LONDON MATHEMATICAL SOCIETY
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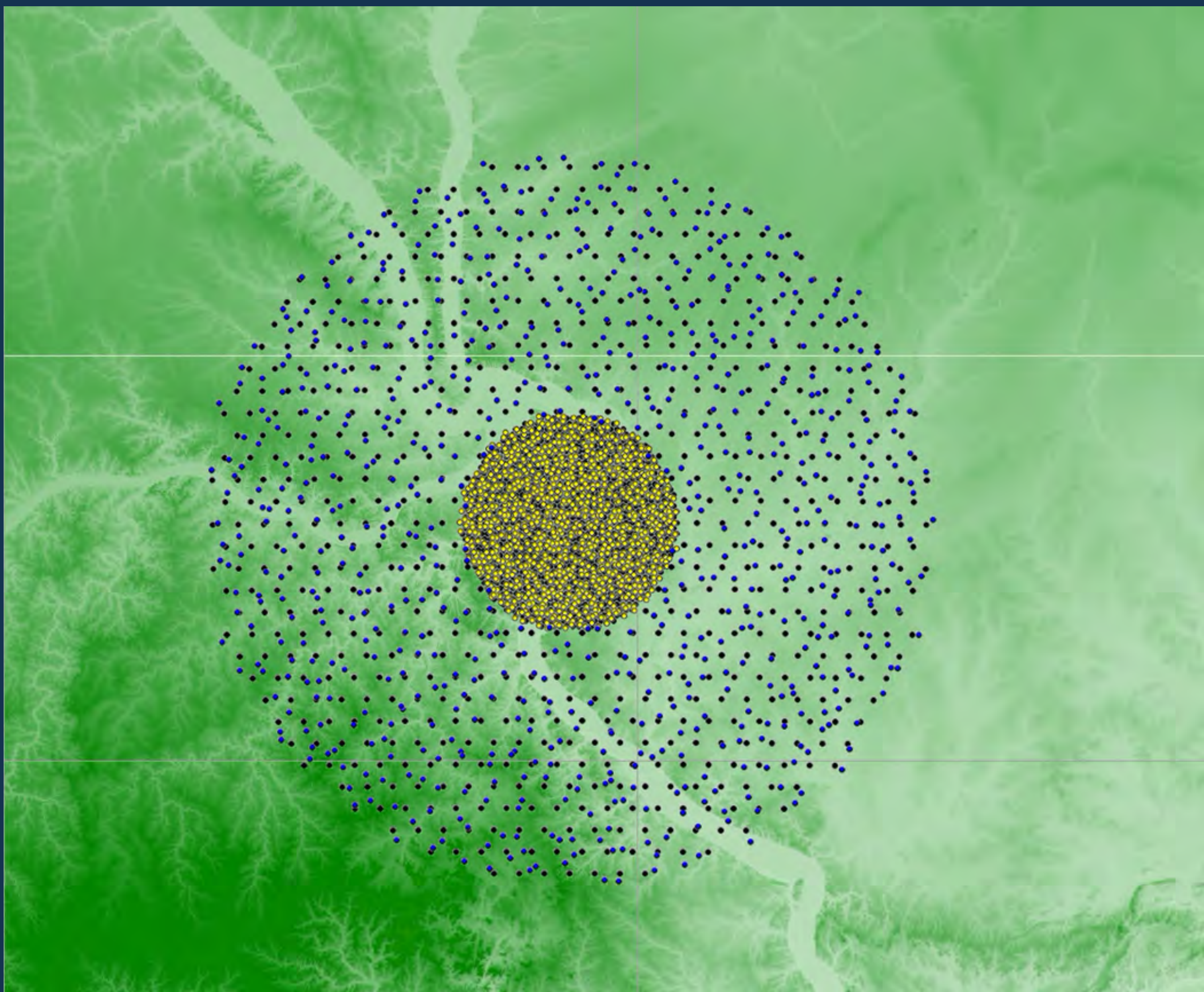
NEWSLETTER

Issue: 493 - March 2021



MATHEMATICS OF FLOATING-POINT ARITHMETIC

RANDOM LATTICES IN THE WILD

MARRIAGES, COUPLES, MATHS CAREERS

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COVER IMAGE

One step in a Monte-Carlo simulation using Visualyse Professional software. See page 24 for further details.

Do you have an image of mathematical interest that may be included on the front cover of a future issue? Email images@lms.ac.uk for details.

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NEWSLETTER WEBSITE

The Newsletter is freely available electronically at lms.ac.uk/publications/lms-newsletter.

MEMBERSHIP

Joining the LMS is a straightforward process. For membership details see lms.ac.uk/membership.

SUBMISSIONS

The Newsletter welcomes submissions of feature content, including mathematical articles, career related articles, and microtheses from members and non-members. Submission guidelines and LaTeX templates can be found at lms.ac.uk/publications/submit-to-the-lms-newsletter.

Feature content should be submitted to the editor-in-chief at newsletter.editor@lms.ac.uk.

News items should be sent to newsletter@lms.ac.uk.

Notices of events should be prepared using the template at lms.ac.uk/publications/lms-newsletter and sent to calendar@lms.ac.uk.

CONTENTS

Annual Elections to LMS Council

The LMS Nominating Committee is responsible for proposing slates of candidates for vacancies on Council and vacancies on Nominating Committee itself. The Nominating Committee welcomes suggestions from the membership.

Anyone who wishes to suggest someone for a position as an Officer of the Society or as a Member-at-Large of Council (now or in the future) is invited to send their suggestions to Professor Kenneth Falconer, the current Chair of Nominating Committee (nominations@lms.ac.uk). Please provide the name and institution (if applicable) of the suggested nominee, their mathematical specialism(s), and a brief statement to explain what they could bring to Council/Nominating Committee.

It is to the benefit of the Society that Council is balanced and represents the full breadth of the mathematics community; to this end, Nominating Committee aims for a balance in gender, subject area and geographical location in its list of prospective nominees.

Nominations should be received by 16 April 2021 in order to be considered by the Nominating Committee.

In addition to the above, members may make direct nominations for election to Council or Nominating Committee. Direct nominations must be sent to the Executive Secretary's office (nominations@lms.ac.uk) before noon on 1 September 2021. For details on making a direct nomination, see lms.ac.uk/about/council/lms-elections.

