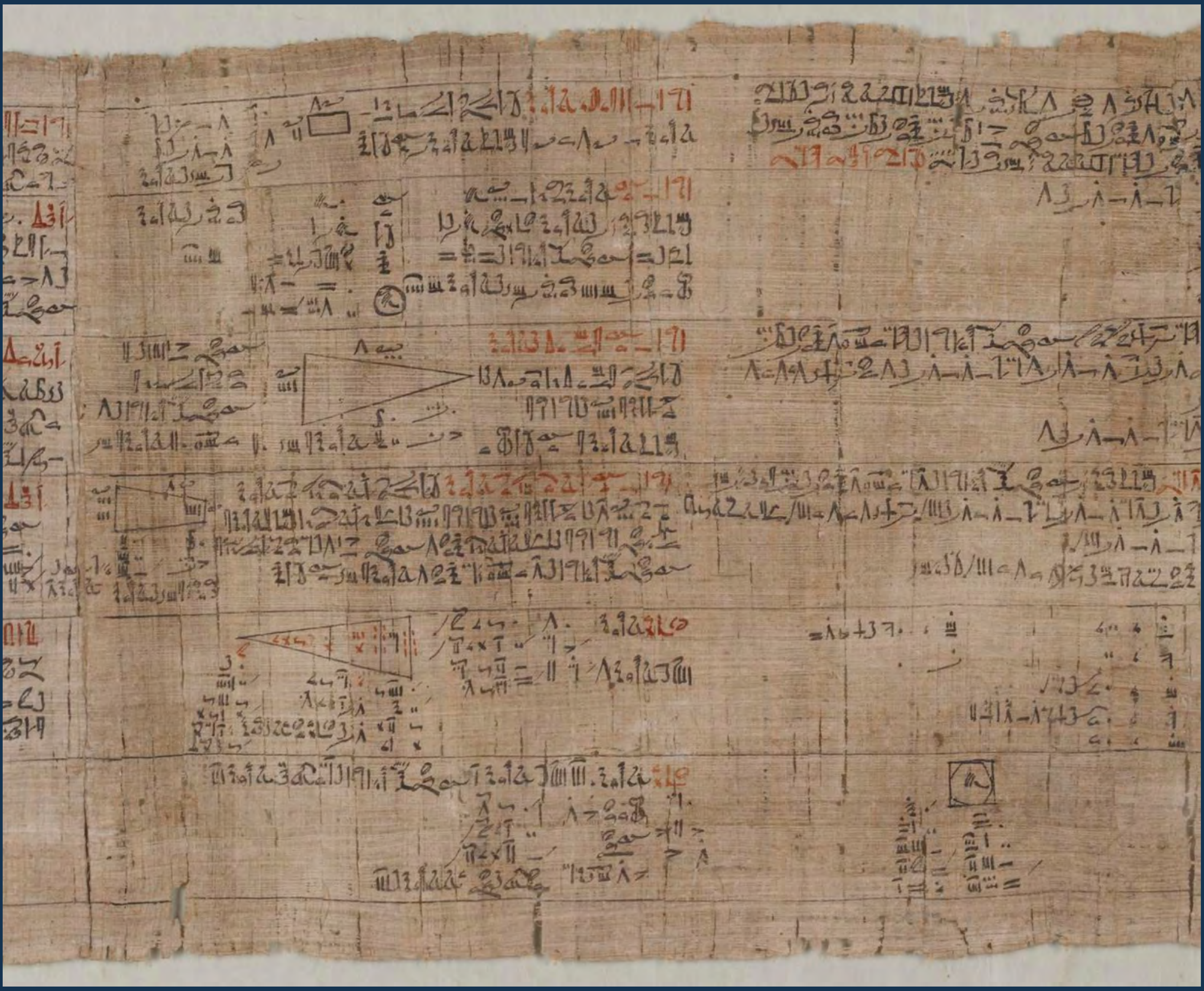
LONDON MATHEMATICAL SOCIETY
EST. 1865

NEWSLETTER

Issue: 490 - September 2020



OLYMPIAD PROBLEMS TO MATHEMATICS

ANCIENT EGYPTIAN MATHEMATICS

ASSESSING MATHEMATICS AT UNIVERSITY

EDITORIAL OFFICE

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Charity registration number: 252660

COVER IMAGE

The Rhind Mathematical Papyrus (© The Trustees of the British Museum) from the article 'Differing Approaches to Ancient Egyptian Mathematics' (page 28).

Do you have an image of mathematical interest that may be included on the front cover of a future issue? Email images@lms.ac.uk for details.

ISSN: 2516-3841 (Print)
ISSN: 2516-385X (Online)
DOI: 10.1112/NLMS

NEWSLETTER WEBSITE

The Newsletter is freely available electronically at lms.ac.uk/publications/lms-newsletter.

MEMBERSHIP

Joining the LMS is a straightforward process. For membership details see lms.ac.uk/membership.

SUBMISSIONS

The Newsletter welcomes submissions of feature content, including mathematical articles, career related articles, and microtheses, from members and non-members. Submission guidelines and LaTeX templates can be found at lms.ac.uk/publications/submit-to-the-lms-newsletter.

Feature content should be submitted to the editor-in-chief at newsletter.editor@lms.ac.uk.

News items should be sent to newsletter@lms.ac.uk.

Book reviews should be sent to reviews@lms.ac.uk.

Notices of events should be prepared using the template at lms.ac.uk/publications/lms-newsletter and sent to calendar@lms.ac.uk.

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FROM THE EDITOR

It is amazing how a week's holiday in North Yorkshire can cast a brighter light on academic life — it must have been the sea air, moor-walking, the complete lack of emails and plenty of reading of 1980s *Reader's Digest* and *Yorkshire Countrywomen's Association* magazines. I hope that you too have had a decent break over the summer months, and have somewhat renewed your reserves for the academic year ahead. At difficult and changeable times like these, academic communities like the London Mathematical Society are more important than ever. Our aim here at the *Newsletter* is to offer some respite from the ongoing drama of the covid-19 changes, by sharing the latest news, business and some items of mathematical interest.

To this end, I am delighted to feature an article by Ben Green on Olympiad problems and research mathematics, and one for history of mathematics and Ancient Egypt aficionados by Christopher Hollings and Richard Bruce Parkinson. Covid-19 isn't totally ignored, as you will see in Paola Iannone's timely work on online assessment. I am also pleased to announce a new 'Notes of a Numerical Analyst' column by Nick Trefethen, and share a short pitch by one of my undergraduates on 'Counting on your Fingers'.

Finally, my bright idea (see Issue 489) about publishing members' mathematical images on the front cover hasn't worked out so far, as we haven't received any submissions — so please do bear us in mind if you have anything suitable. And as always, comments, submissions and suggestions are very welcome.

LMS NEWS

LMS President-Designate

The London Mathematical Society is pleased to announce that Ulrike Tillmann, FRS, Professor of Mathematics at the University of Oxford, as President-Designate. Professor Tillmann would take over from the current President, Professor Jon Keating, FRS in November 2021. Professor Tillmann is known for her leading contributions in algebraic topology and for her many contributions to the LMS and the broader mathematical community.

Professor Tillmann received a BA from Brandeis University in 1985 and was awarded her PhD from Stanford University in 1990 under the supervision of Ralph Cohen. Following her PhD, Professor Tillmann held several positions, first as a Junior Research Fellow at Clare Hall and SERC post-doc of Graeme Segal, FRS in Cambridge. She then took up a Tutorial Fellowship at Merton College and a lectureship at the University of Oxford. This was followed by an EPSRC Advanced Research Fellowship. She has been Professor of Mathematics at the University of Oxford since 2000 and was elected a Fellow of the Royal Society in 2008.

Her research interests include Riemann surfaces and the homology of their moduli spaces. Her work on the moduli spaces of Riemann surfaces and manifolds of higher dimensions has been inspired by problems in quantum physics and string theory. More recently her work has broadened into areas of data science.

Throughout her career, Professor Tillmann has contributed extensively to the work of the LMS: as a Council Member-at-Large (2010–14), a member of Prizes Committee (2007–9 and 2015-18), Nominating Committee (2016–19) and Publications Committee (2008–12). She has contributed to the continued success of LMS publications by giving leadership and valuable input as Managing Editor of the *Journal of Topology* (2007–17). She has also been involved in organising the successful Clay Mathematics Institute–LMS Research Schools as Chair of the Research Meetings Committee (2011–14).

Professor Tillmann has supported the wider mathematical community both nationally and internationally. She is a Fellow of the Alan Turing Institute and serves on the scientific boards of several international institutions, including the Oberwolfach Mathematical Research Institute (MFO). She is a member of Council of the Royal Society and served as an interim Vice-President in 2018. Professor Tillmann has also received several prestigious honours including the LMS Whitehead Prize in 2004. She was elected a Fellow of the American Mathematical Society in 2012 and a Member of the German National Academy of Sciences Leopoldina in 2017.

As previously announced, the Society is in the process of seeking approval from the Privy Council to amend its Standing Orders, which would have an effect on the President-Designate/President-Elect process. The Society is uncertain, in the current circumstances, when the Privy Council will be able to consider this request, and in particular, whether this will be before or after the 2020 Annual General Meeting. Further details will be shared when available.

2020 LMS Prize Winners

The Society extends its congratulations to the following 2020 LMS Prize Winners and thanks to all the nominators, referees and members of the Prizes Committee for their contributions to the Committee's work this year.

Professor Martin Liebeck of Imperial College London is awarded the Pólya Prize for his profound and prodigious contributions to group theory, particularly the subgroup structure of simple groups and probabilistic group theory.

Professor Peter Clarkson of the University of Kent is awarded the Senior Anne Bennett Prize in recognition of his tireless work to support gender equality in UK mathematics, and particularly for his leadership in developing good practice among departments of mathematical sciences.

Professor Thomas Hales of the University of Pittsburgh is awarded the Senior Berwick Prize in recognition of his book 'Dense Sphere Packings: A Blueprint for Formal Proofs', published in the LMS *Lecture Note Series* in 2012.

A Shephard Prize is awarded to **Regius Professor Kenneth Falconer, FRSE** of the University of St Andrews, for his many original and profound results in fractal geometry, particularly the description, occurrence, geometrical properties and dimensional analysis of fractal sets and measures.

A Shephard Prize is awarded to **Professor Des Higham, FRSE** of the University of Edinburgh. Higham has sought to make the theory, application, and insights from network science accessible to wide audiences, with much effort invested in public events and transparent descriptions. He is a natural communicator and presents in an engaging way, highlighting some intriguing paradoxes.

Professor Françoise Tisseur of the University of Manchester is awarded the Fröhlich Prize for her important and highly innovative contributions to the analysis, perturbation theory, and numerical solution of nonlinear eigenvalue problems.

Dr Maria Bruna of the University of Cambridge is awarded a Whitehead Prize in recognition of her outstanding research in asymptotic homogenisation, most prominently in the systematic development of continuum models of interacting particle systems.

Dr Ben Davison of the University of Edinburgh is awarded a Whitehead Prize in recognition of his outstanding contributions to the foundations, the structure and applications of Donaldson–Thomas invariants.

Dr Adam Harper of the University of Warwick is awarded a Whitehead Prize for his deep and important contributions to analytic number theory, and in particular for his work on the value distribution of the Riemann zeta-function and random multiplicative functions using sophisticated ideas and techniques from probability theory.

Dr Holly Krieger of the University of Cambridge is awarded a Whitehead Prize for her deep contributions to arithmetic dynamics, to equidistribution, to bifurcation loci in families of rational maps, and her recent proof (with DeMarco and Ye) of uniform boundedness results for numbers of torsion points on families of bi-elliptic genus two curves in their Jacobians.

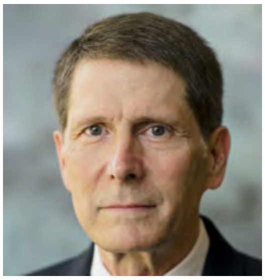
Professor Andrea Mondino of the University of Oxford is awarded a Whitehead Prize in recognition of his contributions to geometric analysis in differential and metric settings. In particular, he has played a central part in the development of the theory of metric measure spaces with Ricci curvature lower bounds.

Dr Henry Wilton of the University of Cambridge is awarded a Whitehead Prize for his remarkable contributions to geometric and combinatorial group theory.

2020 LMS prize winners


Martin Liebeck
Pólya Prize


Peter Clarkson
Senior Anne Bennett Prize


Thomas Hales
Senior Berwick Prize


Kenneth Falconer
Shephard Prize


Des Higham
Shephard Prize


Françoise Tisseur
Fröhlich Prize


Maria Bruna
Whitehead Prize


Ben Davison
Whitehead Prize


Adam Harper
Whitehead Prize


Holly Krieger
Whitehead Prize


Andrea Mondino
Whitehead Prize


Henry Wilton
Whitehead Prize

LMS Elections and AGM 2020

Voting for the LMS Elections for Council and Nominating Committee will open on 16 October 2020. The slate of candidates can be found at tinyurl.com/y8ps5c5c and an online forum for discussion is available at tinyurl.com/ya3od9j2.

Instructions on how to vote in the elections to Council and Nominating Committee will be sent to members by email or post before the ballots open. Members are encouraged to check that their contact details are up to date at lms.ac.uk/user.

This year the AGM will be held on Friday 20 November at 3.00pm (further details to be decided). The results of the Council and Nominating Committee elections will be announced at the meeting.

First LMS Emmy Noether Fellowships Announced

Anne-Sophie Kaloghiros (left) and Milena Hering

The London Mathematical Society is pleased to announce the award of the inaugural LMS Emmy Noether Fellowships to Dr Milena Hering (University of Edinburgh), Dr Anne-Sophie Kaloghiros (Brunel University), Dr Irene Kyza (University of Dundee) and Dr Cristina Manolache (University of Sheffield).

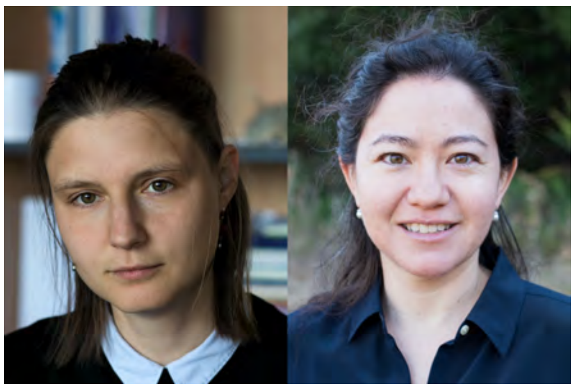
LMS Emmy Noether Fellowships are designed to enhance the research of mathematical scientists who are either re-establishing their research programme after returning from a major break associated with caring responsibilities, or who require support to maintain their research programme while dealing with significant ongoing caring responsibilities. They are sponsored by a generous donation from the Liber Stiftung.

Cristina Manolache (left) and Irene Kyza (photo credit: Kallam Corke)

We had a large number of exceptionally high-quality applications, more than we were able to fund, highlighting the importance of supporting mathematicians with caring responsibilities. Networking opportunities have been offered to all applicants, and, in addition, funded recipients will be assigned mentors in their host department.

We were pleased to receive applications from a wide range of areas of pure and applied mathematics and statistics, and a wide range of areas within the UK. The 2020 LMS Emmy Noether Fellows will be invited to De Morgan House to celebrate their research and to meet the donors once safe travel is possible.

LMS Honorary Members 2020



Maryna Viazovska (left) and Lauren Williams

At the Society meeting on 26 June 2020, the LMS elected Professor Maryna Viazovska, of École Polytechnique Fédérale de Lausanne, and Professor Lauren Williams, Dwight Parker Robinson Professor of Mathematics at Harvard and Sally Starling Seaver Professor at the Radcliffe Institute, as honorary members of the Society.

Professor Viazovska has produced ground-breaking work on sphere-packing problems in eight and twenty-four dimensions, based on innovative use of modular and quasimodular forms, and has made major contributions to spherical designs and Fourier interpolation.

Professor Williams is an algebraic combinatorialist whose outstanding research includes influential contributions across many fields, including cluster algebras, mirror symmetry, tropical geometry, matroid theory, and integrable systems.

2020 Christopher Zeeman Medal

The Councils of the IMA and LMS are pleased to announce that the Christopher Zeeman Medal has been awarded to Matt Parker. Matt is a former mathematics teacher and previous Public Engagement with Mathematics Fellow at Queen Mary, University of London.

Photograph by Rosemary Rance

Matt's 'Stand-up Maths' YouTube channel has half-a-million subscribers and his videos across YouTube have over 100 million views. In 2011 Matt helped found the Numberphile YouTube channel, which is now one of the most successful YouTube channels with over 3 million subscribers.

In 2019 Matt's book *Humble Pi* (Penguin Random House, 2019) was the first ever mathematics book to be a Sunday Times number one best-seller. His previous book *Things to Make and Do in the Fourth Dimension* (Penguin Random House, 2014) won the Euler Book Prize awarded by the Mathematical Association of America.

The full citation can be found at tinyurl.com/y2eol8du.

2020 Louis Bachelier Prize Winner

The 2020 Louis Bachelier Prize is awarded to Professor Mathieu Rosenbaum (École Polytechnique, Paris).

Professor Rosenbaum is internationally recognised as one of the foremost experts in stochastic modelling in finance. Rosenbaum's research, focused on stochastic processes and their applications in finance, impresses through the breadth of topics it has covered and the depth of results obtained on each topic. His research spans theoretical topics in probability and statistics as well as market microstructure, statistical modelling of high frequency financial data and volatility modelling. A full citation can be found at tinyurl.com/ydzbultm.

OTHER NEWS

Shaw Prize 2020

David Kazhdan (left) and Alexander Beilinson

The Shaw Prize in Mathematical Sciences 2020 has been awarded jointly to Alexander Beilinson (University of Chicago) and David Kazhdan (Hebrew University of Jerusalem) for their influence on and profound contributions to representation theory, as well as many other areas of mathematics such as arithmetic geometry, K-theory, conformal field theory, number theory, algebraic and complex geometry, group theory, and algebra more generally. As well as proving remarkable theorems themselves, they created conceptual tools that have been essential to many breakthroughs by others. The exceptionally broad reach of their work has led to significant advances in large areas of mathematics.

Royal Society Guest Editorships

We all know that times are strange right now, and the lives of scientists very different from normal. For those who are missing conference travel, and the opportunity to discuss their work and build collaborations with other research groups, the Royal Society would like to suggest a replacement project: why not edit a themed issue of *Philosophical Transactions* (tinyurl.com/y9tmxcpv)?

Each issue is carefully planned out, so is more like a book than a standard collection of related papers. The broad scope means that you are not restricted in terms of subject area, and you can be inventive with different article types. As Guest Editor, you will have the opportunity to build your network and gain editorial experience, with a high-profile Editorial Board and experienced staff to help you every step of the way.

Read about the benefits of guest editing an issue at tinyurl.com/y9v4fnkq. Then, if interested, feel free to contact the Commissioning Editor, Alice Power, (alice.power@royalsociety.org) with your ideas.

MATHEMATICS POLICY DIGEST

UK Research and Development Roadmap

The government published its UK Research and Development Roadmap in July. The stated long-term objectives for research and development (R&D) are: to be a science superpower and invest in the science and research that will deliver economic growth and societal benefits across the UK, and to build the foundations for new industries. The roadmap sets out to identify:

- the strengths and challenges facing the sector;
- the issues that need to be addressed; and
- how the government wants to work with universities, business, the third sector and across government to cement the UK's reputation as a science superpower.

The Roadmap is published at a time when the government has committed to increasing UK R&D investment to 2.4% of GDP by 2027, building on the announcement in March 2020 of increasing public funding for R&D to £22 billion per year by 2024/25. Further information is available at tinyurl.com/ybh6wv3s.

Research Integrity: A Landscape Study

The research integrity landscape study published by UK Research and Innovation (UKRI) in June provides new insight into the incentives and pressures in the UK research system and their perceived impact on research integrity and wider researcher behaviour.

Key findings include almost unanimous agreement from respondents that personal integrity drives research integrity, and that local culture can have a strong influence on behaviour. Those factors considered to have a strong and positive impact on research integrity include good leadership and management, professional development, sharing research, and the opportunity to collaborate and work with colleagues from other disciplines.

Bullying and harassment had the biggest negative influence on integrity and almost eight out of ten respondents believed that researchers feel tempted or under pressure to compromise on research integrity at times. The full report is available at tinyurl.com/yyy62uqe.

Report on Equity in STEM Education

The All-Party Parliamentary Group (APPG) on Diversity and Inclusion released an inquiry report on Equity in STEM Education in June, highlighting five key findings and six recommendations.

The findings highlight shortcomings across the education system. They include the need for a more joined-up approach by government to tackle the causes of inequity in STEM education.

Other key findings include the need to strengthen STEM-specific teaching, wider access to good careers education, and the inequity schools are reinforcing with their GCSE options, especially in the most disadvantaged areas.

Six key recommendations have been created from the findings. These include calling for a minister responsible for addressing inequity within the education system, making STEM education more relevant to young people, and more action to address teacher shortages in STEM subjects. The other three recommendations include the full implementation and follow up of changes to careers support and guidance, addressing inequities in Double Award and Triple Science at GCSE, and a review of fundamental changes to STEM GCSEs. The full report is available at tinyurl.com/y37ltkma.

Digest prepared by Dr John Johnston
Society Communications Officer

Note: items included in the Mathematics Policy Digest are not necessarily endorsed by the Editorial Board or the LMS.

