fact, according to Newton, the reciprocal gravitational action of the planets and comets generates irregularities which over time accumulate until the system requires a 'Reformation' guaranteed by the providential intervention of God. To search in Nature for a law comparable to that of the conservation of energy, as Huygens and Leibniz did, was anathema for this natural philosopher who ended the second edition of his masterpiece (1713) with a Scholium devoted to a Deus Pantokràtor. Left to herself, Nature will slow down and become chaotic: loss of 'motion' and loss of 'order' can only be mended by the continual Providence of God.

Thinking about mathematical method

Newton's deeply felt revulsion towards the philosophical system of Descartes, and later Leibniz, went hand in hand with his rejection of their mathematics. Both Continental mathematicians had profiled their mathematical methods as radical innovations that belittled the achievements of the Ancients. In the *Géométrie* Descartes had famously stated:

> This is one thing which I believe the ancient mathematicians did not observe, for otherwise they would not have put so much labor into writing so many books in which the very sequence of the propositions shows that they did not have a sure method of finding all.

Descartes' proof of his superiority over Apollonius was an algebraic solution of the Pappus problem. Newton furiously rebutted that this was not at all the case. Descartes, the system builder who claimed he could reject all previous philosophical systems

[3] See [2], pp131–3, and [8], 7, pp588–645

and build up a new philosophy from scratch, was the mathematician who could make a show of his symbolic methods. But:

> To be sure, their [the Ancients'] method is more elegant by far than the Cartesian one. For [Descartes] achieved the result [the solution of the Pappus problem] by an algebraic calculus which, when transposed into words would prove to be so tedious and entangled as to provoke nausea, nor might it be understood. But they accomplished it by certain simple proportions, judging that nothing written in a different style was worthy to be read, and in consequence they were concealing the analysis by which they found their constructions[4].

Newton's statements in praise of geometry are numerous and ubiquitous in his mathematical works. Often, as in the above passage, we find Newton praising the greater elegance and simplicity of geometry. He also underlined that it is preferable to exhibit 'finite quantities and real' such as fluxions (finite instantaneous velocities), 'visible to the eye', rather than symbols, such as those for infinitesimals. He claimed that geometrical objects are best given by the 'manner of their construction' (very much as in the organic construction of conics) rather than by their equation.

All the above statements might appear somewhat arbitrary. They express admiration for the ancient Greeks, and an aversion for modern algebraists. But they are also the expression of deep ideas that Newton was developing in his research. First he had come up with some very interesting ideas in projective geometry. Second, when he developed his mathematized natural philosophy, a geometrical approach allowed him to obtain a very deep representation of force and acceleration. One only has to read the *Principia* to be convinced that an approach to the science of motion in terms of geometrical representation allows one to grasp Newtonian mechanics in a more meaningful way than via equations.

Yet, the work of the mathematical physicist rules out purism. One needs a rich mathematical structure in order to mathematize the natural world: opportunism, not purism, is the answer. And indeed, as readers of the *Principia* know very well, Newton made recourse to calculus techniques (sometime explicitly, mostly in a somewhat hidden way). Algebraic equations, limits, infinite series, 'quadratures' (that is, integrations) occur frequently, so much so that in the preface to L'Hospital *Analyse des infiniment petits* (1696) one can read that Newton's masterpiece is 'almost entirely about the calculus': a statement that in essence is true. To give just one of many possible examples: Newton's study of the trajectories in an inverse-cube force field was achieved via the integration of the pertinent differential equation — as can be proved in detail on the basis of available manuscripts [5]. Publishing these more advanced techniques in the *Principia* would have made the work even more illegible in 1687, when the calculus was in practice still unknown. There are thus tensions and contradictions between Newton's anti-modern pronouncements, the Euclidean façade of the *Principia* and his mathematical practice. The young mathematical practitioner, fabulously dexterous in manipulating symbols for practical ends, was still alive and well in the years of composition of the geometrical *Principia*.

Publishing mathematics

How did Newton deal with these tensions between his rather austere preference for geometry and his multifarious mathematical practice? A recollection from David Gregory is significant. Apparently, Newton told him:

> Algebra is fit enough to find out, but entirely unfit to consign to writing and commit to posterity[5].

Algebra, Newton seems to be saying here, is a heuristic method, not to be published. And indeed, at least until the advent of Leibniz's calculus on the Continent, Newton was happy to communicate his ideas on algebra and the methods of series and fluxions via correspondence and manuscript circulation. When such a powerful competitor appeared on the horizon, Newton changed his policy, and the consequences of his tardiness in publishing the methods, and the subsequent confrontation on priority with the German *homo universalis*, are well known. Once again, statements like the one cited by Gregory in his memoranda might appear to be the fruit of arbitrary taste. But we have to delve a little more deeply into Newton's intellectual career to understand his reluctance to send the methods of series and fluxions to the press.

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[4] [8], 4, p277
[5] [7], 3, p38

First, Newton conceived mathematics as the tool which could allow him to overcome the uncertainties and probabilities of both the mechanistic natural philosophy of Descartes and Baconian experimental philosophy. Many of his contemporaries, Newton regretted, were happy to explain natural phenomena in terms of hypothetical assumptions on corpuscular models. For example, Descartes explained planetary motions by assuming the existence of a vortex of subtle matter circling the Sun. Boyle cautiously spoke of the elasticity of the air — a 'spring' acting between the particles of air — as responsible for the gas laws. Newton disliked these 'conjectures and probabilities that are being blazoned about everywhere'. Via the use of geometry he wished to 'achieve a science of nature supported by the highest evidence'. But — and this is the important point — if mathematics has to endow natural philosophy with 'evidence', the natural philosopher must make recourse to methods that are not open to criticism. Making recourse to infinitesimals and infinite series was not the best choice for the realization of Newton's programme. Such a publication would have certainly exposed him to the kind of criticism that he so strenuously wished to avoid. He had experienced the effects of controversy when his paper on the experimentum crucis of 1672 was attacked by Hooke and others.

Second, from the 1670s to the mid-1690s, Newton tried very hard to find a simple rule for the 'inverse method of fluxions' (integration). This was the great open issue for the mathematicians working at the forefront of the research on calculus. In their correspondences and manuscripts, the historian often finds hopes, failed projects, and false announcements about the finding of the golden rule allowing one to integrate as easily as one can differentiate. As L'Hospital wrote in 1694:

> Leibniz proposed a new kind of calculus, which he calls differential ... but what remains to be done is very difficult, that is to find the inverse of that calculus, that is a general method for describing curves if the property of their tangent is given[6].

Yes, the direct method (differentiation) had been solved: mathematicians now had a set of simple rules. But what about integration? Newton, no matter how incredibly wonderful his early discoveries on fluxions might appear to us, thought to have achieved only its simplest part. The method of quadratures struck him as a confused set of recipes, each valid for a class of curves, that required a great deal of guesswork. The best policy was to delay publication, hoping to attain the big result: a simple rule to integrate all differential equations!

[6] [6], p193

Newton's agenda

Newton's agenda as a mathematician was a majestic one. He excelled in tackling the most advanced and daunting mathematical problems. Being able to solve these problems, and putting these solutions to the service of very concrete practical aims was important for him. But he attributed to his mathematical work a crucial role in the fulfilment of a bold philosophical project, in which he fashioned himself as a restorer of the Ancients and a fierce critic of the irreligious Moderns.

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The Shape of a Minimiser: Equilibrium Measures for Nonlocal Energies

LUCIA SCARDIA

Nonlocal energies are continuum models for large systems of interacting particles. They have countless applications, from biological systems to granular media, from vortices in superconductors to defects in metals. Motivated by an example in materials science, we analyse the effect of anisotropy on the shape of energy minimisers and unveil some surprising connections with random matrices and fluid dynamics.

Nonlocal energies

What are *nonlocal* energies? They are continuum models for discrete energies

$$E_n(x_1, \dots, x_n) = \frac{1}{n^2} \sum_{i,j=1}^{n} W(x_i - x_j), \qquad (1)$$

describing a system of particles at $x_1, \dots, x_n \in \mathbb{R}^N$ that interact pairwise via a potential $W : \mathbb{R}^N \to \mathbb{R}$, under the assumption that every particle interacts with every other particle, no matter how far apart they are. Such interactions are called *long-range*. Formally, the behaviour of the discrete energy E_n for a large number of particles is well captured by a functional of the form

$$E(\mu) = \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} W(x-y) d\mu(x) \right) d\mu(y), \qquad (2)$$

defined on measures μ in $\mathbb{R}^N$ representing the density of particles.

The functional (2) is *nonlocal* since, intuitively, the inner integral is a convolution, $(W * \mu)(y)$, which at any point y depends on the behaviour of both the potential W and the measure μ on the whole space, and not just in the proximity of y. Note that the convolution term $(W * \mu)(y)$ is the macroscopic effect of the long-range interactions in (1); in other words, the long-range interactions in (1) are the microscopic origin of the nonlocality of E.

The formal connection between E_n in (1) and E in (2), for large n, can be made rigorous in many interesting cases. So we can effectively consider E as an 'average' particle energy.

Energies of the form (1) and (2) are ubiquitous. Crowds of people, flocks of birds, shoals of fish, atoms, defects in metals, vortices in superconductors, are examples of particle systems — different in nature and in scale — whose interactions are of this form. It is the explicit expression of W that depends on the application: For atomistic interactions W is the Lennard-Jones potential $W_{\mathrm{LJ}}(x) = \frac{a}{|x|^{12}} - \frac{b}{|x|^6}$ (where $a, b > 0$), for charges it is the Coulomb potential W_{C}, for linear springs the square of the distance.

The potential W can describe attraction, repulsion or a combination of both (e.g., repulsion at small distance and attraction at a larger distance, which is natural for crowds, or for animals). It can be isotropic (i.e., $W(x) = W(|x|)$), as in the case of charges, or *anisotropic*, to model interactions depending not only on the distance, but also on the mutual orientations of the particles. The isotropy assumption is indeed often unrealistic, especially in biological applications, where the limited field of vision of the individuals introduces a preferred direction of interaction. For example, we interact differently with a person in front of us or behind us, even if they are at the same distance from us. Moreover W can be a smooth function, bounded at zero, like in the case of linear springs, or it can be singular. *Singular* interactions are not mathematical artefacts: requiring $W(0) = +\infty$, for example, corresponds to modelling strong repulsion, as it prevents two particles from occupying the same position.

A fundamental question in applications is the characterisation of *minimisers* of the interaction energies (1) and (2). Such minimisers correspond to 'aggregation patterns' and they are observed experimentally in self-assembled biological systems as well as in materials science.

Aggregation patterns can be very diverse: depending on W particles may prefer to spread (in the repulsive case) or to concentrate (in the attractive case), and the aggregation patch may be radially symmetric (for isotropic W) or elongated along some preferred direction (in the anisotropic case).

The effect of anisotropy on energy minimisers is not yet well understood. We will illustrate it in the case of defects in metals (*dislocations*), starting from the classical case of Coulomb interactions.

The classical case of the logarithm

For many particle systems the potential is singular at zero, and its behaviour close to zero is asymptotically close to that of the Coulomb potential W_C. In two dimensions ($N = 2$), this corresponds to $W_C(z) = -\log|z|$.

This classical case has been studied intensively in the last century, and has important connections with vortices in superconductors, Coulomb gases, interpolation theory, and the theory of random matrices. Indeed, the distribution of the eigenvalues of a random matrix whose entries are independent and identically distributed Gaussian random variables can be obtained by minimising the functional

$$I_n(x_1, \ldots, x_n) = -\frac{1}{n^2}\sum_{i,j=1}^{n} \log|x_i - x_j| + \frac{1}{n}\sum_{i=1}^{n} |x_i|^2,$$

which is exactly as in (1), with $W = W_C$, up to the additional quadratic term, which corresponds to the Gaussian ensemble. In the limit $n \to \infty$, namely for larger and larger matrices, the *average* distribution of the eigenvalues is well captured by the minimiser of the functional

$$I(\mu) = -\iint_{\mathbb{R}^2\times\mathbb{R}^2} \log|x-y|\,d\mu(x)d\mu(y) + \int_{\mathbb{R}^2} |x|^2 d\mu(x), \qquad (3)$$

which is the continuum limit of I_n.