The slate as proposed by Nominating Committee, together with any direct nominations received up to that time, will be posted on the LMS website in early August.

New Editor-in-Chief sought for LMS *Newsletter*

The LMS seeks someone with broad mathematical interests, who is passionate about communicating mathematics and supporting the Society, to become Editor-in Chief of the LMS *Newsletter*, starting in May 2021.

The LMS *Newsletter* has several purposes. It aims to provide a sense of identity, community, and connection for the Society's members. It is a channel for communicating the power, beauty and value of mathematics and mathematical research by disseminating new mathematical ideas and information. It also seeks to make transparent the Society and its workings.

The main duties of the Editor-in-Chief, who is ultimately responsible to the Society's Council, are:

- overseeing the commissioning of content for the mathematical features section
- overseeing the sourcing of material for other sections
- signing off on the content and layout for each bi-monthly issue
- chairing, leading and coordinating support from the Newsletter Editorial Board.

In addition to the Newsletter Editorial Board, the Editor-in-Chief also works closely with, and is further supported by, members of the Society's staff.

This role is unremunerated, although reasonable expenses will be paid. The role requires a time commitment of approximately one day a week on average, year round.

Back issues of the *Newsletter* are available on the Society's website at bit.ly/36h6iUL.

For further information about this role, including attributes sought and how to apply, please contact the Executive Secretary Caroline Wallace at caroline.wallace@lms.ac.uk. Applications will close on 1 April 2021.

Plan S Update

In September 2018 several research funders formed an international consortium, cOAlition S, to launch the Plan S initiative. The intention of Plan S is to require that scientific publications arising from research supported by consortium members must be published immediately open access in compliant journals or platforms. The timeline for implementation of these requirements varies by funder (see bit.ly/2MuZ1d5), but in general only grants awarded by members of cOAlition S beginning after 1 January 2021 will include these requirements. It should be noted that UKRI will not be publishing their open access policy until Spring 2021.

For those subject to Plan S requirements the three main ways compliance can be achieved are:

- Publication of the final typeset version of record (VoR) with a CC BY licence in an open access journal or platform registered in the Directory of Open Access Journals (DOAJ).
- Open access publication of the VoR with a CC BY licence in a hybrid journal covered by a transformative arrangement approved by cOAlition S (only until the end of 2024).
- Publication of the VoR with restricted access in a subscription journal but at the same time depositing the Author's Accepted Manuscript (AAM) with a CC BY licence in an institutional or subject repository, with immediate open access. The AAM is the final author-created version of the manuscript as accepted for publication by the journal, including any changes made during peer review. The use of a CC BY licence is to permit others to distribute, remix, adapt, and build upon an article, even commercially, as long as they credit the original author(s).

If the costs of publication of the VoR open access are supported through an article processing charge (APC) then funders will only pay this if the journal is fully open access or denoted a 'transformative journal'. Alternatively, authors can comply by making their VoR open access in journals covered by 'transformative agreements' that have been negotiated between their institution and a publisher and cover the cost of open access. A list of compliant venues can be found on the cOAlition S website.

Whilst most mathematics journals impose no restriction on posting early versions of a paper to a repository such as arXiv, the standard licences signed with publishers typically impose conditions on depositing the AAM, including an embargo or use of a non-commercial licence.

All of the Society's hybrid journals offer transformative agreements to a growing number of authors, negotiated between our publishing partners and academic institutions. This has resulted in growth in the proportion of papers where the version of record is available open access in all of the journals managed by the Society. For example, close to one fifth of the articles published online in 2020 in the *Bulletin, Journal, Proceedings* and *Journal of Topology* were open access.

It should be noted that the amount of content supported by APCs is taken into account in setting subscription prices (see bit.ly/2MrLbrX).

The Society also publishes a fully open access journal, the *Transactions*. Through the appointment of a new Editorial Board (as described in the January LMS Newsletter) the Society intends to offer authors a high-quality publication venue for those whose funders support open access publishing.

The publications landscape is changing rapidly. The Society is looking to adapt to the changing requirements of authors and funders in a way that allows its publication activities to continue on a sustainable basis while enabling the Society to return all surplus income to support mathematicians and mathematics research.

John Hunton
LMS Publications Secretary

De Morgan Donations and Other Gifts

Launched in 2019, De Morgan Donations are gifts to the Society of £1,865 or more. Named after the Society's first President in 1865, Augustus De Morgan, these donations have started to play a significant role in helping the Society in its support of the mathematical community.

Our funding of mathematical research, typically close to £700,000 annually, is one particularly important way in which the Society meets its objectives to promote, disseminate and advance mathematics. In 2020, the Society drew on its reserves to invest over £120,000 in a second round of LMS Early Career Fellowships, to mitigate some of the impact of the covid-19 pandemic on the academic careers of Early Career Researchers. This, together with a most generous donation from the Heilbronn Institute for Mathematical Research, enabled us to support 22 more Early Career Fellows than we normally would have done.

Going forward, the Society continues to explore new ways in which it can support members of the mathematical community who have been significantly affected by the pandemic. For example, with a generous gift from the Liber Stiftung we are providing higher levels of support to mathematicians whose work has been impacted by caring responsibilities, and with one from Dr Tony Hill we are establishing the new Levelling Up Scheme to support pupils from under-represented backgrounds studying A-level Mathematics.

The De Morgan Donations we have received so far have made a highly significant contribution to these initiatives and we are most grateful to the donors for their generosity. We are also extremely grateful for the many other generous gifts and bequests we receive in support of our charitable work.

De Morgan Donations can be made either via lms.ac.uk/content/donations#Donate or by contacting me (president@lms.ac.uk) for further details.

The Society is also grateful to its many volunteers for the time and energy they contribute to our various activities, especially at a time when many are so busy.

Jon Keating FRS
LMS President

Atiyah UK–Lebanon Fellowship 2021–22

Maciej Dunajski (left) with Michael Atiyah

The Atiyah UK-Lebanon Fellowships were set up in 2019 as a lasting memorial to Sir Michael Atiyah (1929-2019) in the form of a two-way visiting programme for mathematicians between the UK and Lebanon, where Sir Michael had strong ties.

The LMS is delighted to announce that the Atiyah Fellowship for the academic year 2021-22 has been awarded to Professor Maciej Dunajski of Cambridge University.