UPDATE FROM THE EPSRC SAT AND EARLY CAREER RESEARCHER FORUM

A joint meeting of the EPSRC Mathematical Sciences Early Career Forum (ECRF) and Strategic Advisory Team (SAT) took place online over the week 6–10 July 2020.

SATs exist to provide Theme Leaders at EPSRC with strategic advice that will assist them to develop, implement and modify their plans. The ECRF allows EPSRC to interact in a structured way with early career researchers, discuss strategy with them, and provides networking opportunities. The ECRF currently has 20 members, and the SAT comprises 11 academic mathematicians and 3 members with current or past business affiliations. Members are appointed for a term of three years and are expected to act as 'generous generalists' rather than as representatives of their own organisation or research area. Open calls for applications to join both groups are usually advertised annually via the EPSRC website and mailing lists.

Members of the Early Career Researchers Forum

Usually the ECRF and the SAT meet separately, but this time the EPSRC team designed a combination of joint and separate two-hour online sessions distributed over the week. Meeting jointly felt particularly valuable in our discussions over Discipline Hopping, and in Forward Strategy sessions trying to identify UK strengths, and emerging research opportunities. The Forward Strategy sessions aimed to help draw out clear messages that could be used within, and by, EPSRC to help explain mathematical sciences research to UK Research and Innovation (UKRI) audiences and by doing that, promote the Mathematical Sciences Theme.

Discipline Hopping is a funding scheme which has been operated in the ICT and Healthcare Technologies Themes and provides researchers with opportunities to spend time in a different research or user environment and to develop new skills and collaborations. Could a similar scheme work for Mathematical Sciences? What would be the obstacles to its success? Our wide-ranging discussions touched on issues including the possibility of 'discipline hops' within the mathematical sciences, the existence of similar models elsewhere, e.g. Royal Society Industry Fellowships, and the difficulties of immersive engagement with other research fields.

As you might expect, the issues around Peer Review in Mathematical Sciences were mentioned, along with the rather overwhelming response to the New Horizons call which will be peer reviewed over the summer. Timely issues of PhD student and post-doc support in light of covid-19, and UKRI's response, were also discussed.

Members of the Strategic Advisory Team

We noted that an online survey on Net Zero Agriculture had just opened (closing on 4 September); individuals with ideas as to where mathematical sciences research can inform and assist with achieving net zero emissions for the agriculture sector are strongly encouraged to contribute to it at tinyurl.com/netzeromaths.

Finally, the meeting considered the advantages and disadvantages inherent in online meetings, and what opportunities there might be for the mathematical sciences community to develop a model of blended engagement with the EPSRC Mathematical Sciences team in the coming months. The team always welcomes phone calls or emails and we are keen to meet as large a cross-section of the community as possible. Details can be found via the EPSRC website.

Katie Blaney (Katie.Blaney@epsrc.ukri.org)
Head of Mathematical Science, UKRI EPSRC

Jonathan Dawes (J.H.P.Dawes@bath.ac.uk)
Chair of the EPSRC Mathematical Sciences SAT

30th Anniversary Meeting

This year the EMS celebrates 30 years of activity in support of the mathematical sciences across Europe. To mark the occasion, the EMS, with the support of the LMS and of the Edinburgh Mathematical Society, will hold a day of mathematical talks, reminiscences and discussions, each chaired by a past president of the EMS at the ICMS in Edinburgh on 29 October. Further details are available at https://euro-math-soc.eu.

EMS Executive Committee

The EMS Council was meant to meet in Bled (Slovenia) on 4–5 July 2020, but because of the covid-19 pandemic instead met online on 4 July 2020. The following new members of the Executive Committee (EC) have been elected by the EMS Council, and will take office at the beginning of 2021.

- Vice-President: Jorge Buescu (Portugal)
- Secretary: Jíí Rákosník (Czech Republic)
- Members-at-large: Fréderic Hélein (France), Barbara Kaltenbacher (Austria), Luis Narváez (Spain), Beatrice Pelloni (UK), Susanna Terracini (Italy)

At that meeting it was also decided that the 9th European Congress of Mathematics in 2024 will be held in Seville, Spain.

Ernest Vinberg

Professor Ernest Borisovich Vinberg, Moscow State University and Independent University of Moscow, who worked in the area of discrete subgroups of Lie groups and representation theory, died on 12 May 2020 at the age of 82.

2020 Princess of Asturias Award

The mathematicians Yves Meyer, Ingrid Daubechies, Terence Tao and Emmanuel Candès have received the 2020 Princess of Asturias Award for Technical and Scientific Research in recognition of their 'immeasurable, ground-breaking contributions to modern theories and techniques of mathematical data and signal processing'. Their work rests on the use of wavelets and compressed sensing or matrix completion, and their techniques have played a fundamental role in sharpening images from the Hubble Space Telescope and in the detection by LIGO of gravitational waves. For further details visit tinyurl.com/yx9nngw9.

8ECM: Minisymposia and Satellite Conferences

The organisers of the 8th European Congress of Mathematics have reopened the calls for Minisymposia and Satellite conferences. The congress has been rescheduled and will take place 20–26 June 2021 in Portorož, Slovenia. Deadline for applications is 31 January 2021. For further details visit www.8ecm.si/.

EMS News prepared by David Chillingworth
LMS/EMS Correspondent

Note: items included in the European Mathematical Society News are not necessarily endorsed by the Editorial Board or the LMS.

OPPORTUNITIES

LMS Prospects in Mathematics

UK departments are invited to submit Expressions of Interest to host the LMS Prospects in Mathematics Meeting 2021 to the Prospects in Mathematics Meeting Steering Group.

Up to £7,000 is available to support the annual two-day event (usually taking place in September) for Finalist Mathematics Undergraduates who are considering apply for a PhD after they have completed their current studies. This includes funding to support an in-person event to cover fares and accommodation for up to 50 students, travel and accommodation for speakers and subsistence for participants including a social event. Proposals for hybrid events that include online participation are also welcome.

LMS Prospects in Mathematics Meetings should feature speakers from a wide range of mathematical fields across the UK who discuss their current research and what opportunities are available to prospective PhD students.

Expressions of Interest (maximum one A4 side in length) should include the following details:

- Department's confirmation of support to host the LMS Prospects in Mathematics Meeting.
- Reasons to host the LMS Prospects in Mathematics Meeting.
- A description of the event programme, including any online participation for a hybrid event(s).
- A provisional list of speakers who are representative of the UK research landscape both geographically and scientifically.
- Speakers from under-represented groups should be included and women speakers should account for at least 40% of the invited speakers.
- Confirmation that prospective organisers have read and understood the terms and conditions in the Guidelines for Organisers (available from tinyurl.com/y9yn2ryo).
- Willingness to attend both the upcoming online LMS Prospects in Mathematics Meeting hosted by Bath from 10–11 September 2020 and the traditional event, which is planned to be held at the University of Bath in December 2020, but may be subject to change in light of the current covid-19 pandemic, to get an idea of the event.

The Expression of Interest should be sent to the Prospects in Mathematics Steering Group (ECR.grants@lms.ac.uk) by 15 September 2020.

For further details about the LMS Prospects in Mathematics Meetings, please visit: tinyurl.com/y9yn2ryo.

Atiyah UK–Lebanon Fellowships

Set up in 2020 in memory of Sir Michael Atiyah (1929–2019), whose father was Lebanese and who retained strong links with Lebanon throughout his life, the Atiyah UK–Lebanon Fellowships operate in partnership with the Centre for Advanced Mathematical Sciences at the American University of Beirut (tinyurl.com/yycx6ytv).

Objectives

These Atiyah UK-Lebanon Fellowships provide for an established UK-based mathematician to visit Lebanon as an Atiyah Fellow for a period of between one week and 6 months, or alternatively for a mathematician from Lebanon of any level, in particular promising advanced level students from the AUB, to visit the UK to further their study or research for a period of up to 12 months.

For visits from the UK to Lebanon, the Atiyah Fellowship will cover:

- From the LMS, up to £2,000 towards expenses for travel and related expenses, and will pay accommodation and subsistence at £1,000 per month pro rata for up to 6 months.
- In addition, AUB will cover accommodation and provision of office space and logistical support. This will be independent of the host institution.
- There is the possibility of additional subsistence/payment for agreed teaching.
- Consideration may be given for additional support to Fellows travelling with a family.

For visits to the UK from Lebanon, the Atiyah Fellowship will cover:

- From the LMS, up to £2,000 towards expenses for travel and related expenses, and will pay accommodation and subsistence at £500 per month pro rata for up to 12 months.
- Additional support will be available for PhD or MSc candidates from AUB in either mathematics or mathematical physics.

Further information and queries

Further information, including how to apply, is available on the LMS website at tinyurl.com/tvweckc. Queries should be addressed to fellowships@lms.ac.uk. The Chair of the Fellowship Panel is Professor Caroline Series, FRS. The application deadline is 31 October 2020.

Abel Prize 2021: Call for Nominations

The Norwegian Academy of Science and Letters has issued a call for nominations of candidates for the Abel Prize 2021. The prize, which amounts to NOK7.5 million, recognises outstanding scientific work in the field of mathematics, including mathematical aspects of computer science, mathematical physics, probability, numerical analysis and scientific computing, statistics and applications of mathematics in the sciences.

The Abel Prize may be awarded to one single person, or shared for closely related fundamental contributions. The Abel Prize was first awarded in 2003. For laureates up until 2020, see abelprize.no/.

The nomination should be accompanied by a description of the work and impact of the nominee/nominees, together with names of distinguished specialists in the field of the nominee/nominees who can be contacted for an independent opinion. When nominating, it is a requirement to take into account that the nominee has adhered to general guidelines for research ethics. The letter of nomination should be sent no later than 15 September 2020. For further information and the nomination form see tinyurl.com/y6u5xdll.

Ladyzhenskaya Medal in Mathematical Physics

Olga Alexandrovna Ladyzhenskaya (1922–2004) was a leading mathematician whose work on the partial differential equations of mathematical physics and related areas influenced generations of mathematicians, especially in Russia. A new prize, the Ladyzhenskaya Medal in Mathematical Physics, has been established in her honour, to be awarded for the first time at a special event dedicated to her centenary year during ICM 2022 in St Petersburg. The medal will be awarded every four years to recognise revolutionary results in, or with applications to, mathematical physics and neighbouring fields of mathematics. The winner receives a medal and a cash award of 1 million rubles. If the main work is in collaboration with several people, the committee may consider a shared prize.

Nominations should be submitted to the Chair of the 2022 Prize Committee, Professor Giovanni Felder at giovanni.felder@math.ethz.ch. Each nomination should contain a detailed description of the work of the candidate and where it fits in the overall development of the field, including references. Nominations are confidential, and must not be disclosed to the candidate. The deadline for nominations is 1 December 2021. More details can be found at tinyurl.com/yahmgqfm.

Further information about Ladyzhenskaya's extraordinary life and career may be found in a multi-author article in *Notices Amer. Math. Soc.* 51 (2004), no. 11, 1320–1331.

VISIT

Visit of Rafael Alcaraz Barrera

Dr Rafael Alcaraz Barrera will be visiting the University of Birmingham from 4 to 24 October 2020. Dr Barrera is a member of the Dynamical Systems research group based at the Universidad Autónoma de San Luis Potosi. His recent research activity concerns the study of open dynamical systems. During his visit he will give lectures at the University of Birmingham, 7 October (contact Simon Baker: S.Baker.2@bham.ac.uk), the University of Manchester, 13 October (contact Thomas Kempton: thomas.kempton@manchester.ac.uk) and the University of Liverpool, 15 October (contact James Waterman: James.Waterman@liverpool.ac.uk). For further details contact Simon Baker. The visit is supported by an LMS Scheme 5 grant.

HEILBRONN ANNUAL CONFERENCE 2020

Location: Online
Date: 10-11 September 2020
Website: https://eur.cvent.me/eGyG

The Annual Conference of the Heilbronn Institute for Mathematical Research (HIMR) is the Institute's flagship event. Taking place over two days, this online conference will cover a broad range of discrete mathematics, including algebra, combinatorics, data science, geometry, number theory, probability and quantum information.

Confirmed speakers:
Maria Chudnovsky (Princeton)
Adam Harper (Warwick)
Özlem Imamoglu (ETH Zurich)
Kurt Johansson (KTH Royal Institute of Technology)
Ailsa Keating (Cambridge)
Hendrik Lenstra (Universiteit Leiden)
Ulrike Tillmann (Oxford)
Ronald de Wolf (CWI and Universiteit van Amsterdam)

See the Heilbronn Annual Conference website for more information and to register.

Challenges and Recent Advances in Mathematical Physics

Fry Conference Series

Date: 20-22 January 2021
Website: https://bit.ly/2YAL28I

This is the first in a series of high-profile conferences to celebrate the move of the University of Bristol's School of Mathematics to the historic Fry Building. Taking place over three days, this conference will cover a broad range of topics in mathematical physics, including quantum information, quantum field theory, integrable systems, random matrix theory and statistical physics.

Confirmed speakers:
Alexander Bobenko, Technische Universität Berlin
Alexander Bufetov, Aix-Marseille Université
Harry Buhrman, CWI & QUSOFT
Tom Claeys, UCLouvain
Margherita Disertori, Universität Bonn
Eva-Maria Graefe, Imperial College London
Subir Sachdev, Harvard University
Sylvia Serfaty, New York University

See the conference website for more information and to register.

Annual LMS Subscription 2020–21

Members are reminded that their annual subscription, including payment for additional subscriptions, for the period November 2020–October 2021 is due on 1 November 2020 and payment should be received by 1 December 2020. Please note that payments received after this date may result in a delay in journal subscriptions being renewed.

LMS Membership Subscription Rates

The annual subscription rates to the London Mathematical Society for 2020–21 are:

Ordinary membership	£92.00	US$184.00

Concessions on Ordinary membership:

Reciprocity	£46.00	US$92.00
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Associate (undergraduate) membership	£11.50	US$23.00

*These rates are by request and subject to agreement by the Treasurer.

Access to LMS Journals

The Society offers free online access to the *Bulletin*, *Journal* and *Proceedings* of the London Mathematical Society (provided by Wiley) and to *Nonlinearity* (provided by the Institute of Physics) for personal use only. If you would like to receive free electronic access to these journals, please indicate your choices either on your online membership record under the 'Journal Subscription' tab or on the LMS subscription form. The relevant publisher will then contact members with further details about their subscription.

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Members also have the option to pay their European Mathematical Society (EMS) subscription via the LMS and subscribe to the Journal of the EMS. LMS Members enjoy a 50% discount off the EMS membership fee through their LMS membership. If you would like to subscribe to the EMS and JEMS via the LMS, please indicate either on your online membership record under the 'Journal Subscription' tab or on the LMS subscription form. The EMS membership fee quoted online and on the subscription form includes the discount.

Payment of membership fees for EWM

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Online renewal and payment

Members can log on to their LMS user account (lms.ac.uk/user) to make changes to their contact details and journal subscriptions and to make payment either by card via WorldPay or by setting up a direct debit via GoCardless, under the 'My LMS Membership' tab. Members can also renew their subscription by completing the subscription form and including a cheque either in GBP or USD. We regret that we do not accept payment by cheques in Euros. Please note that there may be a delay in processing cheque payments as this will depend on staff presence at De Morgan House to receive and bank the cheques.

LMS member benefits

Members are reminded that their annual subscription entitles them to the following range of benefits: voting in the LMS Elections, free online access to selected journals, printed and/or online copies of the bi-monthly *Newsletter*, among others. For a full list of member benefits, see lms.ac.uk/membership/member-benefits.

Elizabeth Fisher
Membership & Grants Manager
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LMS Grant Schemes

See lms.ac.uk/grants for application forms and full details of these grant schemes.

Research Grants

The deadline is 15 September 2020 for applications for the following grants, to be considered by the Research Grants Committee at its October meeting.

Conferences Grants (Scheme 1): Grants of up to £7,000 are available to provide partial support for conferences held in the United Kingdom.

Visiting Speakers to the UK (Scheme 2): Grants of up to £1,500 are available to provide partial support for a visitor to the UK, who will give lectures in at least three separate institutions. Awards are made to the host towards the travel, accommodation and subsistence costs of the visitor.

Joint Research Groups in the UK (Scheme 3): Grants of up to £4,000 are available to support joint research meetings held by mathematicians who have a common research interest and who wish to engage in collaborative activities, working in at least three different locations (of which at least two must be in the UK).

Research in Pairs (Scheme 4): For those mathematicians inviting a collaborator to the UK, grants of up to £1,200 are available to support a visit for collaborative research either by the grant holder to another institution abroad, or by a named mathematician from abroad to the home base of the grant holder. For those mathematicians collaborating with another UK-based mathematician, grants of up to £600 are available.

Collaborations with Developing Countries (Scheme 5): For those mathematicians inviting a collaborator to the UK, grants of up to £3,000 are available to support a visit for collaborative research, by a named mathematician from a country in which mathematics could be considered to be in a disadvantaged position, to the home base of the grant holder. For those mathematicians visiting a collaborator's institution, grants of up to £2,000 are available.

Research Workshop Grants (Scheme 6): Grants of between £3,000–£5,000 are available to provide support for Research Workshops held in the United Kingdom, the Isle of Man and the Channel Islands.

African Mathematics Millennium Science Initiative (AMMSI): Grants of up to £2,000 are available to support the attendance of postgraduate students at conferences in Africa organised or supported by AMMSI.

Grants to support Computer Science Researchers

Computer Science Small Grants (Scheme 7): Grants of up to £1,000, plus an additional £200 for caring expenses, are available to support visits for collaborative research at the interface of Mathematics and Computer Science. The deadline for applications is 15 October 2020.

Grants for Early Career Researchers

The deadline is 15 October 2020 for applications for the following grants, to be considered by the Early Career Research Committee in November.

Postgraduate Research Conferences (Scheme 8): Grants of up to £4,000 are available to provide partial support for conferences held in the United Kingdom, which are organised by and are for postgraduate research students. The grant award will be used to cover the costs of participants.

Celebrating new appointments (Scheme 9): Grants of up to £600 are available to provide partial support for meetings held in the United Kingdom to celebrate the new appointment of a lecturer at a UK university.

Travel Grants for Early Career Researchers: Grants of up to £500 are available to provide partial travel and/or accommodation support for UK-based Early Career Researchers to attend conferences or undertake research visits either in the UK or overseas.

LMS Council Diary — A Personal View

Council met via video conference on Friday 26 June 2020, before the General Meeting and the Hardy Lecture by Peter Sarnak later that day. The President's business included notification of the award of several Emmy Noether Fellowships and an update on a planned virtual forum for research by black mathematicians that is due to take place in October 2020, with input from the SLAM Committee Chair, the MARM Board Chair and the Chair of the Women in Mathematics Committee. This event will be held jointly with the Institute of Mathematics and its Applications and the British Society for the History of Mathematics. Council then agreed unanimously that Ulrike Tillman, FRS should be nominated as the Society's next President Designate/Elect.

The next major item of business was a report from Vice-President Gordon on the activities of the Covid Working Group, which included: several initiatives to support PhD students, particularly those starting their studies, and Early Career Researchers; support for the Teaching and Learning Mathematics Online events that are being co-organised by the Education Secretary with the other learned societies; and discussions on ongoing equality, diversity and inclusivity and career progression issues related to covid-19. It was reported that the LMS Representatives had discussed the planned activities during their video conference meeting on 8 June and had been very supportive.

There followed a discussion of the virtual Big Mathematics Initiative event on 11 June, which had been concerned with the possible formation of a National Academy for Mathematical Sciences. The advantages of engaging with the ongoing debate via involvement with the Special Interest Group tasked with taking the issue forward were noted, and the President encouraged members of Council to participate.

It was then reported by the President that funds had been offered to the Society to develop online tutoring in A-Level Mathematics for underprivileged students, and Council agreed that the Education Secretary and the Chair of the Women in Mathematics Committee would work with the donor to develop a pilot scheme.

Under financial matters, we heard a summary of the Third Quarter Financial Review from the Treasurer, including the fact that gifts and donations had been higher than anticipated, and, due to the covid-19 circumstances, there had been significant underspends on grants. In part this has enabled the Society to put extra resources into the initiatives developed by the Covid Working Group mentioned above, so that the overall support for mathematics continues at the normal level during this time. We also discussed committee membership; an update to the Committee General Terms of Reference; and a discussion paper on Ethics in Mathematics, which it was agreed to consider further at a later date; and the proposal to introduce a sliding scale of fees for Ordinary membership, which was agreed.

The meeting concluded with the President thanking everyone for their contributions, and noting that he was looking forward to seeing members of Council again for the General Meeting and Hardy Lecture later in the afternoon.

Elaine Crooks
Member-at-Large

Levelling Up Scheme

Thanks to a generous donation from Dr Tony Hill, the LMS has started work on a new venture to support the provision for online tutoring for A-Level mathematics students who come from backgrounds that are under-represented in the maths community.

Working with three or four universities initially, the pilot aims to bring together undergraduate student tutors with A-Level mathematics students who wish to improve their grades to enable them to read a STEM degree. Each university would identify nearby schools with significant percentages of students from under-represented backgrounds and enlist the help of undergraduate students to tutor and engage with their A-Level tutees.

Through enhancing the A-Level students' engagement with mathematics, the Levelling Up Scheme aims to increase the diversity of students pursuing STEM degrees and, eventually, careers. Through providing mentoring to the tutors from experienced teachers, the scheme also aims to enrich the tutors' teaching skills and knowledge and improve the foundation on which they build the next step in their careers.

The scheme therefore aims both to benefit tutors and tutees and to widen participation in mathematics.