The problem of minimising I was solved back in 1935: Frostman [3] characterised the unique minimiser of I, μ_0, and showed that

$$\mu_0 = \frac{1}{\pi}\chi_{B_1(0)}, \qquad (4)$$

namely μ_0 is the characteristic function of the unit disc. (See “The Euler-Lagrange conditions”.) This uniform distribution, called the *circle law*, means that on average eigenvalues like to be at equal distance from each other, which in two dimensions is of order $1/\sqrt{n}$.

The Euler-Lagrange conditions

Heuristically, it is not surprising that μ_0 is the minimiser of I, and one way to see this is by looking at the Euler-Lagrange conditions associated to I. These are the infinite-dimensional equivalent of the familiar stationarity condition ‘$\nabla I = 0$’, and have to hold true for minimisers of I. For the functional I in (3) the Euler-Lagrange conditions are

$$-(\log|\cdot| * \mu_0)(x) + \frac{|x|^2}{2} = c, \qquad (5)$$

where c is a constant, which has to be satisfied on the support of the minimising measure μ_0 (i.e., where μ_0 is non-zero), and

$$-(\log|\cdot| * \mu_0)(x) + \frac{|x|^2}{2} \geq c,$$

which has to be satisfied everywhere in $\mathbb{R}^2$ (up to a set of *capacity* zero). By taking the Laplacian of (5) we obtain the (still necessary) condition

$$-\Delta(\log|\cdot| * \mu_0)(x) + 2 = 0. \qquad (6)$$

Since $-\Delta(\log|x|) = -2\pi\delta_0$, (6) reduces to

$$\mu_0 = \frac{1}{\pi}.$$

Namely, the minimiser μ_0 has to have constant density $\frac{1}{\pi}$ on its support. Moreover, the radiality of both $\log|x|$ and $|x|^2$ suggests that the support has to be a disc.

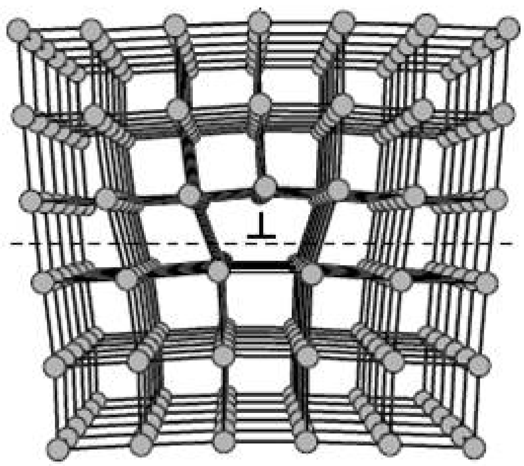
Anisotropic interactions: The case of dislocations

The equilibrium measure μ_0 in (4) has a number of properties: It is *radial*, *uniform*, and has *maximal dimension*, i.e., the support of μ_0 is two-dimensional in $\mathbb{R}^2$.

It is natural to expect that minimisers would no longer be radial if either the interaction potential or the confinement in I were replaced by non-radial functions. What about their dimensionality? Can the dimension of the support of the minimiser change as a consequence of the anisotropy?

A positive answer to this question comes from an application in materials science, more precisely in the theory of dislocations. Dislocations are defects in the atomic structure of metals which collectively, at the macroscopic scale, determine how metals deform. At a microscopic scale, two identical positive (edge) dislocations with Burgers vector $b = e_1$ interact via the potential

$$W_1(x) = -\log|x| + \frac{x_1^2}{|x|^2} = W_C(x) + \frac{x_1^2}{|x|^2}.$$



Positive edge dislocation with Burgers vector $b = e_1$. In 3d it can be represented as an extra plane of atoms in the crystal structure of the metal. *Adapted from Fig. 4.3 Callister 7e.*

In particular, the *glide* force exerted by a dislocation at the origin on a dislocation at position $x \in \mathbb{R}^2$, i.e., the component of the force along the Burgers vector $b = e_1$, is

$$F_{\rm gl}(x) = -(\nabla W_1(x)) \cdot e_1 = \frac{x_1(x_1^2 - x_2^2)}{|x|^4}.$$

It is immediate to see that $F_{\rm gl}(0, x_2) = 0$, and the sign of the force suggests that $x_1 = 0$ could be a stable equilibrium. Indeed, the general belief in the engineering community is that vertical walls of dislocations are minimum energy arrangements for the discrete interaction energy (1), with $W = W_1$ (see, for example, [2, 6]).

This conjecture is what triggered the analysis in [7], where we considered the functional

$$I_1(\mu) = \iint_{\mathbb{R}^2\times\mathbb{R}^2} W_1(x-y)\,d\mu(x)d\mu(y) + \int_{\mathbb{R}^2} |x|^2 d\mu(x), \qquad (7)$$

which represents the mean-field interaction energy for dislocations. The additional quadratic term can in this case be interpreted as a *confinement*, a force that prevents dislocations from escaping at infinity. Indeed, without such a term, the interaction energy could be made arbitrarily small by letting dislocations be infinitely far from one another, due to the repulsive nature of the potential W_1. The specific choice of a quadratic confinement, on the other hand, has only been made to facilitate the comparison with the classical logarithmic energy (3).

In [7] we proved that the minimising measure μ_1 of I_1 is unique, has a one-dimensional support, and is given by

$$\mu_1 = \frac{1}{\pi}\delta_0 \otimes \sqrt{2 - x_2^2}\,dx_2,\ \ x_2 \in (-\sqrt{2}, \sqrt{2}). \qquad (8)$$

(See “A one-dimensional minimiser”.) Our result hence proves the conjecture that dislocations like to form vertical structures, since μ_1 is a vertical, wall-like configuration. The effect of the anisotropic term in W_1 is quite dramatic: unlike μ_0, μ_1 has one-dimensional support, and its density is non-constant.

It turns out that the vertical marginal of μ_1, i.e., the one-dimensional measure

$$m_1 = \frac{1}{\pi}\sqrt{2 - t^2}\,dt,\ \ t \in (-\sqrt{2}, \sqrt{2}), \qquad (9)$$

is a well-known distribution in the theory of random matrices, and is called the *semicircle law*. The connection between our result and random matrices, however, is unclear. In the context of random matrices, the semi-circle law m_1 represents the distribution of eigenvalues in the Hermitian case, and Wigner in [8] obtained it as the unique minimiser of the logarithmic energy in one dimension,

$$I_{\log}(\mu) = -\iint_{\mathbb{R}\times\mathbb{R}} \log|x-y|\,d\mu(x)d\mu(y) + \int_{\mathbb{R}} |x|^2 d\mu(x). \qquad (10)$$

Note that the energy (10) is exactly (3), restricted to one-dimensional measures, which corresponds to the special case of symmetric matrices. It is therefore the one-dimensionality of the *energy* that causes the minimiser to be one-dimensional.

A one-dimensional minimiser

To show that μ_1 is the equilibrium measure for I_1 we can partly follow Wigner's classical proof of the minimality of the semi-circle law m_1 in (9) for $I_{\log}$. Indeed, for every $x_2 \in \mathbb{R}$,

$$(W_1 * \mu_1)(0, x_2) = -(\log|\cdot| * m_1)(x_2), \quad (11)$$

and hence μ_1 satisfies both the Euler-Lagrange condition

$$(W_1 * \mu_1)(x) + \frac{|x|^2}{2} = c$$

on the support of μ_1, and

$$(W_1 * \mu_1)(x) + \frac{|x|^2}{2} \geq c \quad (12)$$

on $\{0\} \times \mathbb{R}$, directly from Wigner's result for $I_{\log}$. Unfortunately, this result does not help to show that (12) is satisfied in the whole plane, since outside $\{x_1 = 0\}$ the simplification (11) is no longer true, and we need to deal with the anisotropy. The way we do it is by showing that the function $x_1 \mapsto (W_1 * \mu_1)(x) + \frac{|x|^2}{2}$ attains its lowest value on $x_1 = 0$ for every x_2, so that proving (12) on $\{x_1 = 0\}$ is enough. The situation for (7) is completely different, since the energy is still two-dimensional, and therefore the support of the minimising measure need not be one dimensional. In this case it is the anisotropic term which causes the loss of dimension of the minimiser! In particular, the special form of the anisotropy, which penalises the horizontal dimension only, causes the concentration on the vertical axis. And since the anisotropic term vanishes on measures with support on the vertical axis, the functional I_1 in (7) *coincides* with the energy $I_{\log}$ in (10) on those measures. Hence I_1 and $I_{\log}$ have the same minimiser, although for different reasons!

Note that, in general, the Euler-Lagrange conditions are only necessary for minimality. In this case, however, they are also sufficient, since we were able to show that I_1 is strictly convex. Proving strict convexity of nonlocal functionals of the form (2) is tricky: even for a quadratic interaction $W = |\cdot|^2$, the corresponding E is not convex in μ (it is, in fact, concave!).

Tuning the anisotropy

What is so special about the anisotropic term $\frac{x_1^2}{|x|^2}$ in W_1 to cause the loss of dimension of the minimiser? To give an answer to this question, in [1] we *tune* the anisotropy in W_1, and consider, for $\alpha \in [0,1]$, the interaction energy

$$I_\alpha(\mu) = \iint_{\mathbb{R}^2\times\mathbb{R}^2} W_\alpha(x-y)d\mu(x)d\mu(y) + \int_{\mathbb{R}^2} |x|^2 d\mu(x), \quad (13)$$

where

$$W_\alpha(x) = W_C(x) + \alpha\frac{x_1^2}{|x|^2}.$$

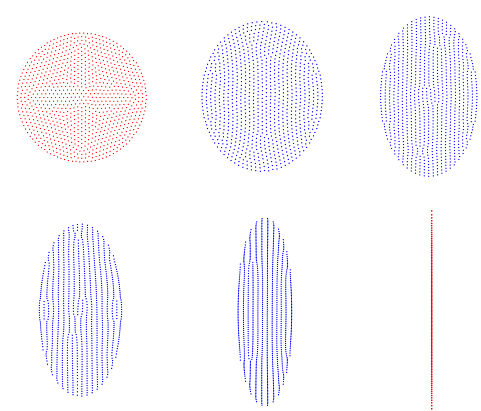
We have seen already that I_0 is minimised by the circle law (4) and I_1 is minimised by the the semi-circle law (8). But what is the minimiser μ_α of I_α in (13), for $\alpha \in (0,1)$? Is it fully dimensional and uniform like μ_0 or is it lower dimensional like μ_1?

In [1] we proved that μ_α is the characteristic function of the region enclosed by an *ellipse* with semi-axes $\sqrt{1-\alpha}$ and $\sqrt{1+\alpha}$, namely

$$\mu_\alpha = \frac{1}{\pi}\frac{1}{\sqrt{1-\alpha^2}}\chi_{\Omega_\alpha}, \quad (14)$$

where

$$\Omega_\alpha = \left\{x \in \mathbb{R}^2 : \frac{x_1^2}{1-\alpha} + \frac{x_2^2}{1+\alpha} < 1\right\}.$$



Minimisers of the discretised energy for $\alpha = 0$, $\alpha = 0.2$, $\alpha = 0.4$, $\alpha = 0.6$, $\alpha = 0.8$ and $\alpha = 1$. The ellipses shrink and elongate to a singular limit for $\alpha = 1$

What helps in proving the minimality of the ellipse is a surprising connection with rotating vortex patches in fluid dynamics. A vortex patch is the solution of the vorticity form of the planar Euler equations in which the initial condition is the characteristic function of a bounded domain. Kirchhoff proved over a century ago that ellipses are rotating vortex patches (or V-states), namely their evolution is nothing but a rotation with constant angular velocity of the initial state (see [5]). In particular, they can be described by means of an equation involving the stream function of the initial patch, which is formally similar to the Euler-Lagrange equation

$$(W_\alpha * \mu_\alpha)(x) + \frac{|x|^2}{2} = c \tag{15}$$

on the support of μ_α. The connection with fluid dynamics is more apparent if we take the gradient of the potential $W_\alpha * \mu_\alpha$ and rewrite it in complex variables, obtaining

$$(\nabla W_\alpha * \mu_\alpha)(x) = \left(-\frac{1}{\bar{z}} + \frac{\alpha}{2}\frac{1}{z} - \frac{\alpha}{2}\frac{z}{\bar{z}^2} \right) * \mu_\alpha. \tag{16}$$

Indeed, the first two terms in the right-hand side of (16) are essentially the Cauchy transform of the ellipse Ω_α,

$$\mathscr{C}(\chi_{\Omega_\alpha})(z) = \frac{1}{\pi}\int_{\Omega_\alpha} \frac{1}{z-\xi}\, d\xi,$$

which was computed explicitly in [4] to show that the *Kirchhoff ellipse* is a rotating vortex patch. As for the term $\frac{z}{\bar{z}^2} * \mu_\alpha$, it can be reduced to the first two up to an anti-holomorphic function that can be determined.