Professor Dunajski's research is in mathematical physics, in particular the interplay between differential geometry, integrable systems, general relativity and twistor theory. His main research achievement is the solution (joint with Robert Bryant and Mike Eastwood) of the metrisability problem posed more than 120 years ago by Roger Liouville. He plans to make one or two short visits to the Centre for Advanced Mathematical Sciences at the American University of Beirut (AUB), as well as Notre-Dame University-Louaize, where he will lecture on twistor theory.

The photograph shows Professor Dunajski with Michael Atiyah. It was taken in Trinity College during what may perhaps have been Sir Michael's last visit to Cambridge in the summer of 2017.

The 2020-21 Fellows were Professor Mark Wildon (Royal Holloway, University of London) and Professor Ahmad Sabra (AUB). Professor Wildon will be visiting AUB and Professor Sabra will be visiting the University of Sussex as soon as the covid-19 situation permits.

For further information about the Fellowships and information on how to apply, see bit.ly/39CDKra. It expected that applications for Fellowships to be held the academic year 2022-23 will open in September 2021.

Privy Council Approval of Amended LMS Standing Orders

At its meeting on 16 December 2020, the Privy Council agreed to proposed amendments to the London Mathematical Society's Royal Charter and Statutes which, with the By-Laws, are known collectively as the Standing Orders.

The Society's Council set up the Standing Orders Review Group in 2014. It was felt that the original wording of the documents, agreed in 1965 when the Charter was granted, was no longer appropriate and in places was out of date, not least with the way the LMS now operated. Linguistic changes were required to remove sexist and ageist wording, and there were several relatively minor procedural changes. The Standing Orders Review Group had been willing to be bold, but increasingly realised in their deliberations that in general the documents had been extremely well drafted in 1965. The proposed changes received close scrutiny by Council on several occasions and were subject to a consultation with the whole LMS membership in 2018-19.

LMS members voted overwhelmingly to approve the proposed changes to the Standing Orders at the 2019 Annual General Meeting. Following this, the changes to the Royal Charter and Statutes were formally submitted to the Privy Council in February 2020 (the

By-Law changes do not require Privy Council approval). Unfortunately because of the covid-19 pandemic, the Privy Council did not consider such proposals from any organisations at its meetings for some months. Liaison between the Society and the Privy Council Office continued throughout 2020 and notification was received in early December that the Privy Council advisors raised no objections to the proposals. Her Majesty the Queen approved the two Orders in Council at the December Privy Council meeting in Windsor. The revised Standing Orders are now posted on the Society's website and are being implemented.

The Society would like to thank all those members who have been involved in the review of the Standing Orders, including especially those who sat as members of the Standing Orders Review Group (Caroline Series, Simon Tavaré, Terry Lyons, Alexandre Borovik, June Barrow-Green, John Toland and Fiona Nixon) — and in particular Stephen Huggett, who as General Secretary ensured that the process was carried out efficiently and in a robust manner, taking into account all viewpoints and 'future proofing' the Standing Orders to take the Society forward.

Further information on the Standing Orders Review, including the rationale for the changes, can be found at lms.ac.uk/about/lms-standing-orders-review.

Forthcoming LMS Events

The following events will take place in forthcoming months:

LMS Women in Mathematics Day: 24 March, online (tinyurl.com/y5uwol5f)

Society Meeting at the BMC-BAMC: 8 April, online (tinyurl.com/yarpowdo)

LMS Spitalfields History of Mathematics Meeting: 14 May, online (tinyurl.com/y3kpv6ye)

Midlands Regional Meeting and Workshop: 2–4 June, Lincoln (tinyurl.com/y5vtaytx)

Summer General Society Meeting: 2 July, London

Invited Lecture Series 2021: August, online (tinyurl.com/y2gyehr4)

Northern Regional Meeting: 6–10 September, University of Manchester (tinyurl.com/yamy8uvq)

A full listing of upcoming LMS events can be found on page 71.

OTHER NEWS

Suffrage Science Awards 2020

Jewellery designed by students at Central Saint Martins–UAL

For the participants, one of the most uplifting moments of 2020 was the virtual celebration of the 2020 Suffrage Science Mathematics and Computing Awards. The Suffrage Science Scheme celebrates women in science for their scientific achievements and for their ability to inspire others. It aspires to encourage more women to enter scientific subjects, and to stay. It was founded in 2011 with awards to an initial cohort of 11 women in Life Sciences, and in 2013 the Scheme expanded with a new cohort of women in Engineering and Physical Sciences. In 2016 the Scheme expanded again to include a Mathematics and Computing cohort.

The awards are beautiful pieces of jewellery designed by students at the art and design college Central Saint Martins–UAL, inspired by the jewellery worn by the suffragettes and by conversations with scientists. The jewellery designed for mathematics and computing awards are a simple circular silver bracelet with a pearl bead that traces out the unit circle, with the equation $e^{i\pi}+1=0$ inscribed on the inside, and a brooch made of gold punched tape encoding suffragette messages. Every two years, awardees are asked to nominate their successors — and the jewellery pieces are passed on in a ceremony.

The 2020 ceremony was held online, and featured video clips from the nominator–nominee award pairs, as well as a panel discussion on diversity in mathematics, statistics and computing. It was inspiring to hear the reasons why each woman had chosen her successor, as well as the awardees' visions for equality in the future of their disciplines. There was also a lively panel discussion, with discussion around concerns about the disproportionate impact of covid-19 on women, as well as the relevance of the Black Lives Matter movement to mathematics, alongside positive stories.

From a personal point of view, this has been a special award, in that it aims to develop a supportive network of women. Gwyneth Stallard was one of the inaugural mathematics awardees and loved wearing the bracelet — she was delighted to pass it onto Eugenie Hunsicker who has done an amazing job as her successor as Chair of the LMS Women in Mathematics Committee. In turn, Eugenie has passed it on to IMA and LMS Women and Diversity Committee member Sara Lombardo. At the same time, jewellery passed this year from Nina Snaith to Apala Mjumdar and from Vicky Neale to Anne-Marie Imafidon, as well as along lines of statisticians and computer scientists. We three authors are currently working together with EPSRC on ways of improving equality and diversity in relation to funding, with Gwyneth and Sara acting as Diversity Champions for the Mathematical Sciences Strategic Advisory Team.

More details and photos of the awardees and the jewellery can be found at suffragescience.org.