Society Meetings — Update

IMA–LMS Joint Meeting

The London Mathematical Society and the Institute of Mathematics and its Applications are planning to hold their Annual Joint Meeting online on 1–2 October 2020 via Zoom. This year's topic is Topological Methods in Data Science. The current confirmed speakers are: Gueorgui Mihaylov, Vidit Nanda, Kathryn Hess, Ulrike Tillmann and Ran Levi. Further details will be available soon.

IMA–LMS–BSHM Joint Meeting Black Heroes of Mathematics

The IMA, LMS and BSHM are planning to hold a Joint Meeting to celebrate Black Heroes of Mathematics from 26–27 October online via Zoom (with support from the ICMS), coinciding with Black Mathematicians Month.

Regional Meeting postponements

In light of the covid-19 pandemic, the three Regional Meetings (Northern, Midlands and South West & South Wales) have been postponed.

The organising committee for the Northern Regional Meeting has decided to postpone this meeting and conference for one year, provisionally commencing 6 September 2021. It is possible that one or two talks may still take place on 10 September 2020, which was the original date the Regional Meeting, to mark Bill Crawley-Boevey's 60th birthday but this will not be a formal Society Meeting. Members are advised to check the meeting website at tinyurl.com/yamy8uvq.

The Midlands Regional Meeting was due to be held in December 2020 at the University of Lincoln and the organisers are planning to host the event in 2021.

The South West & South Wales Regional meeting in Swansea was planned for 5–7 January 2021 but this has been postponed .

Further details on these postponed meetings will be available in due course.

Annual General Meeting 2020

It is still planned to hold the LMS AGM on 20 November 2020. Further details will be available in due course, and we ask that members note the date in their diary.

Records of Proceedings at LMS Meetings
General Meeting: 26 June 2020

Over 130 members and visitors were present for all or part of the meeting, which was a virtual meeting held using Zoom software and kindly hosted by ICMS, Edinburgh. The meeting began at 3.30pm with the President, Professor Jon Keating, FRS in the Chair.

Minutes of the Annual General Meeting, held on 29 November 2019, had been made available 21 days prior to the General Meeting. The President invited members to vote by an electronic poll to ratify these Minutes. The Minutes were ratified by a majority.

On a recommendation from Council, it was agreed to elect Professor Charles Goldie and Professor Chris Lance as scrutineers in the forthcoming Council elections. The President invited members to vote by an electronic poll to ratify Council's recommendation. The recommendation was ratified unanimously.

The President, on Council's behalf, proposed that the following two people be elected to Honorary Membership of the Society: Professor Maryna Viazovska, of École Polytechnique Fédérale de Lausanne and Professor Lauren Williams, Dwight Parker Robinson Professor of Mathematics at Harvard University and Sally Starling Seaver Professor at the Radcliffe Institute. This was approved by acclaim. The President read a short version of the citations, to be published in full in the Bulletin of the London Mathematical Society.

The President then announced the awards of the prizes for 2020:

Pólya Prize: Professor Martin Liebeck (Imperial College London)
Senior Anne Bennett Prize: Professor Peter Clarkson (University of Kent)
Senior Berwick Prize: Professor Thomas Hales (University of Pittsburgh)
Shephard Prize: Regius Professor Kenneth Falconer, FRSE (University of St Andrews); Professor Des Higham, FRSE (University of Edinburgh)
Fröhlich Prize: Professor Françoise Tisseur (University of Manchester)
Whitehead Prizes: Dr Maria Bruna (University of Cambridge); Dr Ben Davison (University of Edinburgh); Dr Adam Harper (University of Warwick); Dr Holly Krieger (University of Cambridge); Professor Andrea Mondino (University of Oxford); Dr Henry Wilton (University of Cambridge)

The President announced Professor Ulrike Tillmann, FRS as the President Designate for 2021–2022.

There were no nominations for elections to membership at this meeting. There was no Members' Book to sign at this meeting, so no admissions will be made to the Society until November 2020.

The President introduced the 2020 Hardy Lecture given by Professor Peter Sarnak (Princeton) on *Gap Sets for the Spectra of Cubic Graphs*. At the end of the meeting, the President thanked Peter Sarnak for his lecture, as well as thanking Péter Varjú for his lecture Random Polynomials and Random Walks at the 19 June Graduate Student Meeting, and the ICMS for helping the LMS deliver the meeting virtually.

From Olympiad Problems to Research Mathematics

BEN GREEN

This is a text version of the International Mathematical Olympiad (IMO) Lecture I gave at The Forum, Bath on the occasion of the 60th IMO in 2019. For a selection of past IMO problems, I discuss how they link with more advanced topics, issues of current research interest and famous unsolved problems.

Solving Olympiad problems is a very different activity to mathematical research. However, over the years there have been a number of Olympiad problems with close connections to interesting current research or to deep unsolved problems. In this article I have selected seven past IMO problems with this property.

I have tried to make the article as accessible as possible. In one or two places I use a little algebraic number theory and some Galois theory, but these sections can be skipped over if desired.

1983 Q5

Is it possible to choose 1983 *distinct positive integers, all less than or equal to* 10^5*, no three of which are consecutive terms of an arithmetic progression?*

I selected this problem to begin with, since it is the closest to my own research interests. Defining $r_3(N)$ to be the size of the largest subset of $\{1,\dots,N\}$ without a 3-term progression (3AP), the question is asking whether or not $r_3(10^5) \geq 1983$.

In thinking about the problem a researcher in the area has the great advantage of knowing that finding upper bounds on $r_3(N)$ is notoriously difficult, involves techniques outside the scope of the Olympiad, and moreover the bounds obtained are rather weak. I pity any contestant who spent time trying to show that the answer to the question is 'no'.

Therefore we need to show that the answer to the question is 'yes'. Here, research mathematics is not directly helpful, since the focus is on the asymptotic (i.e. large N) behaviour of $r_3(N)$. Nonetheless, it seems likely that the techniques used there will be relevant to the IMO problem. For big N, the largest known subsets of $\{1,\dots,N\}$ free of 3APs come from a geometric construction due to Behrend.

Behrend's construction relies on two observations. The first is that any set of points on a sphere (in any dimension) is free of 3APs, simply because no line intersects a sphere in three points. The second observation is that we may use a set of points on a sphere to make a set of integers in the following manner:

Suppose $A \subseteq \{0,1,\dots,n-1\}^d$ lies on a sphere in $\mathbb{R}^d$. Then the image of A under the map

$$(x_1,\dots,x_d) \mapsto x_1 + (2n-1)x_2 + \cdots + (2n-1)^{d-1}x_d$$

is a set of integers, free of 3APs (exercise) and contained in $\{0,1,\dots,M\}$ where

$$M = (n-1)(1+(2n-1)+\cdots+(2n-1)^{d-1}) = \frac{(2n-1)^d-1}{2}.$$

Marcin Kuczma, composer of 1983 Q5 and five other IMO problems, member of the Polish team in 1961 and 1962 and six times team leader

To solve 1983 Q5 we need only choose d, n and A such that $M < 10^5$ and $|A| \geq 1983$ (we can then add 1 to everything to get positive integers $\leq 10^5$). Small d does not work. For instance, when $d = 2$, to have $M < 10^5$ we need to have $n \leq 224$, so $A \subseteq \{0,1,\dots,223\}^2$. Hence if A is contained in a circle then, since vertical lines meet a circle in at most two points, $|A| \leq 448$, far short of requirements.

What about the opposite extreme, where n is small and d large? Take $n = 2$. Then $M = \frac{1}{2}(3^d-1)$ and $A \subseteq \{0,1\}^d$. In fact, we may take $A = \{0,1\}^d$ since all these points already lie on a sphere. The largest permissible value of d (to ensure that $M < 10^5$) is $d = 11$, in which case $M = 88573$. Then $|A| = 2^{11} = 2048 > 1983$, so this solves the problem.

Note that A is precisely the set of integers whose base 3 expansion contains only zeros and ones, and indeed I am certain that those who solved the problem at the competition did so by making the observation that this is a good choice. Why, then, the digression on high-dimensional spheres?

The answer is that for larger numbers the construction using ternary expansions does not provide the best bounds. Roughly speaking, it leads to the lower bound $r_3(N) \gtrapprox N^{\log 2/\log 3}$ for general N. By instead taking spheres of intermediate dimension ($d \approx \sqrt{\log N}$ is a good choice) one may obtain the much stronger bound $r_3(N) \geq Ne^{-c\sqrt{\log N}}$. Note that this is eventually larger than $N^{0.999}$, for instance.

Determining the actual value of $r_3(N)$, even very roughly, is a major unsolved problem. Many people (including me) believe that Behrend's construction is close to optimal. However, upper bounds have proven hard to come by. In 1953 Klaus Roth showed that $r_3(N) \leq CN/\log\log N$. The current record is that $r_3(N)$ is less than or equal to a constant times $\frac{N(\log\log N)^C}{\log N}$; the first bound of this type, with $C = 6$, was obtained by Sanders in 2010, and in May 2020 Schoen made a preprint available in which he showed that any $C > 3$ is permissible.

Erdős conjecture

Conjecture: if $\mathscr{A}$ is an infinite set of integers with $\sum_{a\in\mathscr{A}} \frac{1}{a} = \infty$, $\mathscr{A}$ contains arbitrarily long arithmetic progressions.

The upper bounds on $r_3(N)$ mentioned in the text fall just short of confirming this conjecture in the case of 3-term progressions.

Another very recent development saw a remarkable advance on a finite field variant of the problem: in 2016 Ellenberg and Gijswijt, building on work of Croot, Lev and Pach, showed that any subset of $(\mathbb{Z}/3\mathbb{Z})^n$ of size at least $(2.756)^n$ contains a 3AP. Note that if $N = 3^n$ then $(2.756)^n = N^{0.922\ldots}$, so this work shows that there is no analogue of Behrend's example in this high torsion setting.

There is scope for a further question along the same lines in 2049, but it may be incredibly difficult!

1987 Q6

Let A be an integer greater than or equal to 2. Prove that if $x^2 + x + A$ is prime for all integers x such that $0 \leq x \leq \sqrt{A/3}$, then $x^2 + x + A$ is prime for all integers x such that $0 \leq x \leq A - 2$.

Vsevolod Lev, composer of 1987 Q6 and author of Croot-Lev-Pach

At first sight, this question looks utterly remarkable: given a formula which produces a few primes, show that it produces many primes. On further contemplation, one wonders whether the hypothesis that x^2+x+A is prime for all integers x such that $0 \leq x \leq \sqrt{A/3}$ is so strong that it is only satisfied for a few A, which one might then check by hand. However, there are certainly nontrivial examples which take quite some time to check by hand, particularly $A = 41$. The corresponding polynomial, $x^2 + x + 41$, is Euler's famous "prime producing polynomial" and takes prime values for $x = 0, 1, \ldots, 39$. A successful solution to 1987 Q6 must include a proof of this, starting from the fact that it is true for $x = 0, 1, 2, 3$.

There is close connection, discovered by Rabinowitsch, between values of A such that $x^2 + x + A$ is prime for $x = 0, 1, 2, \ldots, A - 2$ (occasionally referred to as "Euler lucky numbers") and class numbers of imaginary quadratic fields. Indeed, A is an Euler lucky number if and only if $4A - 1$ is squarefree and if the integers in $K = \mathbb{Q}(\sqrt{1-4A})$ have unique factorisation, that is to say K has class number 1. By the solution of the Class number one problem, there are only six Euler lucky numbers: $A = 2, 3, 5, 11, 17, 41$.

Having a complete list of Euler lucky numbers does not really help with 1987 Q6. However, having the connection with class numbers in mind, the problem becomes an exercise in undergraduate algebraic number theory, as follows:

Let $\alpha = \frac{1+\sqrt{1-4A}}{2}$. Then $\mathbb{Z}[\alpha]$ is the ring of integers $\mathcal{O}_K$ in K, and the minimal polynomial of α is $f(X) = X^2 + X + A$.

Class number one problem

If d is positive and squarefree then $\mathbb{Q}(\sqrt{-d})$ has class number one only when $d \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}$. A proof was claimed by Heegner in 1952, but not widely believed at the time. The result was proven by Baker in 1966 using very different methods, and independently by Stark in 1967 using methods close to those of Heegner. Nowadays Heegner's proof is regarded as essentially correct.

Suppose that $x \leq A - 2$ and that $f(x)$ is not prime. Then it is divisible by some prime $p \leq \sqrt{f(x)} < A$, which means that $f(X)$ is a reducible polynomial $\bmod p$. By Dedekind's criterion, the ideal (p) in $\mathcal{O}_K$ splits into two ideals of norm p. Neither of these ideals can be principal, since there are no elements in $\mathcal{O}_K$ with norm p because the equation $N_{K/\mathbb{Q}}(u + v\alpha) = u^2 + uv + Av^2 = p$ has no solutions in integers u, v (since $p < A$). Thus $\mathcal{O}_K$ is not a principal ideal domain.

Dedekind's criterion

Suppose K is a number field whose ring of integers $\mathcal{O}_K$ is $\mathbb{Z}[\alpha]$ for some α with minimal polynomial m_α. Let p be a prime. Then the ideal (p) is inert (does not split as a product of smaller prime ideals) if and only if $m_\alpha(X)$ is irreducible $\bmod p$.

Minkowski's bound tells us that there must be some rational prime $q \leq \frac{4}{\pi}\sqrt{A}$ such that (q) splits nontrivially into prime ideals in $\mathcal{O}_K$ (otherwise, $\mathcal{O}_K$ would be a principal ideal domain). A second application of Dedekind's criterion tells us that $f(X)$ is reducible $\bmod q$, which means that $q | f(x)$ for some x with $|x| \leq \frac{1}{2}q \leq \frac{2}{\pi}\sqrt{A}$. In particular, $f(x)$ is not prime.

If it were the case that $\frac{2}{\pi} \leq \frac{1}{\sqrt{3}}$, 1987 Q6 would be solved. Unfortunately, this is not the case! It turns out that Minkowski's bound (which generalises to apply to all number fields) is not quite strong enough. For imaginary quadratic fields, a slightly stronger bound due to Gauss and Legendre is available, though not so well-known: the ideal class group of $\mathbb{Q}(\sqrt{-D})$, where D is positive, squarefree and $3 \bmod 4$, is generated by the prime ideal divisors of the ideals (q), with $q \leq \sqrt{D/3}$ a rational prime. Using this bound instead, and working through the argument as above, we *do* solve the IMO problem.

Minkowski bound

Suppose $K = \mathbb{Q}(\sqrt{-D})$ is a number field, where $D > 0$ is squarefree and $3 \bmod 4$. Then the ideal class group of K is generated by the classes of prime ideals dividing (q), q a rational prime $< \frac{2}{\pi}\sqrt{D}$.

Of course, this was not the solution expected of competitors at the IMO. There is a completely elementary solution, though in some sense it is equivalent to the one just described.

As we said, the imaginary quadratic fields of class number one were determined in the 1950s/60s. The analogous question for *real* quadratic fields remains wide open. Gauss conjectured that (unlike in the imaginary case) there are infinitely many real quadratic fields with class number one, and hence unique factorisation.

Ramanujan's constant

If A is an Euler lucky number then $e^{\pi\sqrt{4A-1}}$ is very close to an integer. Most spectacularly,

$$e^{\pi\sqrt{163}} = e^{\pi\sqrt{4(41)-1}}$$
$$= 262537412640768743.99999999999925$$

This number is called the *Ramanujan constant* for unusual reasons: in a famous Martin Gardner article on April Fools' Day in 1975, Gardner reported that this number had been shown to be an integer, confirming a (fictitious) conjecture of Ramanujan from 1914. In fact, the phenomenon was observed by Hermite in 1858.

In fact, heuristics of Cohen and Lenstra predict that $\mathbb{Q}(\sqrt{d})$ has class number one for 75.446 percent of all primes d! It is not at all easy to even exhibit a prime d for which $\mathbb{Q}(\sqrt{d})$ does *not* have unique factorisation, the smallest being $d = 79$ (where, for instance, $3 \times 5 = -(8 + \sqrt{79})(8 - \sqrt{79})$).

Let us finish this section with an unsolved problem whose statement seems, at first glance, rather similar to that of 1987 Q6: Is there an A such that there are 1000 consecutive values of x for which $x^2 + x + A$ is prime?

2008 Q3

Prove that there are infinitely many positive integers n such that $n^2 + 1$ has a prime divisor greater than $2n + \sqrt{2n}$.

We quickly sketch the solution. The first observation is that one can almost immediately get rather close to the desired bound by using the fact that -1 is a quadratic residue for every prime $p \equiv 1 \bmod 4$ (together with the fact that there are infinitely many such primes). This means that $n^2 + 1 \equiv 0 \bmod p$ is solvable for some n with $-\frac{1}{2}(p-1) \leq n \leq \frac{1}{2}(p-1)$, and hence for some n with $1 \leq n \leq \frac{1}{2}(p-1)$, or in other words $p \geq 2n + 1$.

To solve the problem, we need only improve $2n + 1$ to $2n + \sqrt{2n}$. To give the idea, let us show that we cannot have $p = 2n+1$ for $p \geq 7$. If we did, $\left(\frac{p-1}{2}\right)^2 \equiv -1 \bmod p$. But $4\left(\frac{p-1}{2}\right)^2 = p^2 - 2p + 1 \equiv 1 \bmod p$, and these two statements are incompatible for $p \geq 7$.

A similar argument shows that we cannot have $p = 2n + 3$ for $p \geq 17$. We leave it to the reader to complete the solution from here.

The problem hints at a question from a notorious list of Landau from over a century ago: are there infinitely many primes of the form $n^2 + 1$?

Somewhat curiously for an Olympiad problem, one can solve 2008 Q3 by simply quoting from the best available results in the direction of Landau's conjecture. For instance, Deshouillers and Iwaniec showed that there are infinitely many n such that $n^2 + 1$ has a prime factor $> n^{6/5}$. Had a candidate at the Olympiad cited this paper, it might have given the coordinators a headache.

A truly daring candidate could have quoted Hooley [4]:

> The theorem that, if P_x be the greatest prime factor of $\prod_{n \leq x}(n^2 + 1)$, then $P_x/x \to \infty$ as $x \to \infty$ for which [. . .] we are indebted to Chebyshev. Revealed posthumously as little more than a fragment in one of Chebyshev's manuscripts, [. . .]

Whilst Landau's question remains unsolved, there have been other spectacular advances in relatively recent times, most notably the result of Friedlander–Iwaniec from 1998 that there are infinitely many n and a such that $n^2 + a^4$ is prime. There being $\asymp cX^{3/4}$ numbers of this form of size at most X, this can be considered in some sense halfway to Landau's conjecture.

Landau Problems

At the International Congress of Mathematicians in 1912, Landau listed the following problems about primes. All are still far from resolved.

(1) Goldbach's conjecture: Can every even integer greater than 2 be written as the sum of two primes?

(2) Twin prime conjecture: Are there infinitely many primes p such that $p + 2$ is prime?

(3) Legendre's conjecture: Does there always exist at least one prime between consecutive perfect squares?

(4) Are there infinitely many primes of the form $n^2 + 1$?

1989 Q3

Let n and k be positive integers and let S be a set of n points in the plane such that no three points of S are collinear, and for any point P of S there are at least k points of S equidistant from P. Prove that $k < \frac{1}{2} + \sqrt{2n}$.

The very great majority of Olympiad geometry questions have little relation with current research (and even if they did, I would be poorly qualified to elaborate upon it). This problem is a pleasant exception.

The solution is short, so we give it here. The assumption implies that for each $s \in S$ there is a set $\Sigma(s) \subseteq S$ of k points, all at the same distance from s. Using the notation 1_X for the characteristic function of the set X (that is, $1_X(x) = 1$ if $x \in X$ and 0 otherwise) this implies that

$$\sum_{s \in S} \sum_{x \in S} 1_{\Sigma(s)}(x) = kn.$$

An application of Cauchy-Schwarz yields

$$\sum_{x\in S}\left(\sum_{s\in S} 1_{\Sigma(s)}(x)\right)^2 \geq k^2 n.$$

Write E for the left-hand side here, thus $E \geq k^2 n$. On the other hand, expanding out the square gives

$$E = \sum_{s,s'\in S}\sum_{x} 1_{\Sigma(s)}(x)1_{\Sigma(s')}(x) = \sum_{s,s'\in S}\left|\Sigma(s)\cap\Sigma(s')\right|.$$

Now comes the crucial observation: if $s \neq s'$, $|\Sigma(s)\cap\Sigma(s')| \leq 2$, since $\Sigma(s)$ and $\Sigma(s')$ lie on circles with distinct centres, which intersect in at most two points. Thus the contribution to E from $s \neq s'$ is at most $2n(n-1)$. The contribution from $s = s'$ is kn, and so $E \leq 2n(n-1) + kn$.

Comparing upper and lower bounds for E yields $2n(n-1) + kn \leq k^2 n$, and the rest of the solution is a routine calculation.

An almost immediate corollary of 1989 Q3 is that, given any $n > 1$ points in the plane, they determine $\geq c\sqrt{n}$ distinct distances for some absolute constant $c > 0$. Indeed, there is always one point which determines at least $c\sqrt{n}$ distinct distances to the other points.

This is a weak bound in the direction of an old problem of Erdős.

Erdős distance problem

Given n distinct point in the plane, show that the number of distinct distances between pairs of them is $\geq C_\epsilon n^{1-\epsilon}$, for every $\epsilon > 0$.