What next?

Several interesting and challenging questions — both from the applications and the purely mathematical viewpoint — are yet to be answered.

In analogy with the classical logarithmic case, one could consider different confinements in the dislocation energy (7). This would correspond to testing the stability of vertical-wall structures under different loadings. Alternatively, one could try to find the extremal measure under the constraint that its support is in a given set $G \subset \mathbb{R}^2$, which mimics a grain in metal.

The results in [7] and [1] raise also the intriguing question of understanding the effect of the anisotropy on the dimension of the support of minimisers. More precisely, under what conditions on W is the minimising measure fully dimensional? Or, conversely, what causes the loss of dimension in the case of the potential W_1?

Finally, the three-dimensional case ($N = 3$) is still quite open. In particular, it is not clear whether to expect the minimiser of the analogue of (13) for $N = 3$ to be an ellipsoid that shrinks to a lower-dimensional measure for some critical value of α (representing the 'maximal anisotropy'), as in two dimensions.

Acknowledgements

Lucia Scardia acknowledges support by the EPSRC Grant EP/N035631/1 *Dislocation patterns beyond optimality*.

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Developing Mathematics in East Africa

BALÁZS SZENDRÖI

This article describes the personal journey of its author[1] over the last eight years to help strengthen mathematics, primarily pure mathematics, in East Africa.

The beginning: work in Maseno

It all began in 2010 with the word "quiver". A year earlier, I had applied to be a mentor in the LMS MARM project, responding to not much more than an internal urge to do something exciting. One aspect of MARM is that it acts as a kind of matchmaking service, aiming to find common points of interest between African mathematics departments and prospective mentors. This is not always an easy task. But after an earlier proposal with no clear links to my interests in geometry and algebra, which I decided not to respond to, I was sent another application, from a place right on the Equator in Western Kenya I had never heard of. The list of specialisms of members of staff included Operator theory, PDEs, as well as "quivers". While my work has nothing to do with the first two topics, it has a lot to do with quivers; but it was an odd choice of word, referring to a relatively narrow area of formal algebra, though with links to many other subjects. The person behind this smoking gun turned out to be David Stern, a recent PhD in my field, who had moved to Kenya and taken up a lectureship at the Maseno University following a much stronger urge to do something really exciting. It was clear then that there was common ground, and thus a MARM partnership was established; and with that, a journey for me to Africa.

Together with my graduate student Ben Davison, I paid several visits to Kenya, gave lecture courses on elementary and algebraic geometry and knot theory, and conducted heated discussions with local colleagues on teaching methodologies, the balance of research and teaching, and many other subjects. We hosted in Oxford some of the talented Masters students David was working with at the time; we helped one on his way to a US PhD position. Contacts were established with other Kenyan institutions: the University of Nairobi, the country's oldest and largest, though not necessarily its most innovative, and the dynamic Strathmore University, a Christian private institution with the soft-spoken, highly regarded statistician John Odhiambo as its Vice Chancellor.

I was impressed by the honesty and openness of many of the lecturers and students I met, as I tried (often unsuccessfully) to remember their names. Due to the hospitality and resourcefulness of David and his colleagues, I had the opportunity to visit some of Kenya's beautiful places: lakes, extinct volcanoes, rainforests and the savannah with its non-human inhabitants, carnivores as well as herbivores. But first and foremost, I began to understand better the difficult conditions under which our colleagues in Africa work. The lack of resources is a problem; as is the relative isolation, including distance and cost of travel between, and even within, countries. But the main difficulties arise from the very large teaching and administrative loads that they carry.

Intermezzo: the rapid expansion of University provision in Africa

A campus in the town of Maseno was established in 1990 as a constituent college of Moi University; it became an independent university 11 years later. It soon established its own satellite campus in a nearby town, Bondo; around the time of my first visit, from Bondo Campus arose Jaramogi Oginga Odinga University of Science and Technology. Two branchings within a decade or so on the tree of Kenyan universities — almost exponential growth. Indeed, while in 1980 Kenya had just one public university, in the early 2000s it had six and now it has over twenty. The number of students multiplied even faster. The pattern is replicated elsewhere (though not everywhere) in the region: in the same time span, Ethiopia went from two to eight to over thirty and Tanzania from one to five to eleven public universities.

[1] My work has been supported by various institutions and projects described in Issue 475 of the *Newsletter* (P. Dorey, S. Huggett, J. Hunton and F. Neumann, *Initiatives for Mathematics in Africa*, March 2018), such as the African Institute of Mathematical Sciences (AIMS) as well as LMS schemes including Mentoring African Research in Mathematics (MARM); these will not be described in detail here. I also received warm support, both financial and moral, from the Oxford and Warwick mathematics departments, and many colleagues who have generously offered their time, expertise and advice, for which I am very grateful; most will have to remain unnamed, but you know who you are!

The consequences are easy to imagine. All these universities need Principals, Deans, Departmental Heads, as well as qualified lecturers teaching different fields of mathematics. Education is in massive demand all over Africa[2], and the mathematical sciences are reasonably popular; calls on service teaching further increase the need. But at the moment supply, especially quality supply, is completely unable to keep up with this demand: there is a huge capacity gap. Addressing this gap is perhaps the most important challenge facing anyone interested in the development of mathematical sciences on the continent.

The 2016 Nairobi Workshop in Algebraic Geometry

Mentoring a research group: algebraic geometry in Nairobi

During one of my early visits, I met two ambitious algebraic geometers at the University of Nairobi, Jared Ongaro and Damian Maingi, both with recent PhDs from Europe. In 2013, we agreed to combine forces to organise a summer school in Mombasa that attracted around 40 Masters students from the region. This became the starting point for a long-term collaboration, with yearly workshops on Algebraic Geometry involving a varied group of participants from the region and elsewhere. Laura Costa from Barcelona and Gavin Brown and Miles Reid from Warwick became long-term partners. The LMS helped with MARM and Scheme 5 grants; some senior participants have generously used their own grants while postdocs and graduate students were able to join using support from their departments. The 2018 workshop will be organized in collaboration with the Clay Institute.

Our main success has been to work with a changing cast of Masters students, who often choose their research topics from among the subjects discussed at a recent workshop. In the last few years, Masters dissertations have been written in Nairobi on toric geometry, Kontsevich's recursion for the number of curves in the plane through a fixed number of points, the Hilbert scheme of points of a surface, and the Euler characteristic of the moduli space of curves, reflecting the interests of workshop speakers. After the initial discussions at the workshops, the presence of Damian and Jared on the ground is key to the successful completion of these theses. In 2016, the group graduated their first home-grown PhD, Ben Obiero, supervised by Damian; Ben remains a frequent attendee and speaker at our workshops. Some of the students are teaching at local schools and universities, while others embarked on international PhDs; one, with support from the Oxford Mathematical Institute and the Simons Foundation, is coming to Oxford to start his doctoral studies in October 2018.

On to other foreign lands: work with EAUMP

The 2013 Mombasa school in fact was the yearly edition of an existing series, the Eastern African Universities Mathematics Programme (EAUMP) Summer Schools. EAUMP is a network of five Mathematics departments, started in 2002 by the International Science Programme (ISP) of the Swedish International Development Cooperation Agency. Its members are the departments in Nairobi, Dar es Salaam in Tanzania, Makerere University in Kampala, Uganda, as well as the Universities of Rwanda and Zambia. For the 2013 edition, the School was reorganized, expanded from two weeks to three, and some other innovations were implemented, including a mini-project competition. The collaboration was judged a success, and I was asked to stay involved. Since then, we have co-organised a yearly three-week School in pure mathematics, on subjects as varied as Experimental Pure Mathematics, Functional Analysis and Homological Algebra; the 2019 edition will feature Algebraic Topology, including applications to Data Science.

One early lesson was how popular computer-based demonstrations and projects are at these events. Working with free software that the students can continue to work with is of course essential; SAGE or some variant is now a feature of most Schools. There are always technical challenges to implementation, but the pleasure on students' faces when a computation finally works is worth all the effort.

Core financial support from ISP has been central to the continuing operation of the Schools; CIMPA and the LMS-AMMSI Postgraduate Fund have also

[2]According to a recent survey by the Pew Center, a Washington DC think tank, sub-Saharan Africans rate education as one of their most important concerns, second only to health care and well ahead of government efficiency, access to food, roads or energy.

been very helpful. More recently very substantial funding from ICTP, Trieste has allowed us to expand the invitation list from members of the five EAUMP departments to students from further afield, in particular recent AIMS graduates.

Some workshop participants at the Victoria Falls, 2017

Work with EAUMP has also allowed me to visit the other beautiful countries of the network, and to develop contacts with colleagues there. A workshop in 2017 in Zambia by the Victoria Falls remains a particularly memorable occasion, both for the amount of pointless topology (please read this in the technical sense!) I learned from South African colleagues, and the drenching we got when standing next to the Falls just after the rainy season.

Branching out into new fields: applications

The funding landscape for development-related projects in the UK was dramatically transformed by Government's establishment in 2015 of the £1.5 billion (yes, you read that correctly) Global Challenges Research Fund, managed jointly by the Research Councils and learned societies to help address research challenges faced by developing countries. Pure mathematics is finding it difficult to tap into this source, as the research conducted has to demonstrate direct impact on Sustainable Development Goals[3]. The time seemed right to think creatively about what could be done in this framework.

Help came from the ubiquitous David Stern, whose interests include statistics and software development. In 2016, we successfully applied for a joint grant to implement a recently developed quantitative method to measure corruption risk in government contracting in developing countries. The aim was to bring together my contacts in political science with David's project in statistics programming, in particular work on a new package R-Instat, and take this to an African audience. Together with Elizabeth David-Barrett, a corruption expert, we ran workshops at AIMS Tanzania in 2017 and AIMS Ghana in 2018, with local mathematics students, DfID and World Bank experts, and others; we have invitations to take this work elsewhere. More recently, we also started a small project that aims to use data to help smallholder farmers and farmers' cooperatives, by investigating the effectiveness of certain low-cost innovations.

Epilogue

Mathematics remains in a difficult condition on the African continent, from early education through schools to universities and beyond. In higher education and research, the capacity gap mentioned earlier remains a fundamental challenge. Students on international PhD programmes have a good track record of returning to Africa, but the numbers are small. Local PhD programmes are being developed and should be helped, but there is a shortage of qualified advisors and the quality of the output remains very variable. Perhaps more innovative solutions should also be tried, such as "sandwich" PhD programmes with substantial time spent both in Africa and in the North; such an approach can work, but currently appears very difficult to fund in the UK on anything like the scale it is needed. "Hybrid" PhD programmes, an idea of David Stern, could also be tried, where research is combined with teaching innovation and other aspects of professional development. But one thing is certain: improvements will only come from continuing engagement — external help combined with internal drive to bring about much-needed changes.

Balázs Szendröi

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[3]This point deserves a longer discussion. One can certainly imagine scenarios, say involving cryptography or robotics, where aspects of algebraic geometry become relevant to development challenges. But the basic point remains: pure mathematics, and especially training in pure mathematics, appears difficult to include in GCRF applications given the way the calls are currently configured.

Reciprocal Societies: Mathematical Society of Japan

The Mathematical Society of Japan (MSJ) was founded in 1877, during the Meiji Restoration, as the Tokyo-Sugaku-Kaisha (Tokyo Mathematics Society). Its founding 55 members recognized the importance of mathematics as the basis of all sciences, and promoted European mathematics in Japan. In 1946, the current MSJ was established. Its membership now stands at over 5000, and includes three fields medallists (Professor Kunihiko Kodaira, Professor Heisuke Hironaka and Professor Shigefumi Mori) as well as the first Gauss Prize winner (Professor Kiyosi Itô).