Gwyneth Stallard
Eugenie Hunsicker
Sara Lombardo

PROMYS Europe Connect 2021

PROMYS Europe Connect, a challenging four-week mathematics summer programme online, based at the University of Oxford, UK, is seeking applications from pre-university students from across Europe (including all countries adjacent to the Mediterranean) who show unusual readiness to think deeply about mathematics.

PROMYS Europe Connect is designed to encourage mathematically ambitious students who are at least 16 years old to explore the creative world of mathematics. Participants tackle fundamental mathematical questions within a richly stimulating and supportive online community of fellow first-year students, returning students, undergraduate counsellors, research mentors, faculty, and visiting mathematicians.

PROMYS Europe is a partnership of Wadham College and the Mathematical Institute at the University of Oxford, the Clay Mathematics Institute, and PROMYS (Program in Mathematics for Young Scientists, founded in Boston in 1989).

The programme is dedicated to the principle that no one should be unable to attend for financial reasons. Most of the cost is covered by the partnership and by generous donations from supporters. In addition, full and partial financial aid is available, for those who need it.

The application form and application problem set are available on the PROMYS Europe website promys-europe.org. The closing date for applications is 14 March 2021, and students will need to allow time before the deadline to tackle the application problems. PROMYS Europe Connect will run online from 12 July to 6 August 2021.

Sir Michael Atiyah Conference

The conference on the *Unity of Mathematics* in honour of Sir Michael Atiyah, postponed from 2020, has been rescheduled to take place in the Isaac Newton Institute from 21–23 September 2021. It is the intention of the organisers that this meeting will be either a face-to-face or a mixed face-to-face and online meeting, depending on circumstances nearer the time. It is anticipated that registration will open in late spring. For details and updates please see the conference website newton.ac.uk/atiyah.

Assuming the conference can take place as planned, some accommodation will be available in Murray Edwards College, and limited funding should be available, especially for early career researchers. For advance expressions of interest and notification when registration opens, email Kathryn de Ridder at office@newton.ac.uk, using the subject line 'Sir Michael Atiyah Conference'.

A number of grants funded by the National Science Foundation are available to cover full expenses for PhD students and postdoctoral researchers from the United States. Women and members of other under-represented groups in the mathematics community are particularly encouraged to apply. To record an advance expression of interest, please contact Laura Schaposnik at schapos@uic.edu.

The organisers are grateful to all the sponsors: the Clay Mathematics Institute, Heilbronn Institute for Mathematical Research, London Mathematical Society, National Science Foundation, Isaac Newton Institute, and Oxford Mathematics Department.

LMS Honorary Member Receives Top Honour

Professor Cheryl Praeger (University of Western Australia) has been made a Companion of the Order of Australia in the 2021 Australia Day Honours List for 'eminent service to mathematics and to tertiary education, as a leading academic and researcher, to international organisations, and as a champion of women in STEM careers'.

Professor Praeger has had a long and distinguished career and has made significant contributions to various areas of mathematics, including group theory, permutation groups, combinatorics and the mathematics of symmetry. Her expertise in group theory and combinatorial mathematics has underpinned advances in algebra research and computer cryptography.

Professor Praeger has received many honours and awards during her career. She received the 2019 Prime Minister's Prize for Science, she was the first female President of the Australian Mathematical Society (1992–94) and the Society now awards the Cheryl E. Praeger Travel Awards to female mathematicians. Professor Praeger became an Honorary Member of the London Mathematical Society in 2014.

Visit tinyurl.com/y3h2xnbc for more information.

Clay Research Fellows

The Clay Mathematics Institute has awarded the 2021 Clay Research Fellowships to Maggie Miller, Georgios Moschidis, Lisa Piccirillo and Alexander Smith. Clay Research Fellowships are awarded on the basis of the exceptional quality of candidates' research and their promise to become mathematical leaders.

Maggie Miller obtained her PhD in 2020 from Princeton University, where she was advised by David Gabai. She will be based at Stanford University. Georgios Moschidis obtained his PhD in 2018 from Princeton University, where he was advised by Mihalis Dafermos. He will be based at Princeton University. Lisa Piccirillo obtained her PhD in 2018 from the University of Texas at Austin, where she was advised by John Luecke. She will be based at the Massachusetts Institute of Technology. Alexander Smith obtained his PhD in 2020 from Harvard University, where he was advised by Noam Elkies and Mark Kisin. He will be based at Stanford University.

For more information visit claymath.org.

EUROPEAN MATHEMATICAL SOCIETY NEWS

European Women in Mathematics

Owing to covid-19, the European Women in Mathematics (EWM) Society held its General Assembly online on 6 July 2020. The next EWM General meeting will be in Helsinki in 2022. At the virtual General Assembly, Andrea Walther (HU Berlin) and Kaie Kubjas (Aalto) were elected the new convenors of EWM, succeeding Carola-Bibiane Schönlieb and Elena Resmerita who served as convenors since 2016. EWM has published an open letter to advocate a proactive policy to support current employees in temporary positions and future job applicants in Mathematics in light of the Corona Crisis: see https://tinyurl.com/yxkg9v5n.

EMS News prepared by David Chillingworth
LMS/EMS Correspondent

Note: items included in the European Mathematical Society News represent news from the EMS are not necessarily endorsed by the Editorial Board or the LMS.

MATHEMATICS POLICY DIGEST

UKRI Ethnicity Analysis of Funding Applicants and Awardees

In December 2020 UK Research and Innovation (UKRI) published detailed ethnicity data on funding applicants and awardees, which highlighted disparities between different ethnic groups. These data form part of UKRI's ongoing work to "increase equality and diversity in the research and innovation system, through effective, evidenced intervention". These data are available at tinyurl.com/y4w5odtp.

Funding Boost for Mathematical Sciences Institutes

Three of the UK's leading research institutes will be supported to widen access to Mathematical Sciences and support training through funding announced by UKRI. The investment will allow the Isaac Newton Institute (INI), the International Centre for the Mathematical Sciences (ICMS) and the Heilbronn Institute for Mathematical Research (HIMR) to launch and expand a wide range of activities supporting education and training.

The funding is part of the £300 million government investment in the Additional Funding Programme for Mathematical Sciences announced in 2020. The funding will be delivered by the Engineering and Physical Sciences Research Council (EPSRC), part of UK Research and Innovation, and the Royal Society over a five-year period from 2020/21 to 2024/25. More details are available at tinyurl.com/y3sorkka.