One could not hope for more, as consideration of the lattice points $\{(x,y) : 1 \leq x,y \leq \sqrt{n}\}$ shows. These determine $\asymp Bn/\sqrt{\log n}$ distinct distances for a certain positive constant B (this is closely related to the number of sums of two squares less than or equal to n).

Around ten years ago there was a remarkable breakthrough, resolving the Erdős distance problem and in fact showing that the lattice example is best possible up to a logarithmic term: Guth and Katz showed that any set of n points in the plane determines $\geq cn/\log n$ distinct distances. Their methods are certainly not of Olympiad type, involving both topology and algebraic geometry (albeit partly of a type that might have featured in a 19th century Olympiad). The following quote from the abstract of their paper is illuminating:

> We introduce two new ideas in our proof. In order to control points where many lines are incident, we create a cell decomposition using the polynomial ham sandwich theorem [...] In order to control points where only two lines are incident, we use the flecnode polynomial of the Rev. George Salmon to conclude that most of the lines lie on a ruled surface. Then we use the geometry of ruled surfaces to complete the proof.

Whilst we cannot do justice to their proof here, we can illustrate the flavour by giving an earlier argument of the same authors[1] solving the following question called the Joints problem.

Problem [Joints Problem] *Suppose we are given a set of L lines in three-dimensional Euclidean space. By a* joint *we mean a point v which is contained in (at least) three of these lines, not all lying in a plane. Show that the number of joints is at most $100L^{3/2}$.*

Let us first reduce matters to establishing the following:

Lemma: Given a finite set of lines containing J joints, one of the lines contains at most $10J^{1/3}$ joints.

To solve the joints problem given the lemma, let $J(L)$ be the maximum number of joints. Then the lemma implies that

$$J(L) \leq J(L-1) + 10J(L)^{1/3},$$

which yields the claimed bound by induction and a calculation.

Now we turn to the proof of the lemma. First observe that there must be a non-zero polynomial $P(x,y,z)$ of degree $< 10J^{1/3}$ vanishing at every joint. This is because the number of monomials $x^i y^j z^k$ of degree $i+j+k = D$ is $\binom{D+2}{3}$, which is strictly bigger than J for some integer $D < 10J^{1/3}$ (by some distance – the constant 10 here is very crude).

Thus choosing coefficients $a_{i,j,k}$ such that

$$P(x,y,z) = \sum_{i+j+k=D} a_{i,j,k} x^i y^j z^k$$

vanishes at the J joints involves finding a nonzero solution to J linear equations in $\binom{D+2}{3} > J$

[1] The argument we sketch here is a simplification of the original one of Guth-Katz due to Kaplan-Sharir-Shustin and Quilodrán.

unknowns, something which is always possible by basic linear algebra.

Suppose that, amongst all non-zero polynomials vanishing at every joint, P has minimal degree d. Thus $d < 10J^{1/3}$.

Now suppose the lemma is false. Let $\ell = v + \mathbb{R}w = \{v + \lambda w : \lambda \in \mathbb{R}\}$ be one of the lines. Restricted to ℓ, P is a polynomial in degree $\leq d$ in λ. By assumption, ℓ contains $> 10J^{1/3} > d$ joints, and by construction P vanishes at all of them. Therefore, P must vanish identically on ℓ.

By Taylor expansion (which, since we are dealing with polynomials, is a finite expansion not leading to any issues of convergence) we have

$$P(v + \lambda w) = \lambda(w \cdot \nabla P(v)) + \text{higher order terms}$$

where $\nabla P = \left(\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial P}{\partial z}\right)$.

Thus $w \cdot \nabla P(v) = 0$ whenever $v + \mathbb{R}w$ is a line in our set. If v is a joint then, by definition, there are three linearly independent values of w such that $v + \mathbb{R}w$ lies in our set, and hence for which $w \cdot \nabla P(v) = 0$.

A vector in $\mathbb{R}^3$ orthogonal to three independent vectors must be zero. Therefore ∇P vanishes at all the joints, and hence so do the three partial derivatives $\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial P}{\partial z}$. At least one of them is not the zero polynomial, and they all have degree strictly less than d, contrary to the minimality of $\deg P$. This contradiction establishes the lemma, and the Joints Problem is solved.

Let us conclude by mentioning a related problem of Erdős which remains unsolved, the Unit Distance problem: Given n points in the plane, what is the maximum number of unit distances they can determine? The best upper bound currently known is $Cn^{4/3}$, a result of Spencer, Szemerédi and Trotter; it has not been improved for nearly 40 years.

1972 Q3

We turn now to a problem posed by the UK in 1972. (I have not been able to establish the identity of the composer, and would be interested in more information.)

Let m and n be arbitrary non-negative integers. Prove that $(2m)!(2n)!/m!n!(m+n)!$ is an integer.

As with many Olympiad problems from the early period, this is a fairly straightforward exercise by modern standards. Even in this two-column format it can be solved in one line (modulo details):

$f(m,n) = 4f(m,n-1) - f(m+1,n-1)$. Induction.

A different solution can be squeezed into two lines modulo a few more details:

Apply Legendre's formula and use $\lfloor 2x \rfloor + \lfloor 2y \rfloor \geq \lfloor x \rfloor + \lfloor y \rfloor + \lfloor x+y \rfloor$ at each prime power.

> **Legendre's formula**
>
> If p is a prime and m a natural number, write $v_p(m)$ for the largest j such that $p^j | m$. Then $v_p(n!) = \sum_j \lfloor \frac{n}{p^j} \rfloor$, where $\lfloor t \rfloor$ denotes the greatest integer less than or equal to t.

Given this, one might wonder what topics of current research interest could possibly be related to the question. The answer is that the following is still unsolved:

Classify all sets of integers $a_1, \ldots, a_K, b_1, \ldots, b_L$ with $\sum a_i = \sum b_i$ and $\gcd(a_1, \ldots, b_L) = 1$ such that the ratio $(a_1 n)! \cdots (a_K n)!/(b_1 n)! \cdots (b_L n)!$ is always an integer.

The IMO question corresponds to the fact that $(a_1, a_2, b_1, b_2, b_3) = (2t, 2u, t, u, t+u)$ is always a solution.

The difference $L - K$ is called the *height*. Even for height 1, the classification of such sets of integers was only completed rather recently, by Bober [2]. Despite the elementary-seeming statement of the problem, the solution is certainly not of Olympiad type. Indeed, quoting from the abstract of [2]:

> [...] we completely describe when certain sequences of ratios of factorial products are always integral. Essentially, once certain observations are made, this comes down to an application of Beukers and Heckman's classification of the monodromy of the hypergeometric function $_nF_{n-1}$.

Even more recently, Soundararajan [9] found a more elementary proof. The nature of the classification is surprisingly exotic. There are three infinite families and 52 sporadic examples.

One of the 52 sporadic examples is

$$\frac{(4n)!(6n)!(9n)!(24n)!}{(2n)!(3n)!(8n)!(12n)!(18n)!}.$$

To show that this is an example, it suffices by Legendre's formula to show that

$$\lfloor 4x\rfloor + \lfloor 6x\rfloor + \lfloor 9x\rfloor + \lfloor 24x\rfloor - \lfloor 2x\rfloor - \lfloor 3x\rfloor - \lfloor 8x\rfloor - \lfloor 12x\rfloor - \lfloor 18x\rfloor \geq 0$$

for all real x. In fact, Landau showed this condition is also necessary. The expression on the left is piecewise linear and periodic under $x \mapsto x+1$, so this is a finite check, albeit not a very pleasant one.

2003 Q6

Show that for each prime p, there exists a prime q such that $n^p - p$ is not divisible by q for any positive integer n.

This problem is unusual for an Olympiad problem in that it may be solved quite systematically by those who know some more advanced techniques, whereas an elementary solution is decidedly tricky.

First observe that $n^p - p$ is irreducible over the integers. This is an easy exercise (a special case of Eisenstein's criterion).

Thus 2003 Q6 is a consequence of the following much more general theorem: if $f(n)$ is an irreducible monic integer polynomial with $\deg f > 1$ then there are infinitely many primes q such that $f(n)$ has no roots modulo q.

Eisenstein's criterion

Suppose that $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$ is a monic integer polynomial. Suppose that there is a prime p such that $p|a_i$ for all i, but $p^2 \nmid a_0$. Then f is irreducible in $\mathbb{Z}[X]$ (and hence, by Gauss's lemma, over the rationals).

To prove this, first note that, since f is irreducible, its Galois group $G = \mathrm{Gal}(f)$ acts transitively on its roots.

Next we invoke Burnside's lemma[2]. In our setting, there is one orbit, so the average size of $|\mathrm{Fix}(g)|$ is one. However, $|\mathrm{Fix}(\mathrm{id}_G)| = \deg f > 1$, so there must be some $\sigma \in \mathrm{Gal}(f)$ with $\mathrm{Fix}(\sigma) = \emptyset$.

[2]Also known as the Cauchy-Frobenius theorem, the orbit-counting theorem, or the Lemma that is not Burnside's.

Burnside's lemma

Suppose a finite group G acts on a finite set X. Then the number of orbits is $|G|^{-1}\sum_{g\in G}|\mathrm{Fix}(g)|$, where $\mathrm{Fix}(g) = \{x \in X : g\cdot x = x\}$.

Finally, we apply Frobenius's density theorem from 1880, which states the following: Let $f \in \mathbb{Z}[X]$ be a monic irreducible polynomial of degree n, and suppose we have a partition of n into ℓ parts $m_1, \ldots, m_\ell$. Then the density of primes q for which the irreducible factors of $f(x) \bmod q$ have degrees $m_1, \ldots, m_\ell$ exists and equals the proportion of elements in $\mathrm{Gal}(f)$ with cycle type $m_1, \ldots, m_\ell$.

Analytic density

The notion of density used by Frobenius is a rather weak one, sometimes known as the analytic density or the Dirichlet density. A set $\mathcal{P}$ of primes has density α in this sense if

$$\lim_{s\to 1^+} \frac{\sum_{p\in\mathcal{P}} p^{-s}}{\sum_{p \text{ prime}} p^{-s}} = \alpha.$$

The element σ has no cycles of length 1, and so Frobenius' s result tells us that for a positive density of primes (and hence for infinitely many primes), $f(x) \bmod q$ splits into irreducible factors, none of degree 1. That is, $f(x)$ has no roots modulo q.

For more on Frobenius's density theorem and its more famous cousin the Chebotarëv density theorem, see the excellent article [5].

The question of how an irreducible polynomial $f(X) \in \mathbb{Z}[X]$ reduces modulo a prime q is connected with some of the very deepest issues in number theory linked to the Langlands Programme.

In the setting of 2003 Q6, one can quickly appreciate the complexity of the problem by looking at the cases $p = 2, 3, 5$.

The polynomial $n^2 - 2$ has a root modulo q iff $q \equiv \pm 1 \bmod 8$, these being the primes modulo which 2 is a quadratic residue.

The polynomial $n^3 - 3$ has a root mod$q$ if either $q \equiv 2 \bmod 3$ or $q = 1 \bmod 3$ and $4q = a^2 + 243b^2$

(this was conjectured by Euler and proven by Gauss and Eisenstein).

There is complicated explicit description, due to Emma Lehmer, of q for which $n^5 - 5$ has a root $\bmod q$. This involves congruence conditions on x, u, v, w where $16q = x^2 + 50u^2 + 50v^2 + 125w^2$, $xw = v^2 - 4uv - u^2$.

Coincidentally, a beautiful article by Serre [8] on these topics appeared just two days after the students sat the second paper at IMO 2003, and Q6 is an immediate consequence of Theorem 1 there! Serre's article is recommended for a longer introduction to this topic.

1997 Q6

For each positive integer n, let $f(n)$ denote the number of partitions of n with each part being a power of two. Prove that if $n \geq 3$ then $2^{n^2/4} < f(2^n) < 2^{n^2/2}$.

I will not give the solution to this one, but let me mention that the asymptotic behaviour of f was described by Mahler [6] in the 1940s. (The LMS at the time seemed to favour extremely uninformative titles, see also [7].)

Mahler showed that $f(2^n) = 2^{(1/2+o(1))n^2}$ which, due to the $o(1)$ term, does not actually imply either bound in the Olympiad problem.

A related function providing a richer source of mathematics is the unrestricted partition function $p(n)$, that is to say the number of ways of writing n as a sum of positive integers. The asymptotic behaviour of $p(n)$ was found by Hardy and Ramanujan: $p(n) \asymp (4n\sqrt{3})^{-1}e^{\pi\sqrt{2n/3}}$.

The first few values of $p(n)$ are 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490.

Remarkably, some extremely basic questions about the partition function remain unsolved, for instance

Are half of the values of $p(n)$ even, and half odd?

In fact, our knowledge of this question is extremely poor. The best known lower bound on the number of odd values of $p(n)$, $n \leq X$, is just $\sqrt{X}/\log\log X$, established in a recent paper of mine [1] joint with Bellaïche and Soundararajan.

The following is also unsolved:

Are there infinitely many n with $p(n)$ divisible by 3?

This *is* known to be true with 3 replaced by any other odd primes. For instance, the celebrated Ramanujan congruence $p(5k+4) \equiv 0 \bmod 5$ implies the analogue for divisibility by 5.

Ben Green

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Differing Approaches to Ancient Egyptian Mathematics

CHRISTOPHER D. HOLLINGS AND RICHARD BRUCE PARKINSON

We discuss some developments in the understanding of ancient Egyptian mathematics that took place during the early decades of the twentieth century. In particular, we highlight the differing views of the subject by mathematicians on the one hand and Egyptologists on the other.

The subject of ancient Egyptian mathematics typically features, alongside its Babylonian counterpart, as an almost obligatory component of any general history of mathematics. The Egyptian system of arithmetic, and the extent of their geometrical knowledge, are often presented alongside other rather more questionable details, such as the spurious ‘Egyptian value of π’ or ambitious claims for Egyptian knowledge of calculus, that stem in part from the imposition of modern mathematical ideas and attitudes onto historical materials. Indeed, the contrasting views of mathematicians and Egyptologists have created a tension that has been present since the earliest studies of ancient Egyptian mathematics.

The traditional view has been that mathematics as a deductive system, based upon the notion of proof, originated in ancient Greece. While this remains a rather complicated and contentious issue for present-day scholars [1], the European attitude has therefore been that ‘true’ mathematics started in Greece — despite a tradition passed on by classical authors like Herodotus that geometry was invented in Egypt and then passed from there into Greece. These claims could not be substantiated or tested, however, until 1858, when the Scottish lawyer and excavator Alexander Henry Rhind (1833–1863) purchased two halves of a broken roll of papyrus in Luxor. This, now known as the Rhind Mathematical Papyrus (RMP), remains one of the most complete surviving sources on Egyptian mathematics, and is held in the British Museum (P. BM EA 10057 and 10058). The papyrus, dating from *c.* 1537 BCE and written in a cursive (hieratic) script, was probably placed in a tomb as part of a display of the tomb-owner’s social and cultural status. It is not a treatise on mathematics, but a collection of over 80 arithmetical and geometrical problems and solutions, ranging from the distribution of rations among workers, to the calculation of areas and volumes: see the box at the end of this article.

Figure 1. Thomas Eric Peet (1882–1934) [© Griffith Institute, University of Oxford]

The papyrus was published quickly, but not well. The British Museum facsimile of 1898, for example, badly muddled the arrangement of the text. In the 1870s, a preliminary study of the papyrus was produced by the German Egyptologist August Eisenlohr (1832–1902), but by the 1900s there was a generally felt need for a new edition. This need was eventually supplied by the British Egyptologist Thomas Eric Peet (1882–1934). He had studied both mathematics and classics at Oxford, before turning first to archaeology, and then to Egyptian philology — the major works of the second half of his career were editions of Egyptian papyri. Apparently encouraged by his own mathematical background, Peet began to study the RMP in 1911, and resumed work in 1920, following service in the Mediterranean during the First World War; his new edition was finally published in 1923 [6], by which time he was Professor of Egyptology at the University of Liverpool. His edition provided a detailed transcription of the hieratic script and a translation and a full commentary on the problems, and was much praised by both mathematicians and Egyptologists. It reignited an academic interest in Egyptian mathematics, which had stagnated slightly in the preceding decades, and it also allowed the subject to be studied on its own terms, rather than standing in the shadow of ancient Greek mathematics.

The techniques of Egyptian arithmetic that had first been identified by Eisenlohr were brought out more clearly by Peet. Central to these was the 'two-column' method, which we illustrate in Figure 2, where we multiply 15 by 13. We start by writing down '1' and '15' at the tops of two columns, which we construct by repeatedly doubling both numbers and writing down the result. Our goal is to find numbers in the first column that sum to make 13. Having doubled our numbers three times, we see that the 1, the 4 and the 8 in the first column add to 13, and so the corresponding numbers in the second column, 15, 60 and 120, add to give the required answer of 195. In the same manner as an Egyptian scribe, we mark the relevant rows with a '\'. Alternatively, we could have begun the problem by writing down 1 and 13 with a view to finding 15 as a sum of numbers from the first column. This two-column method was the standard technique for performing multiplication in ancient Egypt (both of integers and of fractions), though doubling was not the only permissible operation: halving, and multiplying or dividing by 10 were among the other options. Division was also possible in a similar way: here we take 1 and the divisor as the tops of our columns, and seek the dividend as a sum of entries in the *second* column, before reading off the corresponding entries in the first, which add to give the quotient. It seems reasonable to suppose that an experienced scribe would develop an intuition for the most useful values to construct within the two columns.

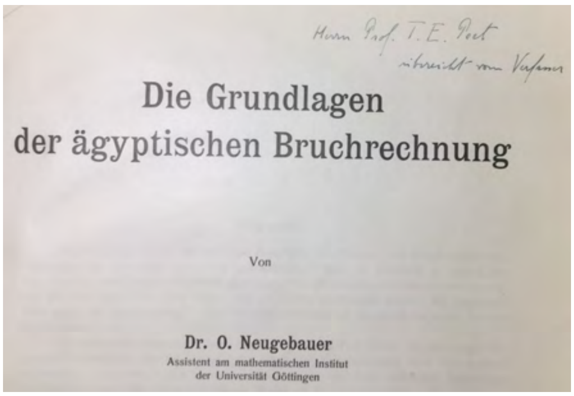
When studying Egyptian arithmetic, one feature that deserves special attention is the well-developed system of fractions. In modern terms, Egyptian scribes employed only unit fractions, with the one exception of $\frac{2}{3}$, which enjoyed a special status. Most (unit) fractions were denoted by the corresponding integer, with an extra mark to indicate that the fraction was intended. In what follows, we will employ a notation that is often used in writings on ancient Egyptian mathematics, and which is designed to reflect Egyptian fractional notation: in place of $\frac{1}{n}$, we write instead $\overline{n}$. The special fraction $\frac{2}{3}$ is written $\overline{\overline{3}}$. Juxtaposition is used to denote addition of fractions. Although the restriction to unit fractions might seem to modern mathematical eyes to be a severe limitation, Egyptian scribes appear to have been well practiced in the manipulation of elaborate fractional expressions, as we shall see shortly. The question of whether they had a *notion* of non-unit fractions even if they did not have a notation for them is a difficult one to answer, and has generated much debate, though we will not go into this here.

\	1	15
	2	30
\	4	60
\	8	120
	13	195

Figure 2. Multiplication of 15 by 13

It appears to have been a known principle among Egyptian scribes that if a fraction of the form $\overline{2n}$ were to be doubled, then the answer would simply be $\overline{n}$. Fractions of the form $\overline{2n+1}$, however, were more problematic: in the absence of a notation for non-unit fractions, it was necessary to express the result of such a doubling (equivalently, a division of 2 by an odd integer) as a sum of unit fractions. Thus, for example, twice $\overline{5}$ would be expressed as $\overline{3}\ \overline{15}$: the sum of two fractions to us, but a single number to an Egyptian scribe. Depending on the numbers involved, the result of such a duplication may not be obvious, and so standard tables of expressions for the duplication of odd fractions were drawn up as works of reference. A few examples of these survive, most notably that which takes up most of one side of the RMP: a table giving unit fraction expressions of $2 \div n$ for odd integers n between 3 and 101, the final expression being $\overline{101}\ \overline{202}\ \overline{303}\ \overline{606}$. Since such representations as sums of unit fractions are in general not unique, there has been much scholarly debate about how the Egyptian scribes chose the representations that they did, and whether there were any general principles underlying those choices. It was the accessibility that Peet's edition afforded the RMP, together with his up-to-date analysis of its contents, that opened the door to much of this speculation.

One scholar who was particularly inspired by Peet's work was Otto Neugebauer (1899–1990), a young student in Göttingen. Neugebauer had already studied mathematics and physics, but was now cultivating an interest in ancient science. His first publication was a review of Peet's edition of the RMP, and his doctoral dissertation [5] concerned the principles of Egyptian fraction reckoning, as reflected in the papyrus. While completing the dissertation in 1926, he was in correspondence with Peet, as demonstrated by two letters that have recently come to light in a library in Oxford. In 1933, Peet became Professor-elect of Egyptology at the University of Oxford and a Fellow of The Queen's College. After his sudden death the following year, his personal library was purchased from his executors and donated to the college where it became part of a new 'Peet Memorial Library', which remains to this day. The library included Peet's copy of Neugebauer's dissertation (sent to him by Neugebauer — see Figure 3), and the two letters from 1926 were found inside this book.[1]


Herrn Prof. T.E. Peet
überreicht vom Verfasser

Die Grundlagen
der ägyptischen Bruchrechnung

Von

Dr. O. Neugebauer
Assistent am mathematischen Institut
der Universität Göttingen

Figure 3. Title page of Peet's copy of Neugebauer's dissertation with the latter's inscription [With the kind permission of The Provost, Fellows and Scholars of The Queen's College, Oxford]

The two letters shed light on the way in which ancient Egyptian mathematics was being re-evaluated in the 1920s and in particular, they show up a contrast between the attitudes of two scholars who approached the subject from quite different directions. Both were competent mathematicians and Egyptologists, and yet one (Neugebauer) put the mathematics first, and made general assertions about the nature of ancient Egyptian mathematics that arguably owed more to modern ideas about how mathematics 'should' be than to the direct evidence of papyri; the other (Peet) brought Egyptological considerations to the fore, drawing conclusions that were more firmly embedded in a knowledge of the cultural context of surviving sources — where Neugebauer saw lacunae in the historical record as gaps to be filled with educated speculation, Peet tip-toed cautiously around them, confining his commentary largely to what was clearly and unequivocally present in the original texts that he was editing and analysing.