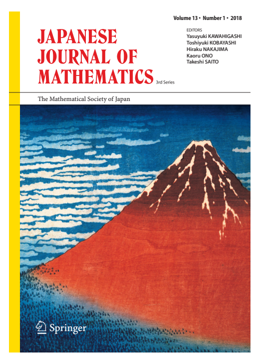
The first MSJ-SI at Kyoto University

The MSJ organizes biannual meetings in spring and autumn. Approximately 1500 members participate in these and more than 400 research papers are presented. In the spring conferences, the Spring Prize, Publication Prize, and Outstanding Paper Prize are awarded. In autumn, the Autumn Prize, Takebe Katahiro Prize, and Seki Takakazu Prize are presented. The biannual meetings include sponsored lectures on mathematics for the general public. The MSJ also holds various symposia in which mathematicians as well as mathematical educators exchange ideas and express opinions.

Apart from these regular meetings, the MSJ coordinates international conferences and scientific meetings. A prominent convention is the Takagi Lectures named after Teiji Takagi, one of the first Fields Medal Committee Members. This is the first series of lectures in mathematics to be crowned with a Japanese mathematician's name. It provides survey lectures by the finest mathematicians from all over the world.

Another international workshop is MSJ-Seasonal Institute (MSJ-SI). This annual meeting is the successor of MSJ-IRI which had been held between 1993 and 2006. The society recognizes the importance of interaction with Asian mathematicians, and so invites promising young Asian students to the seminar.

The MSJ promotes mathematics at all levels. For example, together with the city of Fujioka (where the great Japanese mathematician Seki Takakazu is believed to have been born), it organises an annual 'mathematics for fun' class for the junior high school children of the city. The MSJ also sends mathematicians to elementary, junior and senior high schools to give lectures on fun and interesting mathematics.



The society publishes various journals including, the *Journal of the Mathematical Society of Japan*, *Japanese Journal of Mathematics*, *Sugaku*, *Advanced Studies in Pure Mathematics*, and *MSJ Memoirs*.

Further information about the MSJ can be found on its website, mathsoc.jp/en.

Hideo Kozono
President of the MSJ

Editor's note: the LMS and MSJ have a reciprocity agreement meaning members of either society may benefit from discounted membership of the other.

Microtheses and Nanotheses provide space in the Newsletter for current and recent research students to communicate their research findings with the community. We welcome submissions for this section from current and recent research students. See newsletter.lms.ac.uk for preparation and submission guidance.

Microthesis: Statistical Practice and Replication Success in Behavioural Science

KENNETH T. K. LIM

Behavioural science research findings increasingly influence policy, e.g. the UK's 'Nudge Unit'. However, recent evidence suggests that many published research findings are not replicable. This project looks at how statistics is used in behavioural science research and its relationship with replication success.

Introduction

In 2010, the UK Government established a 'Nudge unit' to use findings from economics and psychology to inform policy decisions. This builds on claims that behavioural science research findings 'justify the need for paternalistic policies to help people make better decisions and come closer to behaving in their own best interest' [1].

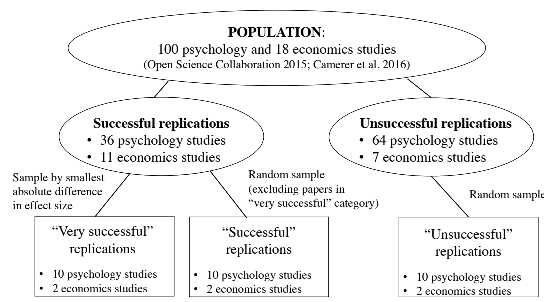
Recently, two teams of researchers tried to independently replicate 100 psychology and 18 economics studies published in the most influential journals. The outcomes were concerning. Only 36/100 psychology studies, and 11/18 economics studies replicated successfully with broadly similar results [4, 5].

This project aims to address two questions. How is statistics used in behavioural science research? Is there a relationship between adherence to recommended statistical practices and replication success? There is some evidence from medicine that methodologically weak studies tend to produce inconsistent results, which are not reproducible [3].

Method

To assess statistical practice, a 100-item checklist was created. The items were mostly informed by recommended statistical practices in psychology [2]. The responses could be categorical or free-text.

The checklist was then used to evaluate a sample of the original 100 psychology and 18 economics studies that were chosen to be replicated. The population of studies was divided into whether they replicated 'successfully' or not, and sampled from three further categories that were created.



Sampling plan of studies to be evaluated

The final sample consisted of 38 studies from 36 papers as one study had multiple parts that were evaluated separately.

Each study was assigned a score based on $k = 42$ checklist items that had categorical responses. The score for the i^{th} study is calculated as

$$\text{Score}_i = \frac{\left(\sum_{j=1}^{k} \text{Y}_{i,j} + 0.5(\text{YS}_{i,j})\right)}{k - \sum_{j=1}^{k} \text{NA}_{i,j}} \times 100\,,$$

where $\text{Y}_{i,j}$, $\text{YS}_{i,j}$, and $\text{NA}_{i,j}$ are indicator functions for the j^{th} checklist item having a response of 'yes', 'yes for some', and 'not applicable', respectively.

The score can range from 0 to 100. A higher score indicates greater adherence to recommended statistical practices.

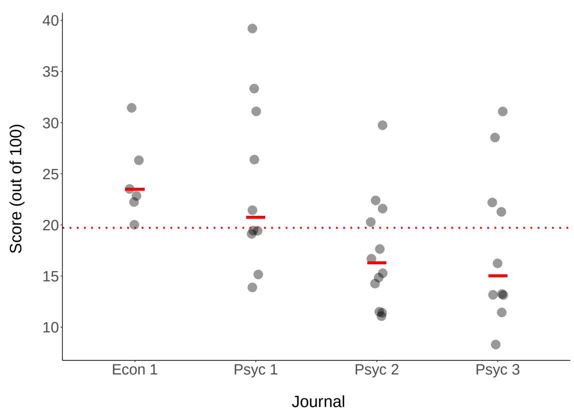
Pilot study results

Selected results are presented in the table below. The checklist items are ordered by the number of 'Yes' responses recorded on the 38 studies evaluated.

Checklist item	Y	N	U
Sample size reported?	38	0	0
Country of study reported?	34	0	4
Evidence data quality was checked?	18	20	0
All outcomes tested reported?	9	11	18
Target population specified?	3	35	0
Sample size planned in advanced?	0	2	36

Selected results. Y: 'Yes', N: 'No', U: 'Unclear'

The strip chart below shows the scores of the 38 studies by journal. Each point represents one observation (score), and is slightly jittered horizontally to improve readability. The median score for all 38 studies is just under 20 out of 100: on average, studies are adhering to 20% of the recommended statistical practices. Across journals, the median scores ranged from about 15 to 23.

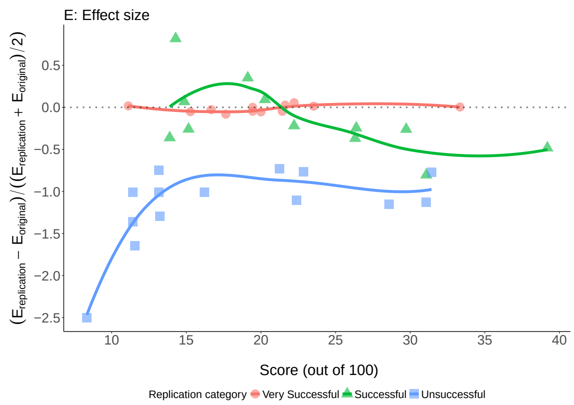


Scores of 38 studies by journal, with median red lines

Eyeballing the scatter plot shows that there is no apparent relationship between scores in this analysis and the normalised difference in effect sizes of the replication and original studies. Smooth trend lines and points are coloured by the replication category that we used for sampling these studies for evaluation. Most of the (blue) unsuccessful replications are around -1.0 on the y-axis, corresponding to a replication effect size of about 1/3 of the original effect size.

Limitations

All studies were evaluated by only one person (KTKL). There is likely to be bias and errors in evaluation. More studies are being independently evaluated by a team of behavioural and mathematical scientists. The information reported in a study may not reflect what happened in practice.



Scatter plot of scores and a measure of effect size

Acknowledgements

The University of Warwick's Bridges-Leverhulme Doctoral Training Programme. Professor Jane L Hutton and Lewis Rendell for feedback.

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So You're a Postdoc. What next?

EMMA COMPTON-DAW

The majority of postdocs will leave their current positions within the next three years, either through choice or, most likely, necessity, as they are usually employed on fixed-term, funding-linked contracts. Where do they go, where do they want to go and, most importantly, how can they be supported to succeed?

This article gives some practical tips for postdocs on how to go about finding, and achieving, their 'What Next?'. It also gives practical tips for PIs and for Institutions on how they can support career independence for their researchers. The advice below is based on my work with postdocs and new academics on achieving their own career progression, as well as on my own experiences of moving out of a cycle of postdocing into a career which I find fulfilling and rewarding. I'll start with some context for why career development and developing independence is critical at the postdoctoral stage.

The background

Career aspirations of postdocs in the UK have been tracked over the last ten years through the *Careers in Research Online Survey* (CROS). Consistently, about 75% of respondents aspire to an academic career and around 60% expect to achieve this [1]. However, if you look around your department you will see that there are more postdocs employed than academic positions coming available. The Royal Society estimated in 2010 that 10%–15% of researchers on fixed-term contracts move into permanent academic positions, and that 1.5% will become professors [2].

This can seem disheartening. However, unemployment amongst PhD holders is low and the evidence suggests that those who have moved into positions beyond academia have high levels of job satisfaction, move readily into leadership positions, and only a minority aspire to return to an academic career [3]. Whatever the career destination may be, it is essential that postdocs are aware of the realities of academic careers and their career options as well as the opportunities to develop the skills and evidence necessary to progress in any career.

Research funders recognise this problem and, in 2008, implemented the *Concordat to Support the Career Development of Researchers* [4], an agreement between the funders and employers to improve the support for researchers (postdocs and related roles) and research careers in UK higher education. In Europe, a *Charter for Researchers and Code of Conduct for the Recruitment of Researchers* [5] addresses similar issues. Institutions implementing these recommendations are able to apply for the HR Excellence in Research Award; currently 98 institutions hold this award in the UK.

During 2017/2018, a review of the Concordat and a community consultation took place, which considered the impact of the Concordat and made recommendations for the future. There has been progress in relation to supporting career development; however, issues persist, particularly in relation to career progression and feeling supported to develop independence. The review recommendations aim to clarify the responsibilities of researchers themselves, their Principal Investigators (PIs), Employers and Funders in supporting the development of their independent careers, regardless of destination.

The key tenet of the 2008 Concordat and of the new recommendations remains the same: researchers and their employers have a role to play in a researcher's career development. Institutions, and PIs, should provide support mechanisms while the researcher must take responsibility for their own career.

A note about Academic vs Non-Academic Careers

We often talk about academic and 'alternative' careers. However, this suggests the two are different and possibly even incompatible: if a postdoc position is an 'academic' track job, surely a postdoc themselves wouldn't be qualified for a career outside academia?

There are some specific differences in the evidence needed when applying for academic roles vs non-academic roles (I'll touch on this later), but the skills you need to excel, and progress, in any career are the same: communication skills, project management, independent and team-working, time management and leadership. As a postdoc, you have these in abundance. The knowledge-base you draw upon may be different beyond academia, but a postdoc position demands that you are a fast and efficient learner so picking up new knowledge shouldn't be a problem.

Advice for postdocs

Start early, plan ahead.

It's very easy to put off thinking about your long-term career goals, particularly if you're on a fixed-term contract. Initially you are focused on getting up to speed in your new position, settling into a new group/institution/city/country, then on producing research outputs. There's never any time to think about 'What Next'. Suddenly the end of your contract is six months away and you only have time to apply for other postdoc positions.

It can be hard to break this cycle and progress to another, 'non-postdoc', role. Finding a position you are confident will be fulfilling and enjoyable will take time and hard work. However, with some planning and commitment you can do it alongside your current role and be ready to make the transition when your contract ends.

The 'What Next' can seem overwhelming and you might even be feeling anxious just reading about it now. You don't have to 'know' at this moment what career you'd like to follow. Set yourself realistic expectations and use the tips in this article to start planning small, achievable steps and goals that will help you progress up your personal career ladder.

The advice in this article is as appropriate for academic careers as it is for non-academic careers — all careers need planning and time to be successful.