Digest prepared by Dr John Johnston
Society Communications Officer

Note: items included in the Mathematics Policy Digest are not necessarily endorsed by the Editorial Board or the LMS.

OPPORTUNITIES

Forthcoming LMS Grant Schemes

Cecil King Travel Scholarships: The LMS administers two £6,000 travel awards funded by the Cecil King Memorial Foundation for early career mathematicians, to support a period of study or research abroad, typically for a period of three months. One Scholarship will be awarded to a mathematician in any area of mathematics and one to a mathematician whose research is applied in a discipline other than mathematics.

Applicants should be mathematicians in the UK or the Republic of Ireland who are under the age of 30 at the closing date for applications, and who are registered for a doctoral degree or have completed one within 12 months of the closing date for applications. The LMS encourages applications from women, disabled, Black, Asian and Minority Ethnic candidates, as these groups are under-represented in the UK or the Republic of Ireland mathematics.

To apply, complete the application form at tinyurl.com/yarns982 and include a written proposal giving the host institution, describing the intended programme of study or research, and the benefits to be gained from the visit.

The application deadline for applications is 31 March 2021. Shortlisted applicants will be invited to interview during which they will be expected to make a short presentation on their proposal.

Interviews will take place in May 2021. Queries may be addressed to Tammy Tran (ecr.grants@lms.ac.uk). In view of the low number of applications received in previous rounds, there is a high chance of success in this scheme.

Computer Science Small Grants (Scheme 7): the deadline for applications in the next round is 15 April. The grants support a visit for collaborative research at the interface of Mathematics and Computer Science. More details at tinyurl.com/y7rbdhpn.

Maximising your LMS Membership

International Connections with Other Mathematical Societies

In forthcoming issues of the *Newsletter* we aim to shine a spotlight on different membership benefits. This month we would like to highlight to members the opportunities to connect internationally with other mathematical societies through your membership of the LMS.

Reciprocal Agreements with 21 International Mathematical Societies

The Society has reciprocal agreements with the following mathematical societies, enabling LMS members to join those societies at a 50% discount on the full membership fee of each society if they are not normally resident in the same country as the society. For example, a UK-based Ordinary Member could join the Mathematical Society of Japan at their reciprocal membership rate. For further information about these societies, see tinyurl.com/y6jwpg8q.

American Mathematical Society
Australian Mathematical Society
Belgian Mathematical Society
Canadian Mathematical Society
Dansk Matematisk Forenig
Deutsche Mathematiker-Vereinigung
Finnish Mathematical Society
Société Mathématique de France
Indian Mathematical Society
Irish Mathematical Society
Unione Matematica Italiana
Mathematical Society of Japan
Koninklijk Wiskundig Genootschap
New Zealand Mathematical Society
Nigerian Mathematical Society
Norsk Matematisk Forening
Real Sociedad Matemática Española
Singapore Mathematical Society
South East Asian Mathematical Society
Svenska Mathematikersamfundet
Swiss Mathematical Society

In return, members of the above societies who are not normally resident in the UK can join the London Mathematical Society as a Reciprocity Member and receive a 50% discount on the Ordinary membership fee (other subscription rates such as Associate Membership are already discounts on the Ordinary Membership rate and the Society does not offer 'double discounts'). If you have any queries, please contact membership@lms.ac.uk.

LMS members based in either Northern Ireland or the Republic of Ireland and who are also members of the Irish Mathematical Society can choose whether to be a Reciprocity Member of either the LMS or the Irish Mathematical Society.

Discounted membership of the European Mathematical Society

Members of the London Mathematical Society can join the European Mathematical Society (EMS) at a 50% discount of the full EMS membership fee: currently €25.00 annually instead of €50.00. Further information about the European Mathematical Society is available at euro-math-soc.eu/.

To join the EMS, members can contact the EMS direct and mention their LMS membership or add EMS membership to their LMS membership record when signing in via the LMS website here: www.lms.ac.uk/user, or by completing and returning the LMS subscription form. Payment for EMS membership can also be made via the LMS and over 150 LMS members already pay for EMS membership alongside their LMS membership. If you have any queries, please contact membership@lms.ac.uk.

LMS members who are students may be interested in the EMS' free membership for students while LMS members who are aged 60+ may be interested in the EMS' lifetime membership. Further details about EMS membership rates are available here: https://euro-math-soc.eu/individual-members.

Option to pay for membership of European Women in Mathematics

Members of the London Mathematical Society who are also members of the European Women in Mathematics (EWM) association have the option to pay for their EWM membership via their LMS Membership account and over 30 EWM members already pay for their EWM membership alongside their LMS Membership.

To do so, you must first be a member of EWM: further details about EWM membership and how to join are available at tinyurl.com/y2ta5xh5. If they wish, EWM members can then add their EWM membership to their LMS membership account either by logging in via the LMS website www.lms.ac.uk/user or by completing and returning the LMS subscription form. If you have any queries, please contact membership@lms.ac.uk.

We hope this spotlight on the benefits of LMS membership in supporting your international connections has been helpful.

Elizabeth Fisher
Membership & Grants Manager

LMS Council Diary — A Personal View

Council met via video conference for a relatively short meeting on the morning of Friday 20 November 2020, before the Annual General Meeting and Naylor Lecture later that day. The meeting began with the President's business, including a report on the successful two-day online Black Heroes of Mathematics Conference, hosted jointly with the IMA and British Society for the History of Mathematics and facilitated by ICMS, which attracted over 250 participants, and an update on the 'Levelling Up' scheme pilot, generously supported by Dr Tony Hill, which aims to support the provision of online tutoring for A-level Mathematics students who come from backgrounds that are under-represented in the mathematics community. Other business included the Publications Secretary giving a brief update on contract negotiations for the future production of the Society's main journals and reporting that *Mathematika* would be made available free to LMS members from 2022 onwards, as well as an update on Committee Membership. In a discussion about Society Membership, the Treasurer noted that a healthy number of new members were due to be elected at the AGM later that day, in the recruiting of which the LMS Representatives in universities had played a significant role. There was also a discussion about the importance of voting in Society elections and how this could be encouraged.

The meeting concluded with the President giving thanks on behalf of Council to the outgoing Council Members, and wishing luck to those members up for election.