The detailed mathematical content of the two letters relates to certain arithmetical calculations (Problems 7–20) that appear early in the RMP. Unlike later problems, which motivate the calculations by reference to real-world situations, these early computations simply give the results of multiplying together particular fractions, without setting them in a practical context. Problem 7, for example, appears in Figure 4. Focusing for the moment on the black numbers only, we see that Problem 7 consists of the calculation $1\ \overline{2}\ \overline{4} \times \overline{4}\ \overline{28}$, with result $\overline{2}$. It is of course by no means obvious that $\overline{4}\ \overline{8}\ \overline{16}\ \overline{28}\ \overline{56}\ \overline{112} = \overline{2}$; we will return to this point below.

\	1	$\overline{4}\ \overline{28}$
		$7\ \ 1$
\	$\overline{2}$	$\overline{8}\ \overline{56}$
		$3\ \overline{2}\ \ \overline{2}$
\	$\overline{4}$	$\overline{16}\ \overline{112}$
		$1\ \overline{2}\ \overline{4}\ \ \overline{4}$
	$1\ \overline{2}\ \overline{4}$	$\overline{2}$

Figure 4. Problem 7 from the RMP

In Peet's view, Problems 7–20, which for technical reasons he dubbed the 'first group of completions', were the results of numerical experimentation, in which the scribe had noted down relationships between fractions that might prove to be useful later on. Neugebauer, however, saw them as being much more systematic and part of a method for constructing the $2 \div n$ table at the beginning of the papyrus. Indeed, Neugebauer proposed an overarching scheme for organising the problems in the first group of completions, and was not above interpolating whole new problems in the RMP (he

[1] Apparently the only letters between Peet and Neugebauer that survive, they have subsequently been transferred to the Egyptological archive of the Griffith Institute, Oxford, where they are catalogued as Peet MSS 4.9. A detailed account of the letters can be found in [2].

added a 'Problem 11b') in order to make his scheme work. He also insisted that all of the problems within the first group of completions had been solved by a uniform method: a somewhat advanced technique within Egyptian mathematics which has been termed the 'method of red auxiliaries' after the colour of ink used for certain numbers ('auxiliary numbers') that could be included to aid working. Indeed, this method was certainly used in Problem 7, as demonstrated by the presence of the red numbers there. In spirit, the method of red auxiliaries is akin to finding a common denominator for a collection of fractions, although the process of multiplying up need not result in integers in all cases. In Problem 7, we notice that each of the red numbers is 28 times the black number immediately above it; we notice further that the resulting quantities are all either integers or reciprocals of powers of 2, and so adding up the red numbers is considerably easier than finding the sum $\overline{4}\ \overline{8}\ \overline{16}\ \overline{28}\ \overline{56}\ \overline{112}$. The total is 14, which we know to be 28 times the desired answer, and some simple arithmetic (either via the two-column method or by inspection — this calculation is omitted from the papyrus) gives us the answer $\overline{2}$. Again, a practiced scribe would probably have developed an intuition for choosing common denominators that yielded numbers that were more easily handled. A slightly more involved example of the method of red auxiliaries, taken from what Peet termed the 'second group of completions', is given in Figure 5, whose interpretation is left as an exercise for the reader. (Hint: in common with many problems in the RMP, the solution to the problem is simply stated, without any derivation, but a verification of this answer follows; recall also that $\overline{\overline{3}} = \frac{2}{3}$.)

$\overline{4}$	$\overline{8}$	$\overline{10}$	$\overline{30}$	$\overline{45}$	Complete into $\overline{\overline{3}}$
11	5	4	1	1	
$\overline{4}$	$\overline{2}$	$\overline{2}$	$\overline{2}$		
	$\overline{8}$				

Therefore $\overline{9}\ \overline{40}$ is what must be added to it, making $\overline{\overline{3}}$

$\overline{4}$	$\overline{8}$	$\overline{9}$	$\overline{10}$	$\overline{30}$	$\overline{40}$	$\overline{45}$	$\overline{3}$
11	5 $\overline{2}$	5	4	1 $\overline{2}$	1 $\overline{8}$	1	15
$\overline{4}$	$\overline{8}$		$\overline{2}$				

making 1

Figure 5. Problem 23 from the RMP

Neugebauer's belief that the first group of completions had a bearing on the construction of the $2 \div n$ table probably stemmed from the way in which its various entries are constructed: by breaking 2 into convenient parts, and then finding the corresponding parts of the divisor. Thus, for example, the resolution of $2 \div 7$ notes that $1\ \overline{2}\ \overline{4}$ is $\overline{4} \times 7$ and $\overline{4}$ is $\overline{28} \times 7$, and so, since $1\ \overline{2}\ \overline{4} + \overline{4} = 2$, we must have that $2 \div 7 = \overline{4}\ \overline{28}$. The decompositions of 2 that are employed elsewhere in the table range from the simple ($1\ \overline{2} + \overline{2}$ in the resolution of $2 \div 9$ as $\overline{6}\ \overline{18}$) to the rather more elaborate ($1\ \overline{42} + \overline{2} + \overline{3} + \overline{7}$ in the resolution of $2 \div 43$ as $\overline{42}\ \overline{86}\ \overline{129}\ \overline{301}$). Most of the resolutions feature a verification of the result, but never any indication of how the particular decomposition of 2 was arrived at, or why the given resolution into unit fractions was chosen. Since several of the problems in the first group of completions involve multiplying different quantities by $1\ \overline{2}\ \overline{4}$ (as we saw in Problem 7), it is tempting to see a connection between these and the $2 \div n$ table, though rather harder to find any part of the table where they have been used directly.

Neugebauer's wider scheme for the first group of completions involved reordering the problems in such a way that they could be derived from each other. Thus, for example, Problem 11 is the calculation $1\ \overline{2}\ \overline{4} \times \overline{7}$, from which we may derive Problem 12 by halving certain quantities in the working (see Figure 6); Problem 14 is obtained by halving again. In particular, Neugebauer sought to halve the red auxiliaries (not reproduced in Figure 6), and so 'restored' these in places where they seemed to him to be 'missing' from the papyrus; in the examples given in the first part of Figure 6, only Problem 14 contains red auxiliaries in the original. In Neugebauer's view, Problem 11 was special, and was the original calculation from which the others were derived, since it has integer auxiliaries for common denominator 28. On this point, however, Peet took a much more straightforward view: where auxiliary numbers are seemingly 'missing' from the papyrus, it is probably because this method was not the one that was employed — in many cases, the calculation involved was too elementary to warrant such an advanced technique. In the case of Problem 11, for example, we note that 7 was a commonly encountered conversion factor within the Egyptian system of units (7 palms = 1 royal cubit), and so calculations involving $\overline{7}$ were probably quite familiar and straightforward. A further level to Neugebauer's scheme took Problem 7 as its original calculation, and derived Problems 9, 13 and 15 by doubling and by dividing by 4 and 8, respectively. His interpolated Problem 11b filled an apparent gap between the calculations of Problems 7 and 13. Later readers have acknowledged the mathematical elegance of Neugebauer's scheme, but have also expressed doubts as to whether it is a true reflection of the methods of ancient Egyptian scribes. Judging by comments that he made in correspondence with other Egyptologists, and by annotations in his copy of Neugebauer's dissertation, Peet seems to have harboured reservations about the certainty with which Neugebauer put forward his conclusions, although he was too reserved to say this directly to Neugebauer.

Problem 11:	$1\ \overline{2}\ \overline{4} \times \overline{7}$	$= \overline{4}$
Problem 12:	$1\ \overline{2}\ \overline{4} \times \overline{14}$	$= \overline{8}$
Problem 14:	$1\ \overline{2}\ \overline{4} \times \overline{28}$	$= \overline{16}$
Problem 9:	$1\ \overline{2}\ \overline{4} \times \overline{2}\ \overline{14}$	$= 1$
Problem 7:	$1\ \overline{2}\ \overline{4} \times \overline{4}\ \overline{28}$	$= \overline{2}$
[Problem 11b:	$1\ \overline{2}\ \overline{4} \times \overline{8}\ \overline{56}$	$= \overline{4}$]
Problem 13:	$1\ \overline{2}\ \overline{4} \times \overline{16}\ \overline{112}$	$= \overline{8}$
Problem 15:	$1\ \overline{2}\ \overline{4} \times \overline{32}\ \overline{224}$	$= \overline{16}$

Figure 6. Neugebauer's rearrangement of the first group of completions

Besides Egyptian mathematics, Peet's interests stretched across the breadth of Egyptology, where his major concern was that the discipline should avoid imposing modern ideas on the subject matter. A month before his death, in his inaugural lecture in Oxford [7], he commented that "no side of Egyptian mental activity has been more discussed or written about in the last fifteen years than mathematics. Writers are insisting more and more on the necessity of tearing ourselves away from all modern conceptions and trying to start with blank minds, if we are to understand the process by which the Egyptians reached their results . . . We are coming to see that what is true of mathematics must also be true of all activities of the mind".

As an Egyptologist, Neugebauer was part of the German philological tradition, with its focus on texts, but his mathematical training also placed him in the Göttingen mathematical school, with a concern for rigorous foundations and general principles. A tendency towards systematisation and idealisation can be seen in his approach to the RMP, and his assertion of an overarching principle is characteristic of this background, and of his subsequent work in the history of the exact sciences in the Ancient Near East. In contrast, Peet remained more cautious and pragmatic, taking what the historian of Egyptology Clare Lewis has termed an 'anti-presentist' stance.[2] Peet's early death had a significant impact on Egyptology, especially at a time when the discipline was being professionalised as an academic area of study. Because of his death, together with the fact that he was the only Egyptologist studying this topic in detail at that time, his influence on the later study of ancient Egyptian mathematics was minimal, while Neugebauer went on to become one of the most prominent historians of mathematics of the twentieth century; his work helped to turn the history of mathematics into an academic discipline. It was not until the 1970s that Peet's culturally-sensitive approach to the history of mathematics began to gain ground once again, and is now increasingly the norm [9, 10]. Neugebauer's approach has come under criticism for providing analysis only in mathematical terms, with little consideration of cultural or archaeological context. The process of recontextualisation that Peet's edition of the RMP exemplified has more recently been championed by scholars such as Eleanor Robson (UCL) for the Ancient Near East [8] and Annette Imhausen (Frankfurt-am-Main) for pharaonic Egypt [3, 4]. Nevertheless, the essential tension between the attitudes of Peet and Neugebauer can still be found in the study of ancient mathematics today. As is the case with any ancient cultural artefacts, this academic tension involves an inevitable and unavoidable balancing act between seeing the past as entirely other and ancient, and interpreting it through our own world and mindsets.

Acknowledgements

This paper is warmly dedicated to Peter M. Neumann, in thanks for inspiring and encouraging the present work. We are grateful to Clare Newton for the use of the image of Peet.

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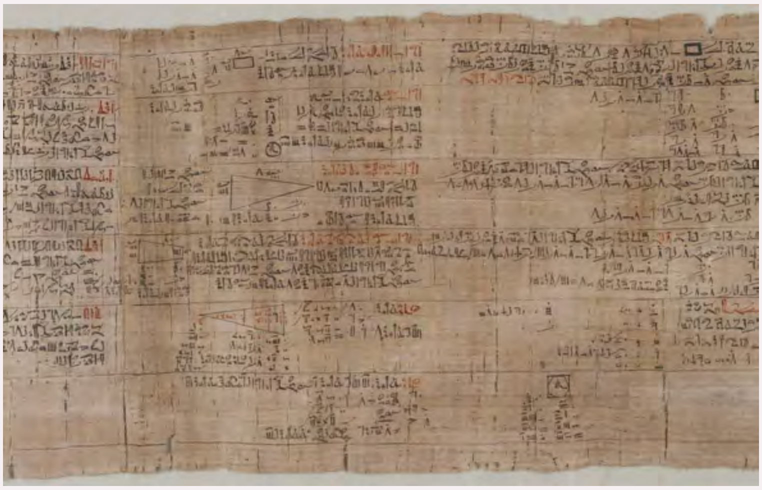
The Rhind Mathematical Papyrus



[© The Trustees of the British Museum]

According to its introductory title, the Rhind Mathematical Papyrus contains "rules for enquiring into nature, and for knowing all that exists, every mystery, ..., every secret". Many of the early problems in the papyrus are arithmetical, such as Problem 40, which seeks to divide 100 loaves of bread among 5 men in an arithmetical progression. Later problems are geometrical, featuring area and volume calculations. Among these, we note Problem 41 on the calculation of the volume of a cylindrical grain silo of diameter 9 cubits and height 10 cubits. In arriving at a final result of 640 cubic cubits, the scribe first finds the area of a circular face of the cylinder by taking eight-ninths of the diameter and squaring it. Indeed, this method appears to have been the standard Egyptian means of finding the area of a circle. Looking at this with modern eyes, we see that the corresponding value of π is $\frac{256}{81} \approx 3.16$. However, the explicit calculation of the ratio of a circle's circumference to its diameter appears nowhere in surviving sources on Egyptian mathematics, so we cannot speak of an 'Egyptian value of π' — to do so would be to impose our own knowledge of circles onto the original writings.

Assessing Mathematics at University: Covid-19 and Beyond

PAOLA IANNONE

This article discusses how mathematics students are assessed and the effect that the COVID-19 pandemic had on assessment. It gives suggestions for questions suitable for open-book exams that also foster conceptual understanding and that could be included in mathematics assessment beyond the emergency of the pandemic.

Introduction

I agreed to write this article about the summative assessment of mathematics at university when none of us really had ever asked what impact a pandemic would have on our work and lives. My original plan was to write about how we assess mathematics students in the UK and what consequences this summative assessment diet (intended to be the whole summative assessment experience that a student has throughout their degree) has on the students' perceptions of and engagement with mathematics.

Instead I am now writing this article in week ten of lock-down somewhere in a roof space in a house in the North East of England. I will most likely not see my students face to face till October and even then interaction will be different from what we are used to. Most of my time since universities were asked to move to remote teaching in March 2020 has been spent collaborating with colleagues in my own institution to organise remote teaching and then remote assessment. In this article I will first give a general picture of what the assessment diet of our students is. I will then reflect on actions that have been taken by colleagues in mathematics departments to radically change the assessment diet and move from the standard closed-book timed exams to open-book remote exams. I will finally draw some conclusions regarding what we can learn from the pandemic experience in terms of summative assessment. But first – an explanation of why it is important to think about summative assessment.

Formative and summative assessment

In a naïve way we could define summative assessment as assessment that carries marks and nothing else, while formative assessment is assessment that carries information for the students on how to go from where they are with their learning (as manifested by their work) to where we, their lecturers, would like them to be. These two types of assessment are not mutually exclusive: a piece of coursework that carries marks administered in the middle of a semester and returned with rich feedback can have both a summative and a formative function. Of course formative assessment is very important in the learning cycle of the students, but my attention as a researcher in mathematics education has been always steered towards summative assessment. This type of assessment is extremely important for many reasons, some of which I discuss here.

The first reason is that what the students perceive summative assessment to require in order to be successful is a very big influence on the way in which they engage with learning and mathematics. This is a known and tested fact (Entwistle and Entwistle, 1991) that does not apply only to mathematics. In short, the idea is that if a student believes that in order to be successful in an assessment item it is enough to memorise parts of the curriculum, they will engage in this activity rather than studying for conceptual understanding. However in order to reverse the engagement from memorising and procedural learning to conceptual understanding, it is not enough to just change assessment method: there are many other factors to be considered. The familiarity that students have with the new assessment (e.g. do the students know what is necessary to be successful?) and whether the students perceive the new assessment to be doable in the given time-frame (e.g. do the students perceive the new assessment to be impossibly hard or unmanageably time-consuming?) are two factors that have been found to be of impact on the success or failure of the assessment change to encourage conceptual understanding. If students are not familiar enough with the new assessment method or think it is too hard and time consuming they will not engage with it and will revert to procedural learning. The second reason is that summative assessment conveys (or should convey) to the students what we (lecturers) value in mathematics and it will contribute to the perceptions that students have of mathematics itself. If students have only been assessed on replication of seen procedures they would be forgiven if they think that this is all there is of value to mathematics. The last reason I will mention here is that of employability. With an increasing emphasis on employability in higher education, and following from the points above, summative assessment should help students in gaining employability skills which will be very useful in the workplace. Indeed authentic assessment, i.e. assessment that resembles in some way situations which a student is likely to encounter in the workplace, is considered to be of paramount importance at university level.

Therefore it is important to have a map of how we assess our students and what consequences this assessment diet has on students' engagement and conceptual understanding of mathematics.

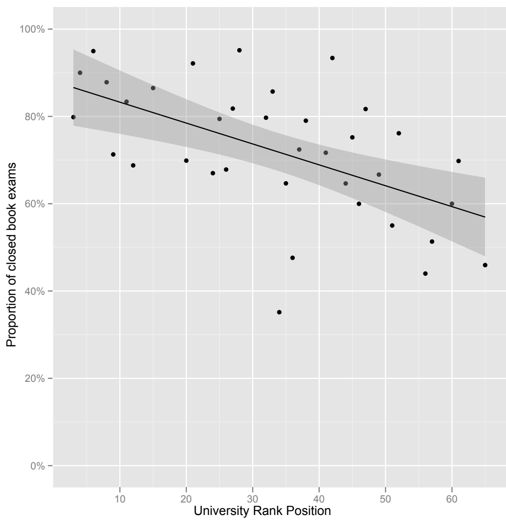


Figure 1. University rank position in a selected league table and proportion within the final degree classification mark assessed by closed-book examination (from Iannone and Simpson, 2012, p. 6).

The mathematics students' assessment diet

In 2011, with Adrian Simpson at Durham University, I carried out a study funded by the Higher Education Academy that investigated the way in which we assess mathematics students in England and Wales (Iannone and Simpson, 2012). We found – not surprisingly – that the most common assessment method in mathematics degrees is the timed closed-book examination, i.e. the timed, unseen exam where students are not allowed to access external material. In fact we found that the higher the university was in a (randomly chosen) league table, the higher was the proportion of credits accrued during the degree by closed-book exams. We found that only four universities in our sample had an average weighted contribution of closed-book exams of less than 50%, while the median contribution across all the universities surveyed was 72% (Figure 1).

There were also differences across years: second-year and third-year modules tended to be examined by closed-book exams more often than first-year modules, probably because in many universities marks accrued in first-year modules do not contribute towards degree classification. Other forms of assessment, such as open-book exams, reports, essays and coursework in general, were present overwhelmingly in the assessment of modules other than mathematics modules (e.g. statistics, history of mathematics and mathematics education, problem solving modules) included in mathematics degrees. Finally most universities had a final-year project. I have collected corresponding data in the winter 2019 (pre-pandemic) and I am in the course of doing the analysis, but preliminary results show that the situation is not much changed. In particular, the differences in closed-book contributions across universities has shrunk relative to 2011, meaning that there has been an increase in uniformity of assessment patterns especially for universities at the top of the chosen league table. There is also, compared to 2011, an increase in the use of computer-aided assessment, but this tends to have the status of coursework and generally bears a small percentage of the total marks for the module.

The closed-book exam

While the closed-book exam is a tried and tested way of assessing mathematics at university, and has been for a long time, it is not without its problems. Studies that have investigated the reasoning skills that are assessed by closed-book exam questions, as implemented in mathematics degree courses, have found that very often it is possible to achieve high marks by employing only procedural understanding (i.e. by being able to remember seen processes and procedures, cf. Darlington, 2014). This appears to be true not only in the UK but also elsewhere, as the study by Bergqvist (2007) shows. It is possible that in this way we are encouraging students to learn procedurally and not to engage with conceptual understanding, as this seems not to be required to achieve good marks. Of course this is not a feature of the closed-book exam per se, but it is an observation on how closed-book exams are currently written.

Oral exams for mathematics

This type of assessment is not very common in the UK while it is widely used in many other European countries. Indeed many Italian mathematics departments have, in the wake of the recent pandemic, cancelled written exams and kept only oral exams as mandatory in mathematics. While the UK higher education system is not designed to have the capacity to hold synoptic oral exams for very large cohorts, this type of assessment could be used for smaller, second-year or third-year cohorts. The advantages of oral exams are that students are more likely to engage conceptually with the mathematics material, as they know that they could be asked to explain anything they say (Iannone, Czichowsky and Ruf, 2020), and to engage in revision strategies that are known to foster conceptual understanding. This type of assessment also allows students to practise communication skills which are otherwise seldom practised in a mathematics degree and are very relevant in the workplace. Finally, given the ease with which it is possible to record and exchange videos of oral exams, these exams can be moderated as easily as exam scripts and samples can be easily shared with external examiners.