Do your research.

Find out more about your options. It's important to be open-minded and try not to make assumptions. Do you really know what is involved in being an academic? Do you really know what someone in industry does day-to-day? Do you know what other options there are?

At first this is about finding out information, not committing to a career. Your research is likely to start out quite broadly and get more focused over time. There are many places to start: researcher and career development teams at your institution will have information on career planning; websites such as Vitae.ac.uk and jobs.ac.uk have excellent information; company/institution websites can give an idea of the sorts of role available; think, and ask, about people who have moved on from your department.

As your focus narrows, start scanning through job adverts to become familiar with the different roles available, the responsibilities that come with them and the skills needed to get them. Find people a couple of years into a position and ask if you could talk to them about their job. You could ask: what do they do day-to-day? how is it different to what they did before? what do they enjoy about it? what don't they enjoy about it? how did they get it? what advice do they have? This will give you invaluable insights that you can't get just by reading.

While you're doing this research, reflect on what is important to you. Which parts of your current job are you passionate about (what are the 'extra' things you add to your role? what topics do you find yourself always talking about?) and will the roles you are considering support this?

Don't do it alone.

Every successful career is supported by a wide and diverse network. This will range from immediate colleagues, friends, and family who can act as a sounding board, to acquaintances whom you could learn from or who could become collaborators. Those you have interviewed about career options may well become an important part of this.

Talking to others about your career plans can seem intimidating, but in reality most people want to help others. Try to be open with those you work with, particularly your PI. If they don't know what career you're thinking about, they won't know how to help you. It's important to talk about this.

Once you have narrowed down your options consider finding an independent mentor: this can be an incredibly powerful form of career development. Many institutions run formal mentoring schemes or you might ask someone outside of a scheme. It's important that your mentor is not part of your line management so they can give you truly impartial advice.

Build your evidence.

You know you can do the job, but how do you prove that to an employer? Whatever career you choose to follow, you will eventually need to provide evidence you can do the job. Looking at job adverts will help you understand what sort of evidence this is, but in general you will need to show that you can independently do the tasks required for the role.

It impossible to say exactly what the criteria will be, although for non-academic careers these tend to be evidencing skills described earlier such as project management, communication, etc. For academic careers it tends to be more focused around research outputs, academic activities (perhaps teaching, knowledge exchange) and — increasingly — evidence that you can bring in funding.

Whatever your career destination, undertaking activities outside of your day-to-day research will support you. There may be role-specific activities you need to demonstrate for a non-academic role (although this is generally far less so than for academic roles) or something that makes you stand out in the increasingly competitive academic job market.

Be open to (and seize) opportunities.

If there's one piece of advice you take from this article it should be this! You can't guarantee where the ideas, people and opportunities which will build your evidence and shape your career will come from, but they are less likely to happen if you stay in your office only talking with your immediate colleagues.

Attend conferences (and talk to people!); if your university has a Research Staff Association join it; go on courses; organise seminars; supervise students; join a committee; do public outreach. You never know where opportunities will arise: I am writing this article after meeting a member of the *Newsletter* editorial board at a course — neither of us went expecting this to be an outcome of the day but we are both very happy it was!

Advice for PIs

Be supportive of career development.

This might sound obvious but it is something that can easily be overlooked when there are pressures for research outputs, grant deadlines, marking, etc, etc. Are you providing an environment that encourages your postdocs to investigate their options AND to build independence?

A good start is to make time to discuss their career plans in a way that is open to all options from the beginning and at regular (e.g., six monthly) intervals. You could use the advice above to guide these conversations — do they have steps planned out? do they know their options? who have they talked with? what opportunities outside of research are there? You should also allow them time to work on these areas. Be supportive of them investigating options, going on courses and building the evidence for whatever roles they might move to. Try not to use language that makes value judgements about different career choices (e.g. 'alternative careers', 'Plan B') or equate success only with gaining an academic position.

There is a tension between the needs of the PI (research outputs, conference presentations) and the needs of the postdoc (skills development, researching options, building independence). It is important to acknowledge this to yourself and your postdoc to build a plan that supports both. From my own experience, I was more dedicated to my research whilst preparing to move to a non-academic role because my PI was supportive and gave me space to develop my independence.

Provide realistic advice.

Be honest with your postdocs about the probabilities of moving to an academic role and the evidence they will need to obtain such a position. The majority aspire to this career, but only the minority will achieve it. Ever increasing numbers of PhD students, and postdocs, mean these positions are becoming ever more competitive and candidates are increasingly being asked for activities beyond just producing research outputs.

More widely, you don't need to be an expert in every career, but you should know who in your institution can give broader advice (researcher development and careers services are good places to start) as well as being familiar with online resources, again, such as Vitae.ac.uk and jobs.ac.uk. Put your own feelings aside — if your postdoc is thinking about a non-academic career it isn't a reflection on you or your career choices, it's about them and their career choices.

Be open and honest about your job. If they are considering an academic career, tell them what you do day-to-day, what are the good things and the bad. Let them read your grant applications, give them opportunities to help you write them and to review papers, and provide time for them to build up independent research. If they are looking beyond academic careers, allow them time to develop activities related to this.

Develop yourself.

If you're not confident about giving this kind of advice or how you might support your staff, take some time to find out how. Most institutions will have courses on people management which should provide you with tools for supporting your staff. You could also talk to your colleagues and ask how they do it.

Advice for Departments

Don't let your postdocs get hidden.

Postdocs are sometimes grouped together with research students and/or new academic staff. However, postdocs have different needs from either of these groups, which can easily be overlooked. There are times when these groupings are appropriate, but step back and ask whether you are supporting all your postdocs or whether they need something separate or additional.

Do you have ways postdocs can represent themselves in your department/faculty/institution? They will be able to help you see areas where they need support which permanent staff might overlook — and identifying these is a valuable career development experience in itself. You could have representatives on committees and/or organise regular (e.g., annual) forums or focus groups.

Provide opportunities for career development.

Be explicit in your support of career development for postdocs. Encourage your PIs to support their postdocs' career development and provide postdocs with time for development opportunities outside of their day-to-day research activities. Consider implementing a policy around how much time this should be.

Whatever career path they follow, your postdocs will need to be able to demonstrate their independence. Are there ways you can involve them in the running of the department? Are there responsibilities they can take on? Are there projects they can get involved with?

One of the most effective approaches to providing career development opportunities is to create (and financially support) a Research Staff Association (RSA). These are groups/committees of postdocs that provide representation and a voice for themselves as well as potentially organising events. It is a fantastic opportunity for postdocs to develop leadership skills and independence outside of their immediate research area and to build a wider network, whilst also contributing to the department.

RSAs take many forms and can reside within a single department or across a whole institution (or even a country). If numbers are small in a department, is there a related area you could join up with or could you organise something less formal such as a regular coffee morning which will bring the postdocs together? As soon as they start meeting, ideas will flow!

FURTHER READING

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Emma Compton-Daw

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Mathematical Research Centres: ICMS and INI

The Engineering and Physical Sciences Research Council (EPSRC) funds two general centres for supporting the mathematical sciences community: the International Centre for Mathematical Sciences in Edinburgh, and the Isaac Newton Institute for Mathematical Sciences in Cambridge. We invited their Directors to tell us about the work of these institutes, and in particular what they offer early career researchers.

International Centre for Mathematical Sciences, Edinburgh

I suspect time is the most necessary resource for most mathematicians: time to think, time to read, time to investigate and time to write. It is also increasingly scarce as so many demands are put upon our time. The idea of a solitary mathematician working in isolation was never really accurate, and contact with other thinkers is a fundamental ingredient to progress. It is the recognition that personal contact between experts in the field can be so much more effective and efficient than exchanging papers (or using emails or Skype) that drives mathematical scientists to attend workshops and conferences. To the extent that scientific infrastructure is the enabler of scientific progress, a mathematical scientist's infrastructure is the opportunity to meet with other experts and to be able to develop new ideas either in collaborations forged through those conversations or individually, given confidence by the support of those conversations. The International Centre for Mathematical Sciences (ICMS) in Edinburgh supports the mathematical sciences community, providing time to develop international collaborations and time to talk with colleagues from across the world.

The ICMS is one of two general centres funded by the Engineering and Physical Sciences Research Council to support the mathematical sciences community, the other being the Isaac Newton Institute in Cambridge. At ICMS we run one-week research workshops, connecting the UK research community to their international colleagues and to the broader scientific and industrial communities. We can also fund smaller groups to undertake focussed research with international or industrial partners (Research in Groups and Research Partnerships with Industry), and some follow-on activity. ICMS can also organize workshops funded from external research grants, allowing researchers to concentrate on the academic side of the organization whilst we deal with the administration.

The new ICMS grant was announced in December 2017, so now is a good time for everyone to think about how they might use the resources available. We are particularly keen to encourage early career researchers (ECRs) and there is funding for workshops with ECR Organising Committees; we offer mentoring from a more experienced colleague if they wish. A new class of strategic workshops has been introduced to allow us to respond quickly to changing research and funding landscapes. We also support diversity in a number of ways. Participants can get help with extra expenses for childcare or other caring responsibilities, and the Programme Committee considers the diversity of participation as part of their criteria for assessing proposals.

Participants at the Harmonic Analysis and Its Interactions workshop, July 2017

ICMS Workshops are successful and popular. Our staff members are professionals — this is what we do — and this means that scientific organisers can concentrate on the science. We cover all areas of the mathematical sciences and their applications, and attract both Fields medallists and PhD students. In September we will move to a new building in the centre of Edinburgh with great views and more space for interaction.

ICMS provides a supportive, inclusive arena that allows the flourishing of research excellence in the mathematical sciences together with all of its possible links, as well as the people who undertake this research. Our website www.icms.org.uk has information about how to apply for funding at ICMS, and the timing of our calls for proposals. We are a resource for the community: use us.

Parts of this article were originally published on the EPSRC Guest Blog, and in Mathematics Today.

Paul Glendinning
Scientific Director of the ICMS

Isaac Newton Institute for Mathematical Sciences, Cambridge

In July 2018, the Isaac Newton Institute for Mathematical Sciences (INI) marks the end of its 25th anniversary year. The INI is an international research institute, based at the University of Cambridge, which runs a series of long-term visitor programmes across the entire spectrum of the mathematical sciences. These four and six-month programmes have covered a vast range of topics — from topology and quantum field theory, to the mathematics of sea ice phenomena, uncertainty quantification and statistical analysis of Big Data — with applications over a wide range of science and technology. The INI's online archive of talks and seminars currently contains well over 7000 videos, totalling over 40 terabytes of data, which have between them garnered over three million views.

Some 2500 participants pass through our building each year. At any given time INI's creative collaborative space is occupied by up to 60 mathematical scientists, with many more in attendance during the regular workshops, exhibitions and other one-off events. Many participants will have come from very different countries and cultures, many will not have met before, and many will not have recognised the relevance of other research to their own work. The INI is especially important as a forum where early career researchers meet senior colleagues and form networks that last a lifetime.

During each scientific programme many new collaborations are made, and ideas and expertise are exchanged and catalysed through lectures, seminars and informal interaction. This is enhanced by the open architecture of the INI building, which was designed specifically to foster such engagement.

Sir Andrew Wiles at the INI's 25^{th} anniversary celebrations

Founded on the 350^{th} anniversary of Sir Isaac Newton's birth, the Institute's name reflects this great scientist's strong association with the University of Cambridge, and his achievements in the fields of mathematics, optics, physics and astronomy. This could not be more fitting, as INI continues the tradition of crossing the boundaries between scientific disciplines; our visiting scientists now cover the mathematical, physical and biological sciences; engineering and data science; the environmental and social sciences; and increasingly are reaching into areas such as economics, policy and law. The Institute's scientific steering committee continue to encourage proposals of the greatest scientific merit, and those most likely to yield the most impactful breakthroughs.

None of this, of course, would be possible without support. Aside from philanthropic donations, INI sits alongside Edinburgh's International Centre for Mathematical Sciences (ICMS) as one of two mathematical centres funded by the Engineering and Physical Sciences Research Council. This six-yearly grant was renewed in the first half of this year, increasing by nearly 30% on the previous award. We are therefore seeking high quality proposals in all areas of the mathematical sciences; I am happy to discuss ideas and offer advice to anyone interested in submitting a proposal.