Elaine Crooks
Member-at-Large

LEVELLING UP

This article forms the second of an ongoing series of updates about this new scheme where, thanks to a generous donation by LMS member Dr Tony Hill, the LMS is developing a new venture to support the provision of online tutoring for A-Level Mathematics students who come from backgrounds that are under-represented in the mathematics community.

For students from under-represented groups it may appear an insurmountable task to achieve the necessary qualifications to study for a STEM degree at university. From the perspective of a Year 12 BAME student attending an under-performing school for example, who is hoping to study engineering at university, there may be no well-defined route to achieving their goal. Students from the school may only rarely attain the necessary grades to consider attending university. Teachers may be too stretched to provide additional academic or personal support. There may be pressure from peers to do other things rather than study and, if the student is the first person in their family to consider going to university, family members may not know how best to offer their support. The covid-19 pandemic has also provided huge new challenges with face-to-face teaching curtailed, leaving the student feeling more isolated, under pressure and unfocused. With high grades required, they must perform well in A-level mathematics to be accepted onto their preferred course. Without extra academic and aspirational support, the student may not achieve their goal. But what if a programme were available to provide tailored support to a student in this situation?

The Levelling Up Scheme is exactly the type of programme that can provide the support required to help such students succeed and achieve their academic goals. The overall goal of the Scheme is to significantly increase inclusion, by boosting students' self-confidence, raising their aspirations and accelerating their academic attainment. The aim is to prepare students to apply for a place on a STEM degree at university, and then to succeed once they are there. The Scheme is also vitally important in helping to provide a diverse pool of graduate talent with the skills to contribute to the UK's long term economic growth.

The pilot scheme will provide eighteen months of academic and pastoral support for students from under-represented groups, starting midway through Year 12 and continuing until the end of Year 13. Students will have the opportunity to take part in a variety of specially designed online subject specific tutorials, in collaboration with Durham and Leicester universities.

See more information at levellingupscheme.co.uk.

John Johnston
Society Communications Officer

Report: Annual General Meeting

The 2020 LMS Annual General Meeting was held on Friday 20 November with LMS President Professor Jon Keating, FRS in the Chair. This was the year of the AGM with a twist: it was held online, through the Zoom video conferencing software. Hosting in cyberspace certainly had no adverse impacts on attendance; the meeting attracted over 100 attendees, none of whom were required to travel to London to take part. The meeting began with Society business. First, a presentation from the Vice-President, Professor Iain Gordon, on Society activities over the course of the year. This was followed by a report from the Treasurer, Professor Rob Curtis, who gave a summary of Society accounts for the year. Other matters included votes on Society resolutions, and announcement of the results of elections to Council and Nominating Committee.

Next, congratulations were recorded for recipients of LMS awards. Each year, the LMS awards a selection of prizes for achievements in and contributions to mathematics. These range from the Senior Anne Bennett Prize, which is awarded for work in relation to advancing the careers of women in mathematics, to the Senior Berwick Prize, awarded for an outstanding piece of research published by the Society. 2020 was no different, and attendees congratulated 2020's twelve LMS prize winners, and also the 276 mathematicians elected to LMS Membership this year.

To conclude Society business, the President, Professor Jon Keating, thanked all retiring members of Council. Of particular note were the departures of two long-standing members: Professors Stephen Huggett and Rob Curtis stepped down as General Secretary and Treasurer, having served eight and nine year terms in those roles respectively. An admirable commitment, and commendable to say the least.

At the conclusion of Society business, the meeting was paused for a short break; even in virtual meetings, coffee breaks are always most welcome. The break was followed by the Naylor Lecture, given by Professor Nicholas J. Higham of the University of Manchester — to whom the LMS awarded the 2019 Naylor Prize — on 'The Mathematics of Today's Floating-Point Arithmetic'. The overarching theme was an investigation into the reliability of low precision floating-point arithmetic, with a particular emphasis on understanding the implications for recent developments in the latest hardware implementing this arithmetic. Professor Higham — the author of an acclaimed book on the topic — gave a fascinating and highly accessible talk, frequently linking together historical work and current developments in the area, as well as regularly highlighting applications in, for example, deep learning.

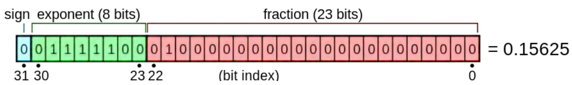
It should be noted how smoothly this AGM ran. It is easy to forget that running an event of this nature in an online setting is never simple, and requires careful planning and preparation. Potential hurdles — of conducting formal votes virtually, and transitioning between presentations and speakers, for example — were made to look non-existent. The meeting was, especially in light of current circumstances, a resounding success.

Matt Staniforth
University of Southampton

Report: LMS Graduate Student Meeting

The LMS Graduate Student Meeting on 16 November 2020 commenced with a lecture from Theo Mary of the Sorbonne entitled *Mixed Precision Arithmetic: Hardware, Algorithms and Analysis*. At the current moment in time, when our entire lives seem to revolve around computers, it is a welcome relief to be reminded of their limitations. To a pure mathematician like myself, it was fascinating to dive into the world of floating-point arithmetic and confront the fact that computers can only work with numbers to finite precisions; numbers, as I imagine them, are Platonic ideals, not finite sequences of bits on a computer.

Theo discussed the trade-off between different standards for floating-point arithmetic: the higher the precision, the slower the speed. But he went on to explain how this trade-off can be overcome by strategically deploying higher-precision arithmetic and lower-precision arithmetic at different stages in the calculation. By using higher-precision arithmetic where it counts, one can achieve both good accuracy and good speed.



A 32-bit floating point number

A primary source of motivation for research into computer arithmetic is the application to deep learning, where intensive computation is required. Consequently, much research is done in this field by big tech companies, who also implement new standards for low-precision arithmetic in hardware. The chair of the session, Ian Short of the Open University, asked Theo the interesting question of how open these companies are when it comes to working on this sort of research. Encouragingly, Theo responded that they did have a good dialogue with several of these companies, though, naturally, some of their information remains strictly proprietary.

The meeting then turned to talks from Graduate Students. A very healthy number of eighteen talks were given. This was more than would be usual for a graduate student meeting, showing that there are advantages to the online format. In order to accommodate such a large number, five breakout rooms were required, each hosting four talks of 15 minutes. The five rooms roughly corresponded to the areas of algebra, combinatorics, analysis, fluids, and mathematical biology. It seems our geometry graduate students must be rather shy.