Some viable alternatives to closed-book exams

There are some viable alternatives to closed-book exams that have been trialled and are currently in use to assess mathematics at university. I have already mentioned computer-aided assessment which, thanks to systems like STACK or NUMBAS [1], is becoming more and more popular. However this assessment type is not dissimilar from the closed-book exam—especially when administered in exam conditions—in that students have a limited amount of time to take the assessment and have no access to supporting material. It is also not clear how these systems could assess conceptual understanding and proof comprehension, for example, and, to the best of my knowledge, it is yet not possible to assess a proof written by a student by the means of a computer-aided system.

Other summative assessment alternatives include the open-book exam, often used for assessing statistics rather than mathematics modules, the project or report, and oral exams. Projects are now in use in many universities in the final year of study, often replacing two modules, and they are often assessed by a written report and either a presentation and subsequent Q&A session or a mini-viva. Oral exams are starting to attract attention again, after they were dismissed in the eighteenth and nineteenth century from English higher education (Stray, 2001), but their implementation requires preparing the students for this type of assessment via, for example, mock oral exams, as students may not have experience of this assessment and may not realise what is required to be successful. However, up to April 2020, these alternatives to the closed-book exam were seldom used to assess modules in mathematics degrees.

The remote open-book exam

In April 2020 it was clear to all universities in the UK (and most universities in the rest of the word) that it was not going to be possible to assess students by traditional exams taken in exam halls, so alternatives had to be found. The majority of universities decided to use the open-book exam administered remotely with varying types of implementations. Two such implementations are the short timed delivery (e.g. the exam would be open for the usual number of hours plus one for uploading and downloading the paper on the university virtual learning environment) and a long timed delivery (e.g. the exam would be available for as many as 23 hours, during which the student could download it, work through it and

[1]STACK and NUMBAS are two open-source systems for computer aided assessment that are widely used in mathematics departments in the UK. For more details see `https://www.ed.ac.uk/maths/stack/` and `https://www.numbas.org.uk`

upload it again). The two types described above have advantages and drawbacks: while it is less likely that the students will resort to external resources and consult each other in the short time implementation, this significantly disadvantages students in time zones far from the UK and students who do not have stable internet access where they are, or a quiet place in their home to focus on this task. With the long time delivery it is much more likely that students will consult with each other and will access resources through the internet. These are however all open-book exams taken remotely, and the questions they contain cannot be similar to the questions that are usually contained in closed-book exams.

It is likely that the remote open-book exam will be a feature also of the coming academic year. Therefore below I suggest some ideas that could help design such exams and hopefully minimise the possibility of plagiarism and collusion while still providing an appropriate and fair test for mathematics.

- Include *why* and *explain* sub-questions to exam questions and make sure that these explanations bear marks in the mark scheme. In this way the students will realise that to be able to write a coherent explanation of how the solution was found is as important as the solution itself.
- Include some questions that instead of asking for a procedure to do something, give the outcome of the procedure and ask the students to explain that outcome, to explain why an alternative outcome could not be possible, or what an alternative outcome would mean for the problem posed. So for example in the context of a question that asks the students to calculate the Fourier series to the first 4 terms of a given function, we could also ask:

  Would you expect a plot of the function given by the first 4 non-zero terms in your answer to look different to a plot of a function with 100 non-zero terms? Give a short explanation.

  This may be more suitable for applied mathematics papers, but the principle could be used for pure mathematics papers too.
- Ask for a narrative to the solution of a question. If the question is of a problem-solving type, the narrative that the students are asked to add could refer to the stages of Pólya's problem solving (Pólya, 1957). The student could be asked to explain what the problem at hand requires; what is the strategy that they are planning to follow in order to solve the problem; once the strategy has been implemented, how they plan to check that the solution is correct and the strategy has been implemented correctly; and finally look back at the solution of the problem and ask whether if there is anything that could be improved or made more effective. This narrative should come with its relevant mark scheme which should ideally be released to the students before the assessment.
- Ask the students to produce an example or a sequence of examples of mathematical objects. For example, in a first-year algebra module students could be asked to give an example of a subspace of dimension 2 of $\mathbb{R}^3$ containing the vector $v = (3,3,5)$. I have included an example sequence at the end of this article.
- One last suggestion is to introduce a truly unseen question to the exam paper, something that can be solved having followed the module material, but that is not similar to any of the question in the exercise sheets or past exam papers. This question could help finding really high-achieving students.

Questions of the type described above not only help the assessors to ascertain whether the work submitted is the student's own, but above all they have been found to encourage conceptual understanding. Just to take an example, through the process of generation of examples with varying characteristics of the same mathematical object, students can come to understand that mathematical object and its characteristics conceptually and can relate that topic to other topics they already have encountered (Watson and Mason, 2005).

What lessons we learned by being unable to use the most common assessment method?

While writing this piece I attended the very interesting TALMO workshop (`http://talmo.uk`) supported by the London Mathematical Society, the Institute of Mathematics and its Applications, and the Royal Statistical Society. This workshop was very well attended and gathered mathematicians from the UK and beyond. At the end of the workshop I reflected on the challenges, with respect to assessment, that we have faced in this pandemic. The first challenge is related to equality, diversity and inclusion (EDI). Each method of implementing remote open-book exams brought with it its own EDI challenges, the biggest of which is that all remote assessment implies the availability of fast internet connections and, in most cases, the availability of a smartphone. Students who do not have access to these resources are excluded from the learning process. Overcoming these challenges has proven to be very difficult. The second challenge is how to write completely new exam questions, different from those that had been written in the past (as the exam type is different) without having been able to prepare the students for such questions. Lastly, much attention is directed to prevent, and detect, plagiarism and collusion, in order for the assessment to be fair for those students (and I believe they are the majority) who choose not to engage in these practices. The discussion of these and other topics related to assessment will carry on on the TALMO website and elsewhere as the coming academic year seems to be bringing more remote assessment. This time however we have time to prepare.

The one big lesson learned is that there are viable alternatives to the closed-book exam that would be beneficial to use even when we will again be able to have large cohorts of students taking exams in the same lecture hall. The suggestions for open-book exams that I have discussed above, as well as the other many suggestions that I have come across in mathematicians' blogs and through talking to colleagues, need not to be forgotten once the pandemic is under control. We could for example adopt open-book exams of the sort described, or just introduce some of the questions suggested in the standard closed-book exams. This way of writing exam questions will hopefully assess conceptual understanding in a better way than some of the 'traditional' questions do. The most important thing we need to remember is that if we change assessment method, or change the type of questions in exam papers, we need to teach accordingly. If we decide to introduce a question in the assessment that requires students to generate examples of some mathematical objects with given characteristics (see the next page for some examples), we need to make sure that the students have had the opportunity to engage with the process of generating examples during the lectures and seminars. The big problem of the assessment we had to implement this academic year was that teaching and assessment were disconnected: we taught with closed-book assessment in mind and then we had to assess the students with open-book remote exams. We need to make sure that this does not happen again – if we can possibly help it – and for the academic year 2020/2021 we have time to make sure this does not happen.

Acknowledgements

I would like to thank my colleagues in the Department of Mathematical Sciences at Loughborough University for their unlimited patience while I asked them many questions about assessment in the middle of a pandemic! I would also like to thank Dave Sibley who suggested the Fourier series example in this paper.

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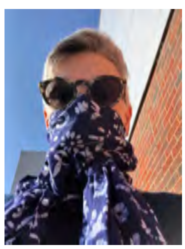

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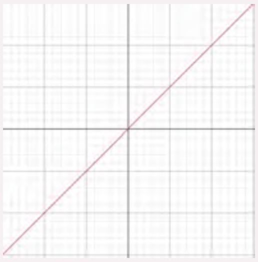
One example of exam questions asking students to generate examples

Questions asking students to generate examples of classes of mathematical objects can be a test of conceptual understanding, pushing the students to think about what are the characteristics of the objects in question. One such sequence of questions encouraging the students to think about injective and surjective functions, could be:

(1) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is injective and surjective.

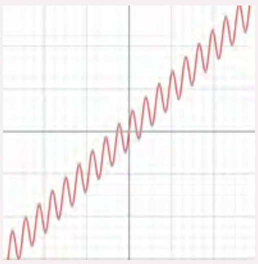


Of course the simplest example is the straight line $y = x$.

(2) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is injective and not surjective.

Again the simplest example is the exponential function $y = e^x$.

(3) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ surjective and periodic but not injective.



This task requires thinking about the combination of a function which is periodic with one which can give surjectivity, the example is $y = x + 8\sin x$.

(4) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is neither surjective nor injective and has a minimum.

This task requires thinking about what simple functions are known that have minimum and are these surjective or injective. Over $\mathbb{R}$ the easy example is a parabola such as $y = x^2 + 11x + 4$.

Of course the sequence of questions can become more and more complex and can require the students to consider not only the definitions of injective and surjective functions, but also other aspects of the definition of functions. Here the students are also asked to think about domain and codomain. Indeed the parabola can be surjective if the codomain is chosen appropriately.

One example of implementation of problem solving stages in an open-book exam

Given the **Definition:** Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. Let $A \subseteq \mathbb{R}$. Then f is preserved on A if and only if $f(A) \subseteq A$. In other words f is preserved on A if and only if $a \in A \Rightarrow f(a) \in A$.

Let A be the closed interval $[-1,0]$. Find an $f : \mathbb{R} \to \mathbb{R}$ such that f has a local minimum in A and is preserved on A. [20 marks]

Questions:

(1) What are the main mathematical objects in the problem and what does the problem require? [5 marks]

The unknown is the function that satisfies the conditions If $A = [-1,0]$ then f preserved in A *and* has a local minimum. The data is that f is a function from $\mathbb{R}$ to $\mathbb{R}$.

(2) Describe the strategy that you will use to solve the problem. [5 marks]

Think of examples! What functions do I know which satisfy the requirements of the problem?

(a) Generic parabola with concavity up
$y = ax^2 + bx + c$ with $a \in N$ and $b, c \in Z$

(b) Preserved in A means that the parabola goes through $(-1,0)$ and $(0,0)$.

(c) Local minimum: The first derivative $f'(x) = 2ax + b = 0$ in A.
Or maybe some other more useful condition - for example the parabola is symmetric with respect to its axis and the minimum point is on such axis - so the minimum point will be something like $(-\frac{1}{2}, m)$ with $m \in [0,-1]$.

(3) Carry out the plan and check the final result against the requirements of the problem. [10 marks]

(a) The parabola goes through $(-1,0)$ This implies that $a - b + c = 0$

(b) The parabola goes through $(0,0)$ This implies that $c = 0$

(a)+(b) implies $a = b$ hence the parabola is $y = ax^2 + ax$ with $a \in N$

(c) Now the point $(-\frac{1}{2}, m)$ belongs to the parabola. This implies $m = \frac{1}{4}a - \frac{1}{2}a$ hence $m = -\frac{1}{4}a$.

(d) Now pick $a = 1$ this implies the minimum point is $(-\frac{1}{2}, -\frac{1}{4})$ and the parabola is $y = x^2 + x$.
Now check:

(i) does the minimum point satisfy the condition that the first derivative is equal to 0?
$f'(x) = 2x + 1$ and $f'(-\frac{1}{2}) = 0$.

(ii) Is it a minimum? Yes - the parabola has concavity up.

Notes of a Numerical Analyst

What We Learned from Galois

NICK TREFETHEN FRS

I am a passionate mathematician, but with my computational perspective, I lie a standard deviation or two away from the LMS mean. I hope you find these columns stimulating, and I would be very glad to hear from you.

Today I'd like to reflect on Évariste Galois, the fiery genius who died in 1832 at the age of 20. As is well known, the quadratic formula was discovered in antiquity and the Renaissance Italians found analogous formulae for roots of polynomials of degrees 3 and 4. It took until the 19th century before Ruffini and Abel proved that there are no such formulae for degrees $n \geq 5$.

A portrait by his brother Alfred

And then came the brilliant Galois. Galois realized that the nonexistence of certain formulae was a consequence of deeper structures, of group symmetries in fields and their extensions that had not been thought about before. The eventual impact (it took a while) was enormous, as profound a paradigm shift in mathematics as you could ask for. Here is how Fernando Corbalán [1] puts it:

"It was the beginning of a true revolution: the end of algebra as understood for centuries (whose main objective was the solution of equations) and the turn to the new problem of the characterization of various structures. This was a step toward modern mathematics."

Or in the words of Michael Harris [2]:

"Galois created a new *point of view*: that what's interesting is no longer the centuries-old goal of finding a root of the equation, but rather to understand the structure of all the roots."

Heady words! It would seem that two things are true: (1) We can't compute roots of polynomials, and (2) There is no need to.

Statement (1) is false. Using standard algorithms implemented in standard software, I can calculate all the roots of a degree 1000 polynomial with random coefficients to 15 digits of accuracy on my laptop in one second. In any but the most artificial sense, roots of polynomials are as computable as π or e.

Statement (2) is false too. After Galois's ideas sank in, did the numerical values of roots of polynomials cease to matter? Of course not. What happened was, rather, that after centuries of trying to develop methods to calculate them, mainstream mathematicians lost interest in the problem. We rewrote our job description. Rather than deciding our field had doubled, we decided it had shifted.

So, for my money, Galois marks not one but two shifts in the history of mathematics. One is the birth of modern algebra. The other is the separation of pure from applied.

I could tell a story about differential equations and Poincaré. . . .

FURTHER READING

[1] Fernando Corbalán, *Galois: Revolución y Matemáticas,* 3rd ed., Nivola Libros y Ediciones, Madrid, 2010.
[2] Michael Harris, *Mathematics without Apologies: Portrait of a Problematic Vocation,* Princeton, 2015.

Nick Trefethen
Trefethen is Professor of Numerical Analysis and head of the Numerical Analysis Group at the University of Oxford.

Association for Women in Mathematics

The Association for Women in Mathematics (AWM) was founded in 1971 when, during a meeting of the 'New Left' Mathematicians Action Group (MAG) at the American Mathematical Society (AMS) Annual Meeting, Joanne Darken stood up and suggested that the women attendees meet afterward to form a caucus of their own. One of these women was the formidable Mary Gray, who became AWM's first president, created its inspiring newsletter, and set its course both ideologically and operationally for decades to come.

AWM presidents would continue to wield great influence and shoulder heavy workloads, with the assistance of an Executive Committee and, eventually, a half-time Executive Director. But AWM's history is one of both strong leaders and legions of volunteers at the grassroots level, creating and running an ever-growing number of programmes to enrich the mathematical lives of women and girls.

During its first decade, AWM sponsored panels at AMS meetings to educate men and women about equality in the profession and to address the difficult employment situation of that era; used its newsletter to help women PhDs find jobs; established a Speakers Bureau; and began to obtain small amounts of external funding. AWM panels have continued to this day at AMS, Mathematical Association of America (MAA), Society of Industrial and Applied Mathematicians (SIAM), and American Association for the Advancement of Science (AAAS) meetings, and at the International Congress of Mathematicians (ICM).

In 1980, AWM founded its annual Emmy Noether Lecture at the AMS-MAA Joint Mathematics Meetings (JMM). It held research symposia in honour of Noether (1982) and Sonya Kovalevsky (1985), and the conference proceedings became AWM's first two books. In conjunction with the 1985 conference, AWM began its Kovalevsky Days for high school girls. Members also formed a Mathematics Education Committee in the early 1980s. By the end of the decade, AWM had funded travel grants to help women attend conferences and meet with research partners.

AWM's first President, Mary Gray (left), and then-President, Jill Mesirov, celebrate AWM's 25th anniversary in 1991

In 1990, AWM established its Alice Schafer Prize for Excellence in Mathematics by an Undergraduate Woman. During the decade, it created two more annual lectures (named for Kovalevsky and Etta Falconer), another prize (this one in mathematics education, named for Louise Hay), two more research symposia (in honor of Julia Robinson and Olga Taussky-Todd), a popular Careers that Count booklet, and annual AWM workshops in which recent PhDs give talks and graduate students present posters.

Together with the European Women in Mathematics (EWM), AWM helped establish the Noether Lecture at ICM (1994) and the Taussky-Todd Lecture at the International Congress on Industrial and Applied Mathematics (ICIAM, 2007).

AWM's second President, Alice Schafer (left), congratulates Schafer Prize winner Jeanne Nielsen at the 1991 MAA MathFest in Orono, Maine.

AWM's creation of new programmes and prizes shows no sign of slowing down. We now sponsor over a dozen different prizes and awards. Since 2011, our biennial research symposia have been held in different parts of the US. Our Student Chapters programme is growing and we recently created prizes for chapter activities. With a 2015–2020 National Science Foundation (NSF) grant, we support 21 research networks, many of which have held specialised research conferences. Our AWM–Springer book series includes 21 volumes so far. We have a Mentor Network and a Policy and Advocacy Committee.

Many women (and men) serve on many AWM committees in order to make all the programmes, publications, prizes, conferences, policy and advocacy happen. My own service to AWM began with its Women in Mathematics Essay Contest in which middle school, high school, and university students interview a mathematician and write an essay about her. Judging essays was my most enjoyable committee service ever! I am now serving on the AWM Executive Committee and co-editing my second book in the AWM–Springer series.

My first contact with AWM was as a new faculty member at my small university in Southern California where AWM's newsletter quickly became a lifeline for me. I remember receiving my first newsletters during the early 1990s and being educated, encouraged, and inspired by the personal stories of Dusa McDuff, Susan Landau and Cora Sadosky, among many others. I have since met many other women mathematicians who also describe AWM's newsletter as their 'lifeline' early in their careers.

AWM's mission is to encourage women and girls to study and to have active careers in the mathematical sciences, and to promote equal opportunity and the equal treatment of women and girls in the mathematical sciences. As we approach our 50th anniversary, we believe AWM has played a central role in increasing the presence, visibility and success of women in the mathematical sciences. But we have far to go in achieving proper representation of BAME women at all educational and professional levels and of all women at the most elite academic institutions.

Left to right: Alexandra Bellow, Roberto Calderón and Mary Ellen Rudin celebrate AWM's 25th Anniversary at the 1991 JMM in San Francisco. Bellow gave the AWM Noether Lecture at the meeting.

To learn more, visit the AWM website: awm-math.org.

Janet Beery
AWM Clerk (Secretary) and Membership Chair
Professor of Mathematics, University of Redlands (USA)

Mathematics News Flash

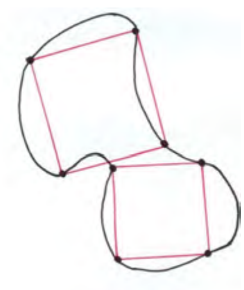
Aditi Kar reports on new path breaking developments in mathematics from the past few months.

In this issue, we report on Green and Lobb's proposed proof of the Rectangular Peg Problem that every smooth Jordan curve contains an inscribed rectangle of arbitrary aspect ratio and on the celebrated proof of the Duffin-Schaeffer conjecture from Number Theory. We describe recent developments in the theory of Non-Commutative Geometry due to Connes and Suijlekom that could hold the key to describing the shape of the universe.

The Rectangular Peg Problem

AUTHORS: Joshua Evan Greene and Andrew Lobb
ACCESS: https://arxiv.org/abs/2005.09193

Green and Lobb recently announced a proof of the Rectangular Peg Conjecture which is an intriguing variation of the Inscribed Square Problem or Toeplitz Conjecture. The Inscribed Square Problem, which was proposed by Toeplitz in 1911, asks whether every continuous simple closed curve in the plane contains an inscribed square. Special cases of the conjecture have been proved but no proof is known of the general case.



An example of the Inscribed Square Problem.
Image credit: Victoria Dixon

As early as 1916, Emch had proved the Problem for curves that were piecewise analytic with only finitely many inflection points and other singularities where the left- and right-sided tangents at the finitely many non-smooth points exist. The common assumption in many successive attempts seems to have been some convexity or smoothness condition. For instance in a more recent paper, Pak proved the conjecture for piecewise linear curves and in 2013, Pettersson, Tverberg and Ostergard used a computer to show that any Jordan curve in the 12-by-12 square grid inscribes a square whose size is at least $\frac{1}{\sqrt{2}}$ times the size of the largest axis parallel square that fits into the interior of the curve. It is worth remarking here that a resolution of the (locally) smooth case does not immediately yield a solution to the original conjecture. Of course a continuous Jordan curve may be approximated by smooth curves. However, there is no guaranteeing that the limit of the inscribed squares isn't degenerate.

While the original version remains intractable, geometers and topologists have been considering variations, the most famous of these is perhaps the Rectangular Peg Conjecture. If instead of squares, one considers rectangles, the problem has a beautiful affirmative solution, first given by Vaughn. Vaughn's argument is topological and so when one specifies that the curve is smooth and moreover, the rectangle has a prescribed aspect ratio, the method fails totally. Green and Lobb overcome these difficulties and employing techniques of Langragian smoothing and surgery, present a proof of the following:

Theorem 1. *For every smooth Jordan curve γ and smooth map $\phi : [0,\infty) \to (0,\pi)$, there exists $r > 0$ such that γ contains the vertices of a rectangle of diameter r whose diagonals meet at an angle of $\phi(r)$.*

The above theorem shows something stronger than a solution to the Rectangular Peg Problem: for every smooth Jordan curve γ and Euclidean rectangle R, there exists a rectangle similar to R which is inscribed in γ.