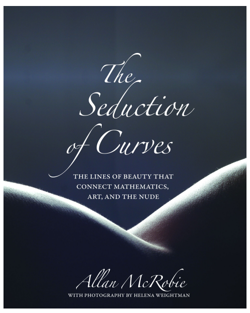
Visit our website at www.newton.ac.uk and discover more about this remarkable place and what it can offer to you.

I. David Abrahams
Director of the INI

The Seduction of Curves: The lines of beauty that connect mathematics, art, and the nude

by Allan McRobie, Princeton University Press, 2017, hardback, 168pp, £27.95, US$ 35, ISBN: 978-0-691-17533-1

Review by Henrik Jeldtoft Jensen



I have a feeling that many mathematicians may have touched on Thom's catastrophe theory but without realising its deep connection to art and the visual experience of observing the human body. McRobie's book is simply straightforwardly very enjoyable, informative and beautiful. Reading it elevates me to a state of wonder that I imagine may have been more commonly encountered back when renaissance people felt it natural to be equally interested in the arts, nature, mathematics, science, astronomy, philosophy, theology and any other curiosity that was considered intrinsic to being human.

McRobie elegantly manages to present the essence of Thom's insights as a natural tool for any careful spectator engaged sufficiently deeply with the surrounding world, and in particular the naked human body, to feel tempted to draw what one is seeing. We are presented with Thom's seven elementary catastrophes as "The Alphabet of Beautiful Curves". And beautiful it is indeed, like when we are presented with the cusp as found when a woman's "cleavage is viewed obliquely" or when we are presented with the swallowtail "nestled between the Paps of Jura". We further learn that three of Thom's catastrophes are impossible to spot as projections of the human body. These are the umbilics which involve projections of manifolds that in three dimensions have to intersect themselves. Thom suggested that the hyperbolic umbilic appears when a wave breaks, which inspires McRobie to include an impressive photograph of a towering turquoise breaking ocean wave.

We are probably all aware that seeing is far from a mechanical automatic process in which somehow the eye functions as the brain's recording device. The eye is an integrated part of the brain and seeing is deeply dependent on our intellectual abilities or as Robertson Davies puts it in Tempest-Tost: "The eye sees only what the mind is prepared to comprehend." McRobie demonstrates how Thom's topology can become for us an expansion of our mind and hence allow us to see deeper. Flickering patterns on the surface of the sea or at the bottom of the swimming pool gain in richness. With an awareness of Thom's catastrophes we may also begin to experience the paintings in the art museums differently. Flickering light and paintings were always beautiful, but McRobie — together with Thom — now make us appreciate that what we are looking at are manifestations in our three-dimensional physical world of robust universality classes residing in the world of mathematical abstraction. And they, i.e. Thom's topological catastrophes, surround us close and far. In the optics of a wine glass, in the rainbow or in the duplication of distant galaxies by the bending of light through gravitational lensing.

The beauty extends far beyond the immediate visual beauty, which of course in itself is of immense value. The beauty and the appeal of the book and its subject is obviously present in the mathematics. It always is for the expert mathematician, but I believe McRobie's book is a rare instance where the beauty, surprises and excitement of mathematics becomes obvious to everyone. We are confronted with catastrophes, but we learn that in fact Thom's theory is about stability. The seven catastrophes describe different ways abrupt changes can arise from smooth geometric forms, but the seven classes themselves are stable and robust and, like all other mathematics, eternal truths.

Go and get hold of this book. Really no matter who you are. It will cost you somewhere about £25, but it is a piece of art, science, culture as well as a good solid genuine critique of platitudes from postmodernism. No matter your background, you'll learn. If you happen to be in charge of the government's department for the school syllabus in mathematics, you'll be surprised that math is much more than multiplication tables, if you are a pure mathematician you most likely will be pleasantly surprised to learn how influential mathematics through Thom can be on artists. We are told about Dalí's profound reaction. If you are an artist, you may very much enjoy how catastrophes links our perception of form to eternal mathematical principles. If you — like myself — planned to become a structural engineer but swapped engineering for mathematical physics, because you found statics a bit too concrete, you will now understand that you never understood the profoundness of mechanical stability.

If just more books like McRobie's existed, we might be able to realise that our world is a whole. That overspecialisation is dreadful because it makes us unaware of how interconnected different disciplines are and how much more wise and richer we all could be if we make an effort to let inspiration flow between mathematics, art and science.

Henrik Jeldtoft Jensen

Henrik Jeldtoft Jensen is a professor of mathematical physics in the maths department at Imperial College London. He leads Imperial's Centre for Complexity Science. His interests straddle effortlessly mathematics, physics, neuroscience, arts, music, theology. Current research projects include a collaboration with Guildhall School of Music and Drama on the effect of improvisation in classical music on performers and audience.

Ten Great Ideas about Chance

By Persi Diaconis and Brian Skyrms, Princeton University Press, 2017, hardback, pp280. £16.99, ISBN 978-0-691-17416-7.

Review by Sally McClean

This book originated in a course taught at Stanford by the authors for about ten years, aimed at students from different disciplines and covering aspects of history, probability and philosophy. The book describes the historical development of probability while embedding these ideas within the appropriate philosophical concepts.

Chapter 1 opens with early thinking about probability dating back to the Ancient Greeks, but it was not until the 16th and 17th Centuries that the idea that chance can be measured emerged. Among these pioneers, were many mathematicians and scientists, such as Cardano, Galileo, Pascal and Fermat, the latter two extending the ideas of measuring probability to developing early concepts of expectation.

In Chapter 2 the focus is on measurement as probabilities, based on coherent judgments. These ideas led to the development of conditional probability and utility. The following chapter provides an interesting discussion of the psychology of probability where individuals may misinterpret probabilities, possibly depending on their degree of risk aversion. Such situations can lead to non-rational decision making. The connection with frequency was established through Bernoulli's Weak Law of Large Numbers paving the way for the determination of chance from empirical data. The Weak (and Strong) Law of Large Numbers led to von Mises putting probability theory on a sound mathematical footing, by postulating infinite sequences that are produced with probability 1 by independent and identically distributed sequences.

Chapter 5 discusses a huge step towards a strong mathematical foundation for probability theory originating in 1933 when Andrei Kolmogorov published a monograph which formalised conditional probability and provided an extension theorem which showed how to build an infinite-dimensional stochastic process from a consistent set of finite-dimensional stochastic processes.

Another major development, known as inverse inference, is described in Chapter 6. This work paved the way for modern statistics and gave a mathematical foundation for statistical inference, from frequencies to chance, following the work of Thomas Bayes, first published in 1763; predictive probabilities, the probability that something will happen next time given the history, could thus be calculated. This topic was taken further by Laplace who developed ideas of updating probability by conditioning. Thus statistical inference can be thought of as part of probability.

While Bayesian analysis is essentially parametric, in Chapter 7, we learn about de Finetti's idea of subjective probability which replaced Bayesian priors with a symmetry condition on degrees of belief. Weaker symmetries, such as those found in Markov chains, can be similarly handled.

The next chapter focuses on computational aspects especially algorithmic randomness and pseudo-randomness, concepts that have found practical application in various settings. This has led to a theory of algorithmic randomness within the Kolmogorov framework and the concept of a (generated) sequence via computational theory, as developed by Martin-Löf and others.

Dating back to Boltzmann in the 1870s, physics has embraced ideas and models from probability theory, leading in the 20th Century to statistical and quantum mechanics. Such topics are still very active within modern physics and pose many interesting philosophical and mathematical problems.

In the final chapter, the authors conclude by revisiting the idea of induction and return to the idea, dating back to David Hume, that it is logically possible to be a consistent sceptic. There are various ways in which inductive scepticism can provide mechanisms for tackling this problem and facilitating induction through coherent belief.

Overall, I thoroughly enjoyed this book which is very readable for those with a background in probability and statistics, with the main ideas presented in an intuitive and accessible manner, with a probability tutorial also provided as an appendix. A reader with a general mathematical background should therefore be able to follow the development without much difficulty. An extensive bibliography is also provided, chapter by chapter, so that the reader is able to extend their knowledge, if so desired. As the title suggests the authors develop their theme through a series of ideas and concepts, some of which follow on from each other while others are more tangential. A number of major breakthroughs in diverse areas are presented: gambling, psychology, mathematics, philosophy, statistics, economics, finance, physics and computer science. I particularly liked the way the concepts were connected, how one led to another and the fascinating insights into how these ideas originated.

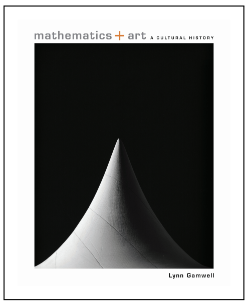
Sally McClean

Sally McClean is Professor of Mathematics in Ulster University with particular interests in Statistics and Probability. Sally was born in Belfast and now lives in Portrush on the north coast of Ireland. She is a long-term member of the Bannside Rambling Club and enjoys trying to play Irish Music.

Mathematics + Art: A Cultural History

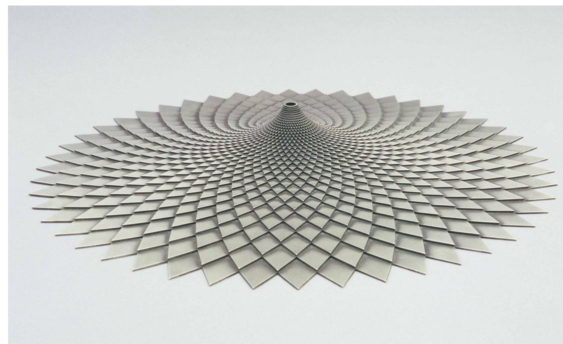
by Lynn Gamwell, Princeton University Press, 2015, hardback, 576pp, £40, US$ 49.50, ISBN: 978-0-691-16528-8

Review by David Singerman



The first thing to say about this book is that visually it must be the most beautiful book with mathematics in its title. It is 550 pages long and every other page is filled with beautiful works of art and mathematics diagrams. The book begins by stating its aims: "Non-specialists who read about mathematics are often disappointed because its secrets are written in a technical language they don't understand. My goal has been to describe in plain English, together with clear symbols and cogent diagrams the ideas that drive mathematics". The author also writes, "to research *Maths and Art* I had to learn maths concepts like calculus, group theory and predicate logic. As a novice struggling to understand these ideas I was struck with the poor quality and confusing content of illustrations in most educational books". Although the book's diagrams are very good, the actual mathematics is sometimes not there. The author tries to avoid formulae and even simple proofs; in her section on irrational numbers she does not give the simple proof that the square root of 2 is irrational. She does tell us that 1.414213562373 multiplied by itself equals 2.0000000011302011499 "but there is no rational at all, no matter how many decimal places when multiplied by itself equals *exactly* 2." When discussing the golden ratio she does not give its formula using the square root of 5; it is just the number 1.618.

However, the section on the golden ratio is interesting. She writes "a common misconception is that artists used the golden ratio since antiquity but it was not associated to art and beauty since the mid-nineteenth century in Germany". She tells us that the golden ratio was first thought of as important for theological reasons. The Renaissance mathematician Fra Luca Pacioli discussed this number in his book *La Divina Proportione*. The ratio is irrational in that it is not expressible as a ratio of whole numbers so, according to Pacioli, it is a symbol for the divine who is beyond reason and whose name is not expressible in words.



Simon Thomas, *Planeliner*, 2005. (Courtesy of the artist.)

He also declared that the dodecahedron whole pentagonal faces are constructed using this ratio symbolises heaven. Pacioli's book was published in Milan in 1509. While in Milan he met Leonardo da Vinci who was seeking a geometry teacher to help him understand linear perspective. In exchange, Leonardo illustrated Pacioli's book including his famous diagram of an open dodecahedron. Never before has a mathematics book had such a great illustrator! As the author states, the golden ratio played no role in Renaissance art (or even in the construction of the Parthenon). However, in the 19th and 20th centuries, some architects and artists, such as Salvador Dáli and Le Corbusier did use it. "Dáli let himself be carried away by the myth that Leonardo had used the golden ratio in his art". Dáli then produced a masterpiece *The sacrament of the last supper*. Not only was this painted on a golden rectangle, it also used an open dodecahedron to symbolise heaven.