I listened to the talks in the combinatorics room, where I also spoke. There was a rich range of talks here, from Ramsey theory to set theory, along with my own talk on polytopal combinatorics. Carl-Fredrik Nyberg Brodda finished us off with a very memorable talk on his efforts to uncover the origins of the B. B. Newman spelling theorem in combinatorial group theory. This was an exciting detective story of missing PhD theses and the like, where everything was happily solved at the end.

Nicholas Williams
University of Leicester

Report: Black Heroes of Mathematics Conference

Growing up, we read about heroes in science and mathematics whose breakthrough ideas led to innovations. Unnoticed by many is the fact that very few of these heroes are black or of black descent. Is this so because black people have not contributed to innovations and discoveries?

A few years ago, I was a PhD student in mathematics and I asked myself similar questions. This is because I needed to see role models who had blazed the trail I was on who would inspire me. My quest led me to an article: ‘Five Famous Black Mathematicians’ by Hazel Lewis with thanks to Dr Nira Chamberlain (https://tinyurl.com/y32kka58). Katherine Johnson was a name that stood out in this article because the successful contributions she made at NASA in doing the calculations that sent the first American to space, beautifully captured in the movie *Hidden Figures*.

The Black Heroes of Mathematics Conference was a virtual conference organised on Monday 26 and Tuesday 27 October 2020 by the British Society for the History of Mathematics, the International Centre for Mathematical Sciences, the Institute of Mathematics and its Applications and the London Mathematical Society. The vision of the conference was “To celebrate the inspirational contributions of black role models in the field of mathematics”. Over 250 participants from over 30 different countries attended all or part of the event online.

Talks covered a range of technical and non-technical topics, with presentations by the following international speakers: Dr Nira Chamberlain, Dr Angela Tabiri, Dr Howard Haughton, Professor Tannie Liverpool, Natalya Silcott, Dr Spencer Becker-Kahn, Professor Nkechi Agwu and Professor Edray Goins. At the end of each day, there was a panel discussion with Professor Clive Fraser, Dr Joanna Hartley, Jonathan Thomas, Susan Okereke and the speakers from that day. Questions discussed included what we can do to increase the representation of black people in mathematics, how can indigenous African mathematics be used to support the learning of mathematics, why should mathematics teachers be interested in Black History Month, and do we really need black heroes of mathematics — to mention a few.

One of my favourite quotes at the conference was by Dr Nira Chamberlain, “You do not need anyone’s permission to become a mathematician”. Other quotes included: “Being the first is not something to be proud of, but is a calling to ensure that one is not the last” — Dr Nira Chamberlain. Enthusiastic comments from participants included “What an utterly brilliant, inspiring event — may there be many more!” and “We hope that one day, we will live in a world where all children feel that maths belongs to them. Until then let us showcase the stories of diverse mathematicians so all of our students feel like they could be a mathematician.”

Videos of talks from the conference can be found at https://t.co/MyGefYZ8Id?amp=1.

Angela Tabiri
African Institute for Mathematical Sciences Ghana

Report: LMS Computer Science Colloquium

The LMS Computer Science Colloquium was held on Thursday 19 November 2020 online via Zoom with the topic of the colloquium being *Algorithms, Complexity and Logic*. A record number of participants enjoyed a compelling series of talks.

The first speaker was Dr Anupam Das, from the University of Birmingham, who gave a broad and bountiful talk across his wide interests in logic, taking in especially, proof complexity, computational complexity and bounded arithmetic. Anchored around these three disciplines, on a beautifully built whiteboard, he described the ways in which the areas interact, culminating in a seminal result of Buss tying together polynomial time computation, the bounded arithmetic theory S_2^1 and induction on notation over NP-predicates. He also explained the relationship of this bounded arithmetic theory to length of proofs in the fundamental propositional proof system called extended Frege.

The second speaker was Professor Nobuko Yoshida from Imperial College, London. She gave a talk titled *Session Types: A History and Applications*. She was introduced and hosted by one of the LMS Computer Science Committee members, Dr Ornela Dardha. More specifically, Yoshida’s talk was about: a history of (multiparty) session types; what are binary and multiparty session types; and several applications on multiparty session types, including: (3-1) runtime monitoring for large cyberinfrastructures; (3-2) robotics; (3-3) code generation in Go; and (3-4) inference of session types from Go code and verification by model checking. Her talk was followed by questions and a lunch break where further interaction continued.

The next speaker was Dr Kitty Meeks from the University of Glasgow, who discussed the interplay between certain decision problems and their exact and approximate counting versions. These were introduced in both monochrome and multicoloured versions. Of course, if one can solve exact counting, then also one can solve approximate counting; and this latter is sufficient to solve the decision problem. The remaining interactions are more subtle. For example, it is known that an oracle for approximate counting does not give rise to an efficient procedure for exact counting (in both the multicolour and monochrome regimes). The principal new result of the talk was that, in the multicolour regime, an efficient procedure for the decision problem does give an efficient procedure for approximate counting. This recent work (with Dell and Lapinskas) is especially remarkable.

The final speaker of the day was Dr Igor Carboni Oliveira from University of Warwick, who presented deep and fascinating questions and results about the descriptive complexity of fundamental notions like prime numbers. Starting from classical descriptive complexity as defined by Kolmogorov in 1963, he rephrased the question whether there are infinitely many Mersenne primes in terms of infinitely many primes of minimum description length. Time-bounded Kolmogorov complexity as introduced by Levin in 1984, can be used to make assertions about the time complexity of deterministically generating n-bit primes. Oliveira then turned to a randomised analogue of time-bounded Kolmogorov complexity introduced by himself in 2019, which can be used to identify short and effective probabilistic procedures that are likely to generate data-like prime numbers. Using this notion, he explained that there are infinitely many primes admitting 'short' and 'effective' probabilistic representations, and that we cannot feasibly distinguish 'structured' from 'random' strings. Throughout the talk he highlighted open problems which demonstrate the fascination and importance this line of research is offering at the intersection of mathematics and computer science.

The colloquium provided the audience with perspectives of many facets of algorithms, complexity and logic. The discussion after the talks was lively. There was something for everyone to take away from the talks and discussion.