Duffin-Schaeffer Conjecture

AUTHORS: Dimitris Koukoulopoulos and James Maynard
ACCESS: https://arxiv.org/abs/1907.04593

Koukoupoulos and Maynard's celebrated proof of the Duffin-Schaeffer Conjecture, first announced around a year ago, appears in the latest edition of the *Annals of Mathematics*. The Duffin-Schaeffer conjecture belongs to the world of Metric Diophantine Approximation. Recall the classical Dirichlet's Approximation Theorem which states that for any (positive) real numbers α and N, there exists a rational number p/q such that $q \leq N$ and $|q\alpha - p| < \frac{1}{N}$. This implies immediately that given any irrational number α there exist infinitely many rationals p/q with $\left|\alpha - \frac{p}{q}\right| \leq \frac{\psi(q)}{q}$, where $\psi(q) = 1/q$ for all positive integers q.

Likewise, as one approximates irrational numbers with sequences of rationals, demands can be made on choosing specific sets of denominators for the rationals. One can also ask for more and more efficient approximations by changing the function ψ. Kinchin's work from 1924 established a dichotomy: given a function $\psi : \mathbb{N} \to [0, \infty)$ such that the sequence $(q\psi(q))_{q=1}^{\infty}$ is decreasing, if $\mathcal{K}$ denotes the set of real numbers α from the unit interval for which the above equation has infinitely many solutions p/q then $\mathcal{K}$ has Lebesgue measure 0 or 1, depending on whether the series $\sum \psi(q)$ converges or diverges.

In 1941, Duffin and Schaeffer followed up with their study of Kinchin's theorem and conjectured a criterion on the efficiency function $\psi : \mathbb{N} \to [0, \infty)$ that would ensure that almost all irrational numbers in the unit interval can be approximated by infinitely many rationals p/q with co-prime integers p, q and the specified $\psi : \mathbb{N} \to [0, \infty)$. Erdős and Vaaler, among others, proved various versions of the conjecture, which has now been resolved by Koukoupoulos and Maynard. They showed:

Theorem 2. *Let $\psi : \mathbb{N} \to [0, \infty)$ be a function such that $\sum_{q=1}^{\infty} \frac{\psi(q)\phi(q)}{q} = \infty$. Let $\mathcal{A}$ be the set of $\alpha \in [0, 1]$ such that the inequality $\left|\alpha - \frac{p}{q}\right| \leq \frac{\psi(q)}{q}$ has infinitely many co-prime solutions p, q. Then $\mathcal{A}$ has Lebesgue measure 1.*

The new technique they introduce is combinatorial. They define the notion of GCD graphs: these encode various characteristics of the specified data and their argument rests heavily on the existence of suitable GCD subgraphs and an iterative argument producing GCD subgraphs with progressively nicer properties.

Spectral Truncations in Non-Commutative Geometry

AUTHORS: Alain Connes and Walter van Suijlekom
ACCESS: https://arxiv.org/abs/2004.14115

If one knows the full spectrum of the Dirac Operator relative to a certain associated function algebra, then currently known techniques of Non-Commutative Geometry can be used to reconstruct the full Riemannian spin manifold. This is especially relevant for Physics, where the structure of curved space-time maybe reconstructed from the eigenfrequencies and eigenfunctions for a fermion transiting through that space-time. At present though, one needs to know the full spectrum in order to reconstruct the manifold. In reality, data for the full spectrum is rarely available, as information we possess or record is undoubtedly influenced by the quality of the gadgets and programs we use.

Alain Connes and Walter van Suijlekom have been working to extend the theory of Non-Commutative Geometry to apply in situations where only part of the spectrum is available relative to the associated algebra. The first of their papers focussing on this theme, entitled Spectral Truncations in Non-Commutative Geometry and Operator Systems was published in the journal *Communications in Mathematical Physics* on 14 July 2020. In this paper, they propose the use of operator systems in Non-Commutative Geometry in place of the traditional C^*-algebras. They argue that these operator systems arise naturally in 'spectral truncation'. They study envelopes, dual operator systems and stable equivalence and further define a propagation number for operator systems. The propagation number is shown to be invariant under stable equivalence and hence can be used to compare approximations of the same space. They deeply analyse the example of the truncated circle in the paper.

It's early days but their theory holds the promise of having far-reaching influence in Physics and Astronomy, where in the absence of a description for the full spectrum of frequencies of the radiation reaching us from beyond the earth, we may in future use it to describe the shape of the universe.

Dr Aditi Kar
Aditi is Senior Lecturer of Pure Mathematics in Royal Holloway University. Her research lies in Geometric Group Theory.

Counting on your Fingers

BRAD ASHLEY

Whilst advanced mathematics can be useful and enjoyable, it can often be fun to step back and play with simpler ideas. Here, we will explore alternative ways to count on your fingers and their surprising connections to mathematical base systems.

When ten fingers just aren't enough

Counting on your fingers! One of the first mathematically abstract things we learn. Want to count up to 10 things? See how many fingers that is equal to, and you know how many things you have. Unfortunately, as we get older, we often find ourselves needing to count more than 10 things, so this old method is no good. Of course we could use our toes, but that's just a bit impractical, or we could use technology in some way, but where's the fun in that? Here, I'm going to show you a few fun new ways to count on your fingers and how they relate to different base systems.

By the dozen

On most hands, there are four fingers, excluding the thumb. Each of these fingers are divided into three sections attached by joints. So, in total, on your one hand that's twelve sections. Using your thumb as a counter, we can count to 12 (see Figure 1), which is already more than we could count with the previous 1–10 system. Bring in your other hand, and that's 24. But there's more we can do with this. For every 12 we count on our left hands, we can move our right hand counter (our thumb) up by one, then start again with our left hand. So we count to 12, twelve times. That's 144. These numbers are getting big quickly.

Unintentionally, we're finding ourselves in the world of 'number bases'. What we're essentially doing here is counting in base-12 (also called the dozenal or duodecimal system), where each 'digit' (finger segment) on our left hand is some multiple of 12^0, and each digit on our right hand is some multiple of 12^1. Now, base twelve has two extra digits than base ten, so let us denote them A and B, which are equivalent to 10 and 11 respectively in base-10. So, we count $1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, 10, 11, 12...$ in dozenal. Where each new column in base-10 represents ascending powers of 10, in dozenal, each new column represent powers of 12. Take 324 in base-10. This would be 3 'hundreds' (3×10^2), 2 'tens' (2×10^1), and 4 'ones' (4×10^0). Instead, in base-12, this number would be written as 230, given by $2\times12^2+3\times12^1+0\times12^0$.

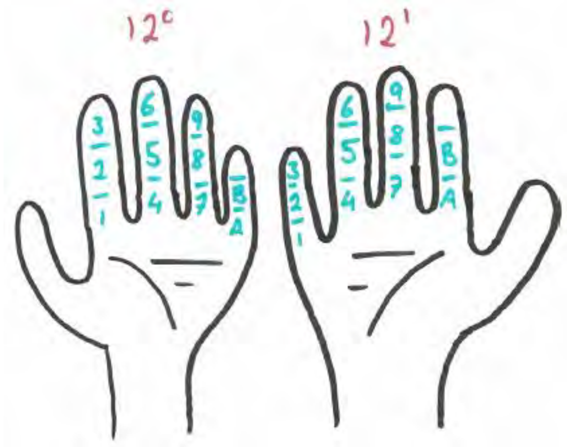


Figure 1. The base-12 digits represented on each hand

So, each digit would look like this on our hands. Note that I am most certainly not an artist, so please excuse the questionable illustrations. This allows us to retrieve the count from our hands. Take the following.

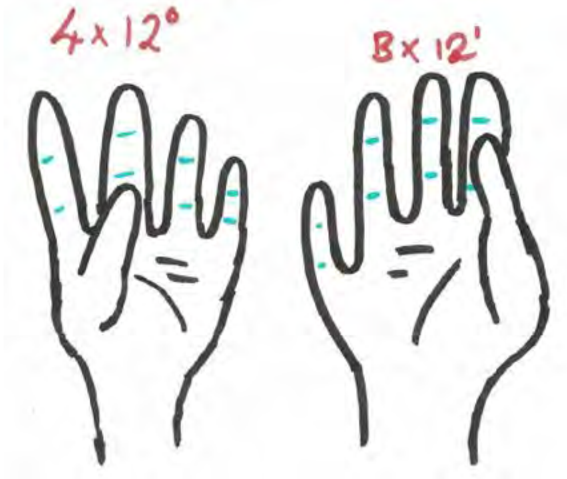


Figure 2. Hands with the 4th segment marked on the left, and the B'th, or the 11th, on the right, representing 136

The left hand has the 2nd finger from left with the bottom segment marked, and the right hand has the far right finger with the middle segment marked. This leads to the following calculation.

$$4 \times 12^0 + B \times 12^1 = 4 \times 1 + 11 \times 12$$
$$= 136.$$

There, we have counted to 136, 13.6 times our previous base-10 limit! This way of counting on your fingers supports an argument made by many who believe we should completely replace our base-10 dependency with base-12. It's a completely valid belief, and would even make fractions easier to understand. Where $\frac{1}{3} = 0.3333...$ and $\frac{1}{4} = 0.25$ in base-10, we have $\frac{1}{3} = 0.4$ and $\frac{1}{4} = 0.3$ in base-12. This is due to 12 having more factors than 10 and such representations are much nicer to work with.

Section 10

We can count much higher, however. Much like before, we're playing with base systems. There exists a way to count up to 31 on one hand, and up to 1023 on two hands, using the magnificent power of base-2, commonly known as binary. Let each finger represent a digit in a binary string (we're including thumbs this time). We can hold each finger in one of two positions; a finger down represents a 0 in our binary string, and a finger up represents a 1. We can then start from the thumb on the right hand, moving left through the fingers, counting in binary.

The issue here, however, is that most of us don't count things using base-2 in our everyday lives, so we need some method to retrieve a base-10 count. Whereas with the base-12 counting system, each hand represented a power of 12, here, each finger represents a power of 2.

Figure 3. The base-2 digits represented on each finger

We can then look at how many of each power of 2 we have, add them up, and get our total. So, if we include both hands, the binary string 1111111111 is the new highest we can count. Converted to base-10, this gives

$$1 \times 2^0 + 1 \times 2^1 + 1 \times 2^2 + 1 \times 2^3 + 1 \times 2^4 + 1 \times 2^5$$
$$+ 1 \times 2^6 + 1 \times 2^7 + 1 \times 2^8 + 1 \times 2^9$$
$$= 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512$$
$$= 1023,$$

an impressively high number. So, using this, we can count up in binary, and retrieve the base-10 total. Take the following hands (and yes, I realise how concerning that phrase would sound out of context).

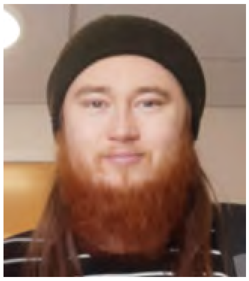
The devil horns are entirely unintentional, I promise. The binary string can be read straight from the fingers giving 0010110011. We can then change the base by the following calculation.

$$1 \times 2^0 + 1 \times 2^1 + 0 \times 2^2 + 0 \times 2^3 + 1 \times 2^4 + 1 \times 2^5$$
$$+ 0 \times 2^6 + 1 \times 2^7 + 0 \times 2^8 + 0 \times 2^9$$
$$= 1 + 2 + 16 + 32 + 128$$
$$= 179.$$

So, with this technique, we can count to much higher numbers than ever before! Theoretically, we could add another position fingers could be in, up, down, and middle. This would give us a base-3 system, and on two hands would allow us to count to 29524. We could keep adding postions to hold our fingers in, however it would eventually be difficult to tell the difference. This might not be the most practical solution to a problem no-one has, but it's interesting at least. You can even choreograph a hand dance to help us remember, which is exactly what "recreational mathemusician" and Youtube personality Vihart did! (Check it out: youtube.com/watch?v=OCYZTg3jahU). So, next time you have the need to count something, but 10 fingers just aren't enough, you'll know exactly what to do!

Acknowledgements

This feature first appeared as a Big Lock-Down Math-Off pitch on *The Aperiodical*: 'a meeting-place for people who already know they like maths and would like to know more', see aperiodical.com/.



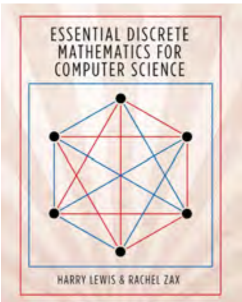
Brad Ashley

Brad is a recent graduate of Sheffield Hallam University and is moving on to an MSc in Mathematics at the University of Sheffield. He has a keen interest in category theory, but he loves maths communication and popular mathematical literature, tweeting mathematical things at `@pogonomaths`. His main interests include craft beer and playing the ukulele.

Essential Discrete Mathematics for Computer Science

by Harry Lewis and Rachel Zax, Princeton University Press, 2019, hardback, 408 pages, £62, US$75, ISBN: 9780691179292

Review by Glenn Hawe



The emphasis on proof, along with the need for abstraction and generalisation, are well recognised as hurdles for new undergraduates in mathematics. However, computer scientists face the same intellectual struggle, for they too have to master these ideas at some point in their studies — or at least understand them enough to pass a module called "discrete mathematics". Often this module will be designed to provide the student with the mathematical knowledge needed by other modules they will study. It is the "fast-paced" discrete mathematics module CS20 at Harvard, a prerequisite for modules on the theory of computation, that the new textbook *Essential Discrete Mathematics for Computer Science* by Harry Lewis and Rachel Zax is based on.

The topics covered can be categorised into: proof (4 chapters), sets (3), logic (4), graphs (6), automata and formal languages (2), order notation and counting (5), discrete probability (4), and modular arithmetic and cryptography (2). Despite the book's title, in the preface the authors suggest four topics that could be omitted by instructors pressed for time. Some topics are less essential than others. One topic surprisingly labelled appropriate for dropping is logic, but presumably only because it would then instead be covered in a separate module (hardware, for example).

The typical length of a chapter is ten to twelve pages, with eight or so pages of explanatory material, a one-page bullet list summary of the key points, and a page or so of exercises. The (more-or-less) consistent length of chapters should make it easier for instructors planning to use the book for teaching. Only three chapters were significantly longer, with about twice as many pages. The explanatory material within chapters could perhaps have benefited from the use of numbered sections (or at least headings), although this was more of an issue in the longer chapters.

Giving an indication of its slant, the book opens with a chapter on the pigeonhole principle, which is really a lesson on mathematical thinking. Things start gentle enough, with the simple example of at least two people from a group of eight necessarily being born on the same day of the week. Just one page later, we arrive at a formulation of the famous principle which requires a basic understanding of functions and sets. After generalising further, the chapter ends with an application to number theory. This first chapter will no doubt be challenging for some students, but the authors help to bridge the gap from secondary school mathematics by asking typical newbie questions such as "why did we choose x_1 and x_2 for the names of the elements of X", no doubt influenced by their experiences in the classroom.

Boosting the accessibility of the book further are the numerous figures positioned in the wide margins (Tufte-style). In my opinion, these visuals were a real strength: the authors have clearly given much thought as to what figures to use to aid understanding of each topic. I especially appreciated the figures in the chapters on proof, a topic many students find difficult. For example, the proof (by mathematical induction) for the sum of positive integers from 1 to n is nicely accompanied by a grid of $n \times (n-1)$ tiles, coloured in a way which instantly helps the reader see, as the ten-year-old Gauss supposedly did, why the sum should be $n(n+1)/2$. Colour is also put to good use in the text in places, for example when keeping track of the different "A"s when making anagrams of the word ANAGRAM in Chapter 22 on Counting. As well as figures, the wide margins are used for occasional short side-notes that would perhaps be too distracting in the main text: truth tables making their first appearance in Wittgenstein's *Tractatus*, for example.

The presentation of material is rigorous, with a significant number of theorems and their proofs. Even the simple concept of an ordered pair is not brushed over in the usual manner. Instead Kuratowski's definition in terms of sets is provided, something which is more typical in textbooks targeted at mathematicians. Proof that this definition implies $(x, y) = (u, v)$ if and only if $x = u$ and $y = v$ is then left for the reader as an exercise at the end of the chapter, typical of the style of exercise in the book. Although the number of exercises is quite low (about ten per chapter) compared to other textbooks in discrete mathematics, it is probably still a sufficient number for students to work on between lessons. Unfortunately, no indication is given of a question's difficultly, and no solutions are provided.

There is already a wide range of textbooks that cover discrete mathematics available, giving an indication of the variety in the modules taught at universities. No single textbook is going to suit everyone. Those (like me) teaching computer scientists their "one and only" mathematics module will likely need something different, to capture other topics the students will need, such as matrices and statistics. Those who would prefer a textbook with a wealth of exercises or worked examples should also look elsewhere. But those teaching computer scientists who take discrete mathematics alongside other mathematics modules such as linear algebra and calculus (as is the case with the CS20 students at Harvard), and who need a book with an emphasis on proof, will likely find this book a very good choice for their students.

Glenn Hawe

Glenn is a Lecturer at Ulster University where he teaches discrete mathematics to computer scientists. While writing this review he attempted to teach his six-year-old daughter how to calculate the sum of integers from 1 to 10 using Gauss's method, but she found it more fun to call Gauss "goose".

What can be computed? A Practical Guide to the Theory of Computation

by John MacCormick, Princeton University Press, 2018, hardback, £70, US$85, ISBN: 9780691170664

Review by Kitty Meeks

As a theoretical computer scientist, I am very conscious of the fact it is not always easy to convince programmers — be they students or professionals — that my subject area has any relevance to their day-to-day work. When I saw the title of this book, I therefore approached it with the unrealistic hope that it might be the holy grail in addressing this issue: one accessible volume that would finally convince the doubters to care about the theory of computation.

Upon opening the book, it became clear that the author actually has a much more achievable goal in mind: the book contains the same content one might expect to see in a theory of computation course in most universities, but instead of presenting the material primarily in terms of abstract mathematical concepts (as is usual for such a course), all of the theory is illustrated with concrete examples of

Python programs. The stated target audience is a class of undergraduate computer science students who have completed at least an introductory programming course, but I think it would be equally appropriate for any competent programmer, for example somebody working in industry or in another scientific discipline, who wants to gain a basic understanding of the core ideas in computation theory. The book aims to be accessible to those with little or no programming experience, and so includes a short introduction to the Python language; this would certainly be a sufficient primer for a reader familiar with other programming languages but new to Python, but a mathematician with no programming experience might feel more comfortable with a more traditional textbook on computation theory.

The book is split into three main sections: Part I covers topics related to computability, Turing Machines and automata; Part II is concerned with complexity theory, leading up to the idea of NP-completeness; Part III provides some historical context for the material, as well as discussing applications to other areas. While the author describes covering this material in a single semester lecture course, this seems to me a fairly ambitious goal (depending, of course, on the students involved and the number of contact hours!) — but there is a helpful guide in the Preface explaining which material may be omitted without causing problems later on.

As with most undergraduate courses in computation theory and complexity, the story of the quest for efficient algorithms ends with concept of NP-completeness; while this notion is undeniably central to the theory of computational complexity theory (and the resolution of the P versus NP problem is undoubtedly a hugely appealing topic for mathematicians), practitioners frequently — and justifiably — complain that the knowledge that the problems they need to solve are NP-complete is of little "practical" use. I always find it a great shame, therefore, that the last thirty years or so of progress in this field, which bring complexity theory much closer to the concerns of real-world programmers, are often omitted from the undergraduate curriculum. It would have been infeasible to include this material in a book that already has such a broad remit, but I would personally love to see a sequel that uses the same approach to present a more advanced course covering topics such as heuristics, randomised algorithms, approximation algorithms, average-case analysis and parameterised algorithms: this would truly be a guide to the issues of computation theory that are relevant to the practitioner!

Putting my unrealistic hopes for the book aside, I think the concept is excellent, and it fills an important gap in the available textbooks on computation theory. It is certainly something I would use in the future when teaching students who are more comfortable with Python code than mathematical proofs. I would also happily recommend it to a physicist colleague who is a far better programmer than me but does not yet understand why I like to talk about computational complexity.

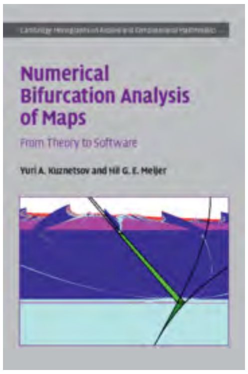
Kitty Meeks

Kitty holds a Royal Society of Edinburgh Personal Research Fellowship in the School of Computing Science at the University of Glasgow, where she works at the interface of combinatorics, algorithms and computational complexity. While she never achieved her childhood dream of becoming a professional cake decorator, she has recently found an alternative outlet for her inner baker by producing themed cakes to represent her research projects.

Numerical Bifurcation Analysis of Maps: From Theory to Software

by Yuri A. Kuznetsov and Hil G.E. Meijer, Cambridge University Press, 2019, hardback, 407 pages. £110, ISBN: 9781108585804

Review by Gavin M Abernethy



This text from Cambridge's *Monographs on Applied and Computational Mathematics* series covers three principal sections: *Theory* (p. 1–216) summarises results from forty years of research to describe and classify the types of bifurcations that can occur in discrete maps as one or two parameters are varied (i.e., codim 1 and codim 2 bifurcations, respectively). *Software* (p. 217–318) describes additional underlying numerical processes operating in the authors' program `MatContM` and provides four short tutorials to get the reader started with using this software. *Applications* (p. 319–388) acts as a series of extended examples, showcasing the kind of bifurcation analysis that can be performed by researchers using the theory and software documented in the preceding sections, and in the case of the generalised Hénon map an impressively-detailed report is provided (including some new results).