A topic that should bridge the gap between art and mathematics is perspective. We are told that linear perspective was founded by the Italian architect Filippo Brunelleschi as a tool to draw three dimensional buildings on a two-dimensional surface while accounting for apparent distortion relative to a viewpoint. "Early Renaissance artists no longer painted saints floating in a golden nest in a faraway place; linear perspective gave them a tool to depict Jesus and the apostles right here." Of course, the study of perspective led to projective geometry. But the next mathematician to be discussed writing on this subject is Poncelet in the 19th century. No mention is made of Desargues. We then jump from Poncelet to Brouwer and it is claimed that Brouwer invented topology as a generalisation of projective geometry! Whatever happened to Euler and Riemann? Also, no mention is made of J.M.W. Turner who besides being a great painter was also Professor of Perspective at the Royal Academy from 1807 to 1837. He gave six lectures on perspective which you can read about in Eric Shanes' book *The Young Mr Turner.* For these lectures Turner presented a number of beautifully drawn geometric diagrams. See bit.ly/2tqGJzC, or Shanes' book.

Another topic that should link art and mathematics together is symmetry. We are told that Group Theory is the mathematics behind symmetry but the term *group* is not defined. There is a nice diagram that exhibits the two, three, and four-fold symmetries of a cube and there is a whole page giving a detailed pictorial description of the Klein four group. What I found valuable about this chapter is that it introduced me to the work of Andreas Speiser. In 1923, Speiser published a group theory text *Die Theorie der Gruppen von endlicher Ordnung* and in his second 1927 edition he added a chapter applying group theory to the decorative arts.

Sylvie Donmoyer, *Still Life with Magic Square*, 2011. (Courtesy of the artist.)

In a later work *Die mathematische Denkweise* he wrote "The oldest examples of surface ornamentation are from Egypt. We do not know whether they had a mathematical theory of groups but their figures are certainly a geometric achievement." He claims that the Egyptian weavers had found all the seventeen plane patterns (wallpaper groups) which I found quite amazing. Speiser's student Edith Müller wrote a thesis on the Islamic tilings in the Alhambra. When Montesinos claimed in his book *Classical tessellations and three manifolds* that all seventeen patterns could be found in the Alhambra it created a bit of a stir as Grünbaum and Shephard had found only thirteen of them. However, Blanco and Harris, *Symmetry groups in the Alhambra* (2011) provided evidence that Montesinos was correct. Note that the ancient Egyptians weaved their patterns around 2000 years earlier than the Alhambra!

There is much, much, more in this book. There are chapters on infinity, formalism (including Hilbert's formal axioms for geometry and consistency), logic, incompleteness of mathematics, computation and computers in mathematics and art. There is almost as much philosophy in this book as mathematics. Some of this goes quite deep and I wonder why the square root symbol cannot be explained whereas the ideas of Kant, Wittgenstein, Kiergkegaard, Hussel and many others can be.

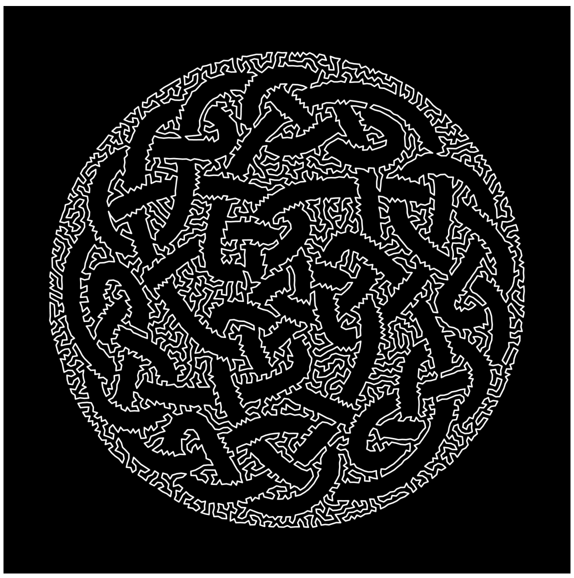
This book is very heavy in weight. It weighs in at over three kilos so you cannot, without difficulty, take it on the train to read. It really is a coffee table book and it has attracted attention on my coffee table. Also, for such a big beautifully illustrated book it seems remarkably cheap. Very much cheaper than a comparative maths book.

The main joys of this book are the paintings and other illustrations, most of which have some mathematical relevance. For someone who clearly does not have a mathematics background (she is primarily an art historian), this is a brave and successful attempt at explaining mathematical ideas to those primarily interest in art.

The main reason for buying this book is the art so I will end by mentioning some of the paintings. The publisher has allowed us to show three in this article. These are as follows.

Sylvie Donmoyer (French, b. 1959), *Still Life with Magic Square*, 2011. Oil on canvas. This contemporary French artist has painted puzzles and geometric objects against the backdrop of two sixteenth-century publications on mathematics. Albrecht Dürer's *Melencholia 1* (1514) and Wenzel Jamnitzer *Perspectiva Corporum Regularium* (Perspective of regular solids 1568).

Simon Thomas (British, b. 1960), *Planeliner,* 2005. Bead blasted stainless steel. Simon Thomas is a young British artist whose typical work, such as this sculpture, is a visualization of a mathematical formula. He studied visual arts at the Royal College of Art in London in the 1980s and went on to create sculpture with striking geometric patterns, serving as artist-in-residence at the University of Bristol, in both the Department of Physics (1993–95) and the Department of Mathematics (2002).



Robert Bosch, *Knot?* 2006. (Courtesy of the artist.)

Robert Bosch (American, b. 1963), *Knot?,* 2006. Digital print. With the development of railroads in the nineteenth century, the topic of finding an optimal route for a journey was of practical interest. The topic entered the mathematics literature in 1930, when the Viennese mathematician Karl Menger described it as the "messenger problem" (das Botenproblem) of finding an optimal delivery route. It was soon dubbed the "travelling salesman's problem" The American mathematician Robert Bosch drew this continuous line based on the solution to a 5000-city instance of the travelling salesman problem. From a distance, the print appears to depict a black cord against a grey background in the form of a Celtic knot. But on close inspection the apparent "grey" is actually a continuous white line.

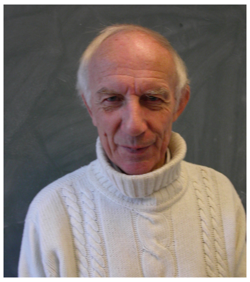
However, one of the loveliest illustrations in the book is just mathematical. It is of the 240 points closest to the origin in the E_8 lattice projected to a plane. You can see this in the article by John Baez, *From the icosahedron to E_8* published in the May 2018 (Issue 476) of the *LMS Newsletter*.

David Singerman

David Singerman is an emeritus professor at the University of Southampton. His main interests have been on Fuchsian groups and Riemann surfaces, in particular the theory of maps (or "dessin d'enfants") on Riemann surfaces.

Obituary of Member

Jan-Erik Roos: 1935 – 2017



Professor Jan-Erik Roos, who was elected a member of the London Mathematical Society on 17 October 1980, died on 15 December 2017, aged 82.

Clas Löfwall (University of Stockholm) writes: Jan-Erik Roos died in his home in Uppsala, Sweden. He was born in Halmstad and he started his mathematical career as a student of Lars Gårding in Lund. He presented his licentiate thesis 1957 about ordinary differential equations. After a period in Paris, during which he met all the leaders in algebra at that time, he returned to Sweden where he started to build a school in algebra. He became professor in Stockholm in 1970 without a doctor's degree; there had never been time for him to get a doctorate! With enthusiasm and openmindedness, he created a warm atmosphere in the department that through the years attracted many guest researchers and doctoral students.

I also came from Lund to Stockholm in 1970 as a student and I had Jan-Erik as my supervisor. I produced a formula for the Poincaré series of a local ring (R, m) with $m^3 = 0$. Seeing this, Jan-Erik was able to prove an analogous result for CW-complexes of dimension at most four, about which he got in contact with a research group in rational homotopy theory. An intense period of cooperation began which culminated with the conference *Algebra, Algebraic Topology and their Interactions* in Stockholm 1983. A major contribution of Jan-Erik's career was bringing together the two research fields of local algebra and rational homotopy.

Jan-Erik started his career in the field of local noetherian categories, and he ended up an expert in experimental mathematics. He was able to find algebraic objects with 'strange' behaviour through computer experiments but was not always so interested in proving his findings! I had the privilege of collaborating with him a couple of times searching for proofs.

One particularly memorable occasion was when our friend David Anick had offered to buy dinner at the Operakällaren restaurant for anyone who could prove a problem he set up about a Lie algebra with quadratic relations and bounded growth. When I saw that Jan-Erik had found such an example experimentally, I offered him the dinner if he could present the proof. We spent a long time working together and at last, we could invite each other to the restaurant, write to Anick and share the bill!

Jan-Erik was always working on a new article and he produced a lot, many of them as Comptes Rendus notes. His last article he sent to arXiv in spring 2017. Even if he spent most of his time doing mathematics, his family was always his first priority. He married Karin in the early seventies and they had a daughter, Sara. Karin also brought to the family two daughters from a previous marriage, Anna and Eva. Jan-Erik and Karin were always very welcoming; guests at the department were always welcome to visit them in their big house in Uppsala, even if it was not so easy to find a free place among all the books and papers!

LMS Popular Lectures

Location: London and Birmingham
Date: 4 July and 19 September 2018
Website: lms.ac.uk/events/popular-lectures

These are free annual events, open to all, which present exciting topics in mathematics and its applications to a wide audience. Speakers: Katie Steckles (*Maths's Greatest Unsolved Puzzles*) and Jennifer Rogers (University of Oxford; *Risky Business*).

Non-associative Algebras and their Applications

Location: Lancaster University
Date: 9–13 July 2018
Website: tinyurl.com/ycvmnob8

This conference will explore four interconnected and overlapping themes: Lie theory, both finite and infinite-dimensional, and its generalisations; New classes of non-associative algebras; Higher structures; and Applications. Supported by an LMS Conference grant.

Quantum Roundabout 2018

Location: University of Nottingham
Date: 11-13 July 2018
Website: tinyurl.com/yc7597ah

This is a postgraduate student conference on mathematical foundations and applications of quantum physics, which will bring together PhD students and postdoc researchers in related areas. Supported by an LMS Postgraduate Research Conference grant.

LMS Invited Lecture Series 2018

Location: University of Warwick
Date: 9–13 July 2018
Website: tinyurl.com/ybz5oq7s

The 2018 Invited Lecturer is Art Owen (Stanford). Supplementary lectures by Nicolas Chopin (ENSAE), Mark Huber (Claremont-McKenna) and Jeff Rosenthal (Toronto). A course of ten lectures will be presented over a week.

Recent Advances in the Statistical Analysis of Extreme Environmental and Actuarial Risk

Location: University of Nottingham
Date: 10 July 2018
Website: tinyurl.com/ybh2yypa

The meeting will focus on state-of-the-art techniques and tools for the statistical analysis of extreme risk in the areas of environmental science and actuarial mathematics. Supported by an LMS Celebrating New Appointments Scheme 9 grant.

Geometry, Derived Categories and Moduli Spaces

Location: University of Plymouth
Date: 12-13 July 2018
Website: tinyurl.com/ybak2aug

This two-day conference will focus on geometry, derived categories and moduli spaces. Supported by LMS Celebrating New Appointments Scheme 9 grant awarded to Nathan Broomhead and Marina Logares.

LMS Meeting at the ICM 2018

7 August 2018; 6.00 – 7.00 pm, ICM, Rio de Janeiro

Website: tinyurl.com/LMSRec2018

This meeting is open to all, including non-LMS members. Members' Book signing will take place during the opening of the meeting and Society Business section.

The LMS lecturer will be Marta Sanz-Solé (Barcelona University), who will give a talk on *From gambling to random modelling*.

The meeting will be followed by the LMS reception for members and guests. LMS members can register for a ticket online.

For further details about the ICM 2018 and to register for a place, visit the ICM website at tinyurl.com/yb4wolad.

Model-theoretic Methods in Number Theory and Algebraic Differential Equations

Location: University of Manchester
Date: 3-5 August 2018
Website: tinyurl.com/ya785h5f

The meeting will connect groups studying applications of model theory and differential algebra to complex exponentiation, real-analytic geometry and number theory/algebraic geometry. Supported by an LMS Conference grant and Manchester University.