This monograph would be a suitable companion to Kuznetsov's well-known textbook [1], providing a reference text that consolidates the work of the authors and others in a compendium of bifurcations in discrete dynamical systems, rather than a teaching or learning aid (it is not an introductory text to bifurcation theory in general, or to maps specifically). Some exercises are provided following each of the software tutorials, but the bulk of the work is more a review in style, and would be challenging to learn from without a significant prior grounding in discrete dynamical systems. Proofs are provided for the normal forms of two-parameter local bifurcations, while numerous references are given to proofs of results in easier cases. Throughout the whole work, there is an abundance of joyfully complex figures depicting various dynamics via phase portrait sketches and bifurcation structures in parameter space.

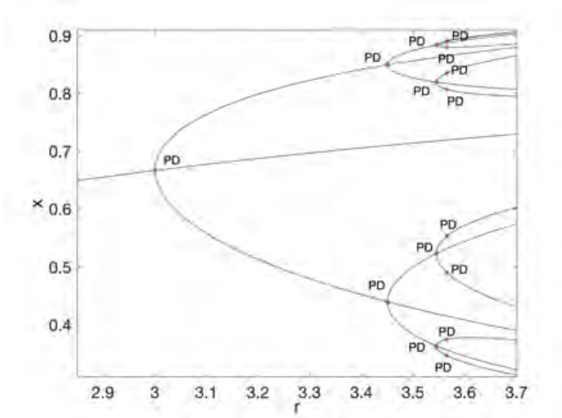


Figure 1. Re-creating the bifurcation structure of the logistic map using `MatContM`. PD denotes that the red stars are specifically period-doubling bifurcations.

The *Software* section forms an excellent cohesive whole with the *Theory* section, given that the program is specialised to apply precisely the ideas discussed in the preceding 200 pages. `MatContM` is a free add-on to `MATLAB` that is simple to download and install, and operated as described with mostly no issues in the *2020a* release. I have worked through the tutorials, which should take you under one hour each (excluding exercises), and I am pleased to report that the walk-throughs provided are very straightforward, and would be accessible to undergraduates even if the main theory of the book is mostly beyond their level. An example of the type of output is shown in Figure 1, where the reviewer has illustrated the classic pitchfork bifurcation structure of the standard logistic map

$$x_{n+1} = rx_n(1 - x_n)$$

up to the point where period-16 cycles would begin to appear. Rather than simply detecting periodic orbits as the parameter space is scanned, this is done by locating, say, an interior fixed-point and tracking it through the parameter space until the conditions are met to detect and classify a bifurcation, at which point one may choose to switch to the new branch of interest. Other functionality demonstrated in the text includes identifying manifolds and calculating the map's spectrum of Lyapunov exponents. Other publicly-available subroutines exist that can do this, but the simplicity of application in this program from such a credible source is certainly an advantage that this reviewer will be employing in the future.

The first half of this book will doubtless be an essential and convenient reference for specialists who already conduct research in this field. Learners seeking an introduction to bifurcation theory in discrete dynamical systems will probably need to look elsewhere first. However, numerical investigations of low-dimensional discrete systems, often of biological motivation, are constantly being published in applied sciences journals, and for this reason the software component and its explanation here could be useful to a much broader base of researchers dipping their toes in this popular area of analysis. The final section does not offer any specific benefit to a general audience, but is inspirational in demonstrating the level of complex detail that a combined analytic and numerical approach using these techniques can achieve.

FURTHER READING

[1] Y.A. Kuznetsov, Elements of applied bifurcation theory, Springer Science & Business Media, 2013.

Gavin M Abernethy

Gavin is a Lecturer in Engineering Mathematics at Sheffield Hallam University, with research interests including complex systems and eco-evolutionary food web modelling. He is currently attempting to finish *Dragon Quest XI*.

Obituaries of Members

Mark H.A. Davis: 1945–2020

Professor Mark Davis, who was elected a member of the London Mathematical Society on 15 November 1996, died on 18 March 2020, aged 74.

Richard Vinter and Harry Zheng write: Mark was a highly regarded scholar in stochastic analysis, stochastic control and financial mathematics who acquired an international reputation through his innovative ideas and powers of exposition. He was an inspiration to many researchers and practitioners in these fields.

Mark was born in Colne, Lancashire in 1945. He was educated at Loughborough Grammar School and Oundle School and went on to Clare College Cambridge, where he obtained a degree in Engineering, specialising in Electrical and Electronic Engineering. He undertook graduate studies in the United States and obtained a PhD at University of California, under the supervision of Previn Varaiya. His PhD research initiated the martingale theory of stochastic control. Its significance was immediately recognised and remains, to this day, one of the main methodological approaches in stochastic control and optimisation.

On returning from the United States, Mark joined the Control Group at Imperial College London. Over the next three decades he frequently visited institutions abroad, including Harvard, the Banach Centre in Warsaw, Stanford, University of Minnesota, University of Oslo, MIT and the University of Vienna, which influenced his ideas and were the basis of fruitful research collaborations.

By the early 1990s Mark had developed a strong interest in applications of stochastic analysis to mathematical finance and had achieved wide recognition in the academic community for his work in this area. Feeling the need for more practical exposure to gain a deeper understanding of this field, in 1995 he accepted the position of Director and Head of Research and Product Development for Mitsubishi Finance (later renamed Tokyo-Mitsubishi International), where he ran a team working on pricing models and risk analysis for financial products.

Mark returned to Imperial College in 2000 to create Imperial's Mathematical Finance group. He launched the College's highly successful MSc in Mathematical Finance, initially designed by Terry Lyons. He served as director for many years, combining these duties with vigorous engagement in his research activities.

Mark was Editor-in-Chief of *Stochastics and Stochastics Reports* for 17 years. During this period he built up the journal into one maintaining the highest standards. He created a wide network of fellow researchers, working in a spirit of cooperation and support and within which participants were often close friends as well as colleagues. He authored six books on stochastic analysis and published over 100 journal articles. His last publication, *Mathematical Finance: A Very Short Introduction*, demonstrates, beyond his academic achievements, his exceptional gifts for synthesis and communication.

Mark had many interests outside his academic activities. With his wife Jessica, a graphic designer and publisher, he was an amateur violinist and, together, they spent many pleasurable hours playing in orchestras and chamber music groups. They had a wide circle of friends, with whom they shared their pleasure in going to concerts, theatre and the cinema. They also enjoyed hiking, swimming, cycling and foreign travel, frequently attached to conferences.

Peter J. Bushell: 1934–2020

Professor Peter J. Bushell, who was elected a member of the London Mathematical Society on 20 March 1969, died on 26 May 2020, aged 86. Professor Bushell was Journal Editor 1994–98.

David Edmunds writes: Born in Iver (near Eton), Peter and his mother moved to Aberdeen in 1940 to be with her family and escape the Blitz. The journey, which Peter remembered vividly, was made by taxi! Peter never really knew his father, an electrical engineer by profession, who tragically died in 1936, but was close to his mother and had happy memories of his childhood in Scotland. In particular he was an enthusiastic member of the mountaineering club.

Having studied at Aberdeen Grammar School (where he overlapped with Bryce McCleod), he began an Engineering degree at Aberdeen but quickly switched to Mathematics after finding the courses on 'Boiler Mechanics' uninspiring. Following this, on the advice of E.M. Wright, he took a BA in Mathematics at Oxford, and in 1961 obtained a DPhil on 'The Mathematics of Quantum Mechanics' under the supervision of Edward Titchmarsh. Peter always gave much credit to David Kendall for emphasising the importance of the functional analytic approach to his thesis topic.

After College Lectureships at Queen's and Magdalen, Peter enjoyed a year as Visiting Lecturer at Dartmouth College in the USA. By this time he had married June, and for their honeymoon they took a train from the East to the West Coast and then drove back.

In 1964 Peter took up an appointment as Lecturer in Mathematics at the relatively new University of Sussex, then known affectionately as 'Oxford by the Sea'. He remained at Sussex for the rest of his career, apart from sabbaticals in Kenya (1966–67), Dartmouth (1971–72, 1984) and Madison, Wisconsin (1971–72). The Kenyan adventure, part of a programme to help the developing world, provided Peter with material for many entertaining anecdotes.

At Sussex, Peter's talent for clear exposition made him a popular lecturer and he taught a wide variety of courses. Early on, he became Warden of the Isle of Thorns, a University of Sussex conference centre, and following this moved into a derelict Georgian townhouse in Brighton, which he transformed into an elegant family home. Peter's organisational skills naturally lent themselves to administrative roles, and he was Chairman of Mathematics from 1980 to 1983 and Dean of the then School of Mathematical Sciences from 1995 to 1998.

Peter's research was characterised by elegance and ingenuity. In his papers on the Cayley-Hilbert metric, he displayed remarkable intuition and insight when establishing the existence of fixed points of certain nonlinear maps and of solutions of nonlinear differential equations. With W. Okrasinski he established an integral inequality, now known as the Bushell-Okrasinski inequality, that is not only of considerable intrinsic interest but has proved to be of great value in diverse applications. He was an absolute master of classical analysis: in our joint paper on p-trigonometric functions and their connection with the p-Laplacian, his encyclopaedic knowledge was indispensable.

Peter had high mathematical standards and above all, was a man of civilised values and integrity. He will be greatly missed by all those who knew him as a mathematician, colleague or friend.

Peter Vámos: 1940–2020

Peter Vámos, who was elected a member of the London Mathematical Society on 18 April 1974, died on 17 March 2020 aged 79.

Rodney Sharp writes: Peter Vámos was born on 6 November 1940, at Székesfehérvár, Hungary. He showed prowess in mathematics at an early age, but he was expelled from secondary school for displaying posters discouraging his fellow students from joining the Communist Youth Party.

He worked for a short while in a chemical factory before being allowed to finish his schooling at a technical college. Although Peter was attracted to chemistry, the pull of mathematics was even stronger, and he managed to secure a place to study mathematics at Szeged University. His undergraduate studies went well, and he wondered about an academic career. However, it was made clear to him that his past anti-communist activities would preclude such a future for him in Hungary. This, together with his increasing distaste for the communist regime in his home country, convinced him that he should leave Hungary.

Peter arrived in Sheffield in the mid-1960s. He had left his entire family (father, mother, sister, and other relatives) in Hungary. He was soon joined by his wife-to-be, Kati, whom he had known in secondary school in Hungary; initially, they did not know anyone at Sheffield, but their outgoing personalities soon changed that. As a couple, Peter and Kati were warmly hospitable and soon Kati was providing wonderful goulashes and other Hungarian dishes at the dinner parties they gave regularly to friends. They quickly built up a wide circle of friends, and not just in the University.

Peter's sharp mind and enthusiasm for mathematics enabled him to build a career in academia in the UK. To begin with, he had postgraduate student status at Sheffield University; he had been attracted to Sheffield by the reputation for excellence in algebra of its Head of Pure Mathematics, Professor Douglas G. Northcott, who nurtured Peter's enthusiasm for things algebraic. It is true that the vast majority of Peter's scholarly publications are in the theory of modules over rings, usually commutative rings. But his mathematical interests were more extensive. For example, inspired by lunchtime conversations with Leon Mirsky and other colleagues, Peter contributed significantly to developments in matroid theory, and one of his papers has the intriguing title 'The missing axiom of matroid theory is lost forever'. Wikipedia has an article about the 'Vámos matroid', which was first described by Peter in an unpublished manuscript in 1968.

Peter was an exciting colleague: his brilliant flashes of mathematical insight would be interspersed with episodes of unexpected logistical complications, but professional life with Peter was fun. He happily broke the unwritten rule that Professor Northcott should not be asked an unexpected question in a seminar. Friends and colleagues helped him in various ways: one colleague lent Peter his car so that Peter could take the driving test, but the outcome was sealed when the car broke down and Peter had to ask the Examiner to push. The next time Peter took the test, he had to ask the Examiner how to put the car (not his, remember) into reverse gear; to his astonishment, this time he passed.

Among the visitors that Peter attracted to the Sheffield Pure Mathematics Department were the famous Hungarian mathematician Paul Erdős and Peter's cousin, Ernő Rubik, inventor of the famous Rubik cube.

In 1983, Peter left Sheffield to become Professor and Head of the Department of Pure Mathematics at the University of Exeter. He loved mathematics and the teaching of it. At Exeter he was given the nickname 'Uncle Bob' by some students, on account of his habit of saying 'and Bob is your uncle' at the end of a proof.

On the other hand, Peter regarded the running of a department, and the associated bureaucracy, as an annoying, but unavoidable, price to pay for a career as an academic mathematician. He loved Apple products, even from their infancy, and he crossed swords (or, more accurately, computers) with Exeter University's IT Department when he bought an Apple Lisa (a precursor of the Apple Mac) for his department, in contravention of the IT Department's hardware strategy.

Peter played a full part in the mathematical life of the UK, and was a member of the Council of the London Mathematical Society for a number of years. He and Kati had the opportunity to design their own house in Exeter; they enjoyed that, and Peter relished his rôle as Project Manager for the construction.

Peter was a man of infectious and inspiring enthusiasms, not only about bits of algebra he had just invented or about new technology he had just invested in, but also about practical developments such as the campervan he and Kati, who made beautiful pots, used to transport her pots around the country to various exhibitions, including 'Pots in the pens', an exhibition of pottery in the cattle market in Penrith. Kati died in 2013; Peter is survived by two sons, Tom and Nick, and four grandchildren.

Death Notices

We regret to announce the following deaths:

- Professor Fred H.J. Cornish, formerly of York University, who died on 15 May 2020.
- Dr Patrick Dolan, formerly of Imperial College London, who died on 29 June 2020.
- Professor Philip P.G. Dyke, University of Plymouth, who died on 4 June 2020.
- Dr Alan R. Pears, formerly of King's College London, who died on 4 July 2020.
- Dr Arthur D. Sands, formerly of Dundee University, who died on 18 May 2020.
- Professor Nigel O. Weiss, FRS, formerly of Cambridge University, who died on 24 June 2020.

Functor Categories for Groups Meeting on Linear Groups

Location: Online
Date: 4 September 2020
Website: tinyurl.com/y4mnkdjx

This afternoon meeting on linear groups will consists of three talks given by Jack Button (Cambridge), Agnieszka Bier (Silesian University of Technology) and Sandro Mattarei (Lincoln). The meeting will take place via Microsoft Teams. To register, email athillaisundaram@lincoln.ac.uk. Supported by an LMS Scheme 3 Joint Research Groups in the UK grant.

Categorifications in Representation Theory

Location: Online
Date: 15–17 September 2020
Website: tinyurl.com/yctrzv38

Categorifications are category-theoretic analogues of mathematical phenomena. This process can engender mathematical progress by allowing problems to be attacked with new techniques. This conference aims to bring together young researchers interested in a variety of categorifications. We encourage talks on a range of subjects.

Heidelberg Laureate Forum 2020

Location: Online
Date: 21–25 September 2020
Website: tinyurl.com/y5ldmw5e

This Virtual Forumwill offer digital programmes where participants can expect panel discussions revolving around eHealth and science communication, plus a platform that enables the laureates and young researchers to exchange ideas and hold interactive discussions.

EMS 30th Anniversary Meeting

Location: ICMS, Edinburgh
Date: 29 October 2020
Website: euro-math-soc.eu

In 2020, the European Mathematical Society celebrates 30 years of activity in support of the mathematical sciences across Europe. To mark the occasion, the EMS, with support from the LMS and Edinburgh Mathematical Society, will hold a day of mathematical talks, reminiscences and discussions, each chaired by a past president of the EMS.

Probability in the North East Lectures

Location: Online
Date: 10, 11, 14 September 2020
Website: tinyurl.com/yyomreh7

Probability in the North East (PiNE) is a collaboration among researchers interested in probability theory and its applications. This is a live-streamed lecture series on Semimartingale Methods for Markov chains, interacting particle systems and random growth models. Supported by an LMS Scheme 3 Joint Research Groups grant.

Imaging Meets Computational PDEs

Location: Online
Date: 17 September 2020
Website: tinyurl.com/imagingPDEs

Imaging and PDE-based numerical modelling play an important role in numerous medical diagnosis tools, such as electrical impedance tomography. This event will give researchers from these communities an opportunity to gather, exchange research ideas and design joint research plans. Registration is free of charge and closes on 14 September 2020.

LMS–IMA Joint Meeting 2020

Location: Online
Date: 1–2 October 2020
Website: tinyurl.com/y2pvq3ur

The London Mathematical Society and the Institute of Mathematics and its Applications are planning to hold their Annual Joint Meeting online via Zoom. This year's topic is Topological Methods in Data Science. Further details will be available soon.

New Challenges in Operator Semigroups

Location: St John's College, Oxford
Date: 12-16 July 2021
Website: tinyurl.com/vga7fd2

This conference will celebrate Charles Batty's ongoing outstanding contributions to the theory of operator semigroups. The meeting will focus on the mathematical theory of operator semigroups and their applications to linear evolution equations and connected fields. A limited amount of funding is available to support UK based PhD students; contact David Seifert (david.seifert@ncl.ac.uk) for details.

Covid-19: Owing to the coronavirus pandemic, many events may be cancelled, postponed or moved online. Members are advised to check event details with organisers.

Society Meetings and Events

September 2020

10-11 Prospects in Mathematics Meeting, University of Bath

October 2020

1-2 Topological Methods in Data Science, LMS-IMA Joint Meeting

26-27 IMA–LMS–BSHM Meeting: Black Heroes of Mathematics

November 2020

19 Computer Science Colloquium, London

20 Society Meeting and AGM, London

September 2021

1-3 Scaling Limits: From Statistical Mechanics to Manifolds, Cambridge

6-10 Northern Regional Meeting, Conference in Celebration of the 60th Birthday of Bill Crawley-Boevey, University of Manchester

Calendar of Events

This calendar lists Society meetings and other mathematical events. Further information may be obtained from the appropriate LMS Newsletter whose number is given in brackets. A fuller list is given on the Society's website (www.lms.ac.uk/content/calendar). Please send updates and corrections to calendar@lms.ac.uk.

September 2020

4 Functor Categories for Groups Meeting on Linear Groups, Virtually at University of Lincoln (490)

10-11 Heilbronn Annual Conference 2020, University of Bristol (490)

10-14 PiNE Lectures on Semimartingale Methods for Markov Chains, Interacting Particle Systems and Random Growth Models (490)

15-17 Categorifications in Representation Theory (490)

16-17 IMA Induction Course for New Lecturers in the Mathematical Sciences, Isaac Newton Institute, Cambridge (489)

17 Imaging Meets Computational PDEs (490)

21-25 Heidelberg Laureate Forum 2020 (490)

October 2020

1-2 Topological Methods in Data Science, LMS-IMA Joint Meeting (490)

7-9 22nd Galway Topology Colloquium, University of Portsmouth (488)

12-16 New Challenges in Operator Semigroup, St John's College, Oxford

12-19 14th International Congress on Mathematical Education Shanghai, China

14-16 IMA Modelling in Industrial Maintenance and Reliability Conference, Nottingham (486)

20-26 8th European Congress of Mathematics, Slovenia

26-27 IMA-LMS-BSHM Meeting on Research by Black Mathematicians (490)

29 European Mathematical Society 30th Anniversary Meeting, ICMS, Edinburgh (490)

January 2021

20-22 Fry Inaugural Series: Challenges and Recent Advances in Mathematical Physics, Heilbronn Institute, Bristol (490)

July 2021

12-16 New Challenges in Operator Semigroups, St John's College, Oxford (490)

August 2021

16-20 IWOTA, Lancaster University (481)

September 2021

19-24 8th Heidelberg Laureate Forum, Heidelberg, Germany

21-23 Conference in Honour of Sir Michael Atiyah, Isaac Newton Institute, Cambridge (487)

July 2022

24-26 7th IMA Conference on Numerical Linear Algebra and Optimization, Birmingham (487)

AMERICAN MATHEMATICAL SOCIETY

CONVERSATIONAL PROBLEM SOLVING

Richard P. Stanley, University of Miami

Presents a dialogue between a professor and eight students in a summer problem-solving camp and allows for a conversational approach to the problems as well as some mathematical humour and a few non-mathematical digressions. The problems have been selected for their entertainment value, elegance, trickiness, and unexpectedness.

Aug 2020 274pp 9781470456351 Hardback £49.95

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Nikolai M. Adrianov, Lomonosov Moscow State University, Fedor Pakovich, Ben Gurion University of the Negev & Alexander K. Zvonkin, University of Bordeaux

The major part of the book is quite elementary and is easily accessible to an undergraduate student. The less elementary parts, such as Galois theory or group representations and their characters, would need a more profound knowledge of mathematics. The reader may either take the basic facts of these theories for granted or use our book as a motivation and a first approach to these subjects.

Mathematical Surveys and Monographs, Vol 249

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GEOMETRIC SET THEORY

Paul B. Larson, Miami University & Jindrich Zapletal, University of Florida and Czech Academy of Sciences

Introduces a new research direction in set theory: the study of models of set theory with respect to their extensional overlap or disagreement. The topic is nearly inexhaustible in its variety, and many directions invite further investigation.

Mathematical Surveys and Monographs, Vol 248

Aug 2020 340pp 9781470454623 Paperback £139.00

ORGANIZED COLLAPSE

An Introduction to Discrete Morse Theory

Dmitry N. Kozlov, University of Bremen and Okinawa Institute of Science and Technology

Provides a gentle introduction into discrete Morse theory. The presentation is broad, ranging from abstract topics, such as formulation of the entire theory using poset maps with small fibers, to heavily computational aspects, providing, for example, a specific algorithm of finding an explicit homology basis starting from an acyclic matching.

Graduate Studies in Mathematics, Vol 207

Sep 2020 312pp 9781470457013 Hardback £88.50

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