33rd British Topology Meeting

Location: The Open University, Milton Keynes
Date: 4-6 September 2018
Website: tinyurl.com/yanykmrs

BTM33 dovetails with workshops at the Homotopy Harnessing Higher Structures programme at the Isaac Newton Institute. Supported by an LMS Conference grant and The Open University. Parents may apply to the LMS Childcare Supplementary Grant Scheme.

Arithmetic Ramsey Theory

Location: University of Manchester
Date: 10-13 September 2018
Website: tinyurl.com/ARTinMCR

Recent advances in our knowledge of structure within large arithmetic sets have used a diversity of techniques, including the polynomial method, ergodic theory and model theory. The purpose of this meeting is to bring together researchers in these fields. Supported by an LMS Conference grant.

Mathematical Foundations of Computer Science

Location: University of Liverpool
Date: 27-31 August 2018
Website: tinyurl.com/ya4gk6wb

This is a high-quality venue for original research in all branches of theoretical computer science. The conference encourages interactions between researchers who may not otherwise meet. Supported by an LMS Conference grant.

BSDEs, Information and McKean-Vlasov Equations

Location: University of Leeds
Date: 10-12 September 2018
Website: tinyurl.com/y7tzbn92

This meeting will disseminate recent results on three popular subjects in stochastic analysis: BSDEs, Information and McKean-Vlasov equations. Supported by an LMS Conference Grant and Leeds University.

Dragon Applied Topology

Location: Swansea University, Singleton Campus
Date: 11-14 September 2018
Website: tinyurl.com/ycauz3su

This conference focuses on the vibrant field of Topological Data Analysis and a family of methods aiming at classifying the shape of the data. Financial support for PhD students, early career researchers and participants from developing countries. Supported by an LMS Conference grant.

LMS–IMA Joint Meeting: Noether Celebration

11 September 2018, De Morgan House, London

Website: tinyurl.com/LMS-IMANoether2018

The speakers will be Katherine Brading (Duke University), Elizabeth Mansfield (University of Kent), Cheryl Praeger (University of Western Australia), Norbert Schappacher (I.R.M.A. / U.F.R. de mathématique et d'informatique) and Reinhard Siegmund-Schultze (University of Agder).

The meeting is free to attend. Please register for your place online.

The meeting includes lunch and will be followed by a reception.

After the reception, the LMS and IMA will host a Joint Society Dinner at a nearby venue. The cost to attend the dinner, including drinks, is £30 per person. If you would like to attend the dinner, please email Elizabeth Fisher (lmsmeetings@lms.ac.uk).

Das Kontinuum — 100 years later

Location: University of Leeds
Date: 11-15 September 2018
Website: tinyurl.com/y7og2738

Weyl proposed in Das Kontinuum (1918) a radically new 'predicative' foundation of analysis, showing how large portions of the subject could be developed assuming only the natural numbers. This conference brings together mathematicians and philosophers working in areas related to Weyl's legacy.

Probability and Nonlocal PDEs: Interplay and Cross-Impact

Location: Swansea University
Date: 17-19 September 2018
Website: tinyurl.com/y6vmvute

The aim of the conference is to present results and methods related to the interconnection between Probability and nonlocal PDEs and to outline further directions of their cooperation and mutual impact. Enquiries to d.l.finkelshtein@swansea.ac.uk.

BCS-FACS Seminar 2018

Location: De Morgan House, London
Date: 1 November 2018

This is an evening seminar organised by the LMS and FACS, the BCS Specialist Group for practioners in Formal Aspects of Computing Science. The speaker, Bill Roscoe (Oxford), will talk on a topic at the interface of mathematics and computer science. To register interest, email lmscomputerscience@lms.ac.uk.

Topology and Applications

Location: Cochin, India
Date: 5-11 December 2018
Website: tinyurl.com/y9s3cg6p

This international workshop and conference in honour of Professor Thekkedath Thrivikraman, who has made important contributions to Topology and Fuzzy Mathematics, will take place at the Rajagiri School of Engineering & Technology.

Combinatorial Algebraic Geometry

Location: University of Bristol
Date: 17 September 2018
Website: tinyurl.com/yawecbh3

This meeting will connect experts in combinatorial algebraic geometry to present their work and explore collaboration opportunities. Anyone interested is welcome to attend: email the organiser (fatemeh.mohammadi@bristol.ac.uk). Supported by an LMS Celebrating New Appointments Scheme 9 grant.

Functor Categories for Groups

Location: Senate House, Central London
Date: 21 September 2018
Website: tinyurl.com/y7wcpmn6

This meeting will focus on the study of Hausdorff dimension for profinite groups, initiated by Abercrombie in the 90s. Limited funding is available for PhD students. To register, email the local organiser (brita.nucinkis@rhul.ac.uk). Supported by an LMS Joint Research Groups in the UK Scheme 3 grant.

Manifolds

Location: Isaac Newton Institute, Cambridge
Date: 3-7 December 2018
Website: tinyurl.com/yb4nvqjh

This workshop will bring together mathematicians to develop new foundational results, pursue applications, and relate these new ideas with more classical work in the topology of manifolds. Closing date for applications is 2 September 2018.

Conclusions and Future Directions

Location: Isaac Newton Institute, Cambridge
Date: 10–14 December 2018
Website: tinyurl.com/y9a66h8b

This concluding workshop will cover progress during the programme, new open problems and future directions. Closing date for applications is 17 September 2018.

LMS Prospects in Mathematics Meeting

7–8 September 2018

Mathematics Institute, Zeeman Building, University of Warwick,
Gibbet Hill Road, Warwick CV4 7AL

All Finalist Maths Undergraduates who are considering applying for a Maths PhD in 2019 are invited to the 2018 LMS Prospects in Mathematics Meeting.

The meeting will feature a range of speakers from a wide range of mathematical fields across the UK who will discuss their current research and what opportunities are available to you.

Speakers:

- **Statistics & probability:** Jason Miller (Cambridge); Christina Goldschmidt (Oxford); Darren Wilkinson (Newcastle)
- **Discrete mathematics:** Nina Snaith (Bristol); Julia Boettcher (LSE)
- **Analysis & applied mathematics:** Federica Dragoni (Cardiff); Ben Leimkuhler (Edinburgh); Felix Schulze (UCL); Ivan Graham (Bath)
- **Pure mathematics:** Rachel Newton (Reading); András Juhász (Oxford)
- **Mathematical biology:** Chandrasekhar Venkataraman (St Andrews)

50 places are available, including overnight accommodation and some funding towards travel costs.

To apply:

Please apply online at https://warwick.ac.uk/fac/sci/maths/research/events/2018-19/lmspm/ or send an email to Dr Stefan Adams (s.adams@warwick.ac.uk) headed Prospects 2018 Application with the statement: "I am on track academically to begin Ph.D. studies in 2019" with evidence of your predicted degree classification. **Application deadline is 18 July 2018.** (Late applications will be considered at the organisers' discretion.)

Society Meetings and Events

July 2018

4 LMS Popular Lecture, London
9–13 LMS Invited Lecture Series 2018, University of Warwick

August 2018

7 LMS Meeting at the ICM, Rio de Janeiro

September 2018

11 Joint Society Meeting with IMA: Noether Celebration, London
19 LMS Popular Lecture, Birmingham

October 2018

9 Joint Society Meeting with the Fisher Trust, Galton Institute, Genetics Society and RSS; Royal College of Surgeons, Edinburgh

November 2018

1 BCS-FACS Evening Seminar, London
9 Society Meeting and AGM, London

December 2018

17 LMS South West & South Wales Regional Meeting, Exeter

May 2019

20-24 LMS Invited Lecture Series 2019, Professor Søren Asmussen (Aarhus University), ICMS, Edinburgh

Calendar of Events

This calendar lists Society meetings and other mathematical events. Further information may be obtained from the appropriate LMS Newsletter whose number is given in brackets. A fuller list is given on the Society's website (www.lms.ac.uk/content/calendar). Please send updates and corrections to calendar@lms.ac.uk.

July 2018

2–4 Calculus of Variations and Geometric Measure Theory at Sussex, University of Sussex (476)
2–5 Low Energy Effective Dynamics of Skyrmions, University of Leeds (476)
2–6 The Mathematics of Multiscale Biology, LMS Research School, Nottingham (475)
2–6 International Statistical Ecology Conference 2018, University of St Andrews (476)
3–6 K-theory, Representation Theory and Hecke Algebras, University of Sheffield (476)
4 LMS Popular Lectures, London (477)
9–13 LMS Invited Lecture Series 2018, Art Owen (Stanford University), University of Warwick (477)
9–13 Homotopy Theory and Arithmetic Geometry: Motivic and Diophantine Aspects, LMS–CMI Research School, Imperial College London (475)
9–13 Non-associative Algebras and their Applications, Lancaster University (477)
10 Recent Advances in the Statistical Analysis of Extreme Environmental and Actuarial Risk, University of Nottingham (477)
11–13 Quantum Roundabout 2018, University of Nottingham (477)
12–13 Geometry, Derived Categories and Moduli Spaces, University of Plymouth (477)
23–26 Young Researchers in Mathematics 2018, University of Southampton (476)
23–27 European Conference on Mathematical and Theoretical Biology, University of Lisbon (475)

24–27 Probability and Statistics: 41st Research Students Conference, University of Sheffield (476)

30–2 Differential Algebra and Related Topics, University of Leeds (476)

August 2018

3–5 Model-theoretic Methods in Number Theory and Algebraic Differential Equations, University of Manchester (477)

7 LMS Meeting at the ICM 2018, Rio de Janeiro (477)

13–17 New Trends in Analytic Number Theory, LMS–CMI Research School, Exeter (477)

13–17 Equivariant and Motivic Homotopy Theory, INI, Cambridge (475)

27–31 Mathematical Foundations of Computer Science, Liverpool (477)

29–31 Simple Groups: New Perspectives and Applications, Bristol (476)

September 2018

2–4 Modern Mathematical Methods in Science and Technology, Kalamata, Greece (475)

3–7 Dynamics Days Europe 2018, Loughborough University (476)

3–7 Model Sets and Aperiodic Order, Durham University (476)

3–7 Renormalisation in Quantum Field Theory and in Stochastic Partial Differential Equations, INI, Cambridge (476)

4–6 33rd British Topology Meeting, The Open University (477)

7–8 LMS Prospects in Mathematics Meeting, University of Warwick (477)

10–12 BSDEs, Information and McKean-Vlasov Equations, University of Leeds (477)

10–13 Arithmetic Ramsey Theory, University of Manchester (477)

11 Joint Society Meeting with IMA: Noether Celebration, London (477)

11–14 Dragon Applied Topology, Swansea University, Singleton Campus (477)

11–15 Das Kontinuum — 100 Years Later, University of Leeds (477)

14–15 Theoretical and Computational Discrete Mathematics, University of Derby (476)

17 Combinatorial Algebraic Geometry, University of Bristol (477)

17–19 Probability and Nonlocal PDEs: Interplay and Cross-Impact, Swansea University (477)

19 LMS Popular Lectures, University of Birmingham (477)

21 Functor Categories for Groups, Senate House, London (477)

24–26 Clay Research Conference 2018, Mathematical Institute, Oxford (476)

24–28 Derived Algebraic Geometry and Chromatic Homotopy Theory, INI, Cambridge (476)

October 2018

9 Joint Society Meeting with the Fisher Trust, Galton Institute, Genetics Society and RSS; Royal College of Surgeons, Edinburgh

22–26 Quantum Field Theory, Renormalisation and Stochastic Partial Differential Equations, INI, Cambridge (476)

November 2018

1 BCS-FACS Evening Seminar, London

9 Society Meeting and AGM, London

December 2018

3–7 Manifolds Workshop, INI, Cambridge (477)

5–11 Topology and Applications, Cochin, India (477)

10–11 3rd IMA Conference on the Mathematical Challenges of Big Data IMA Conference, London

10–12 Mathematical Sciences and Technology 2018, Hotel Equatorial Penang, Malaysia

10–14 Conclusions and Future Directions Workshop, INI, Cambridge (477)

17 LMS South West & South Wales Regional Meeting, Exeter

August 2019

4–9 Theory and Practice: an Interface or a Great Divide? Maynooth University (476)

May 2019

20–24 LMS Invited Lecture Series, Søren Asmussen (Aarhus University), ICMS, Edinburgh (477)