Ornela Dardha (University of Glasgow)
Arnold Beckmann (Swansea University)
Charlotte Kestner (Imperial College London)
Barnaby Martin (Durham University)
Prudence Wong (University of Liverpool)

Records of Proceedings at LMS meetings
Annual General Meeting and Society Meeting of the London Mathematical Society: Friday 20 November 2020

The meeting was held virtually on Zoom, hosted by the International Centre for Mathematical Sciences. About 110 members and visitors were present for all or part of the meeting. The meeting began at 3:00pm, with the President, Professor Jon Keating, FRS, in the Chair.

The President explained that, due to the covid-19 pandemic, the Society had adapted to adhere to UK social distancing measures in order to keep its members, guests and staff safe. In doing so, there had been an impact on the Society's Governance in relation to its Standing Orders. Members and guests were asked to:

- note Council's decision to hold the Annual General Meeting virtually;
- note that the Society was still using the old Standing Orders;
- note that the virtual AGM technically breached the Standing Orders, and in particular that physical voting in person could not take place; and
- note that the Society had followed the Charity Commission's guidelines on this issue and had informed the Commission of our actions.

The Vice-President, Professor Iain Gordon, presented a report on the Society's activities and the President invited questions.

The Treasurer, Professor Robert Curtis, presented his report on the Society's finances during the 2019–20 financial year and the President invited questions.

The President introduced the members' votes on four resolutions. As the voting was open to LMS members only, guests who were not Society members were placed in the 'virtual' waiting room for the duration of the vote, after which non-members were re-admitted to the meeting. This was in keeping with the guidance from the Charity Commission that the Society should have a system in place to ensure that only those eligible to vote could do so.

The minutes of the General Meeting held on 26 June 2020 had been circulated 21 days before the Annual General Meeting and members were invited to ratify the minutes by a completing an onscreen poll. The minutes were ratified.

Copies of the Trustees Report for 2019–20 were made available and the President invited members to adopt the Trustees Report for 2019–20 by completing an onscreen poll. The Trustees Report for 2019–20 was adopted.

The President proposed Moore Kingston Smith be re-appointed as auditors for 2020–21 and invited members to approve the re-appointment by completing an onscreen poll. Moore Kingston Smith were re-appointed as auditors for 2020–21.

The President introduced the fourth poll and advised members that, following a suggestion by the LMS Reps which was subsequently approved by Council in June 2020, the Society would be separating the Ordinary membership subscription rate into three tiers: low, middle and high, based on Members' annual professional salaries, as reported by the Members themselves. For the first membership year in which the new fee would be implemented (2021–22), the high rate would represent an increase of more than 10% over the previous year's rate for those Members affected. Statute 11 of the Society's Standing Orders required the agreement of Members voting at a General Meeting where an increase of more than 10% was proposed. An example of the tiered subscription rates would be:

- Ordinary (high): £120.00 for members earning over £65,000
- Ordinary (middle): £100.00 for members earning between £35,000 – £65,000
- Ordinary (low): £80.00 for members earning up to £35,000.

Members at all tiers would retain the same benefits.

The President advised that Council was recommending the approval of the resolution to increase by more than 10% the subscription rate for those Members paying a 'high' rate under the Society's new three-tiered subscription rate structure.

The President invited members to approve the resolution by completing an onscreen poll. The resolution was approved.

Guests were then re-admitted from the waiting room.

43 people were elected to Ordinary Membership: Babatunde Aina, Murat Akman, Anna Ananova, Eleftheriou Antonia, Ovidiu Bagdasar, Nicholas Baskerville, David Bate, Christopher Birkbeck, Christoph Czichowsky, Remy Dubertrand, Amit Einav, Josephine Evans, Goitom Fessahaye, Fernando Galaz-Garcia, Nicos Georgiou, Noel Giacometti, Andre Henriques, Nick Hills, Ashley Kanter, Tom Kempton, Dawid Kielak, Sergey Kitaev, Angeliki Koutsoukou-Argyraki, Robert Laugwitz, Brendan Masterson, Andrea Mondino, Pieter Naaijkens, Sarah Nagawa, James Newton, Davide Proment, Matthew Pusey, Yogendra Kumar Rajoria, Timothy Reis, Yue Ren, Hayley Ryder, Anuradha Sahu, Nicholas Simm, Ravendra Singh, Terry Soo, Pierpaolo Vivo, Graeme Wilkin, Julian Wilson and Andrew Wilson.

64 people were elected to Associate Membership: Edward Acheampong, Costanza Benassi, Andrea Boido, Alessio Borzì, Stephen Butler, Bromlyn Cameron, Giulia Carigi, Dimitrios Chiotis, Lily Clements, Andrea Clini, Daniel Cocks, Cristina Criste, Joshua Cromwell, Håvard Damm-Johnsen, Thibault Decoppet, Georgios Domazakis, Loki Dunn, Gilles Englebert, Sam Fearn, Samuel Fisher, Yanik-Pascal Foerster, Luca Gamberi, Wissam Ghantous, Katrina Gibbins, Tristan Giron, Sergio Giron Pacheco, Dewi Gould, Elena Hadjicosta, Grant Harvey, Anastasia Ignatieva, Philipp Jettkant, Panagiotis Kaklamanos, Rajesh Kaluri, Daniel Kaplan, Duncan Laurie, Michael Liedl, Abigail Linton, Joseph MacManus, Alexander Malcolm, David Malka, Grace El-Raphaella Matonga, Holly Middleton-Spencer, László Mikolás, Patrick Nairne, John Nicholson, Chimezie Nnanwa, Shreenil Odedra, Chirag Pithia, Daniel Platt, Ellen Powell, Bhairavi Premnath, Michael Savery, Amit Shah, Thomas Sharpe, Eoin Simpkins, Julia Stadlmann, Matthew Staniforth, Samuel Stark, Benedikt Stock, Angela Tabiri, James Taylor, Federico Trinca, Benjamin Ward and Finn Wiersig.

14 people were elected to Associate (undergraduate) Membership: Isobel Baddeley, Bhanu Banerjee, Matthew Bond, David Cawthorne, Max Durrant, William Evans, Rahul Gupta, Lloyd Hughes, Brian Judelson, Lynn Wei Lee, Yasmin Mussa, Glenn O'Callaghan, Qaisar Shah and Jiguang Yu.

12 people were elected to Reciprocity Membership: Daniel Asimov, Iyai Davies, Praveen Kumar Dhankar, Matt Insall, Shobha Lal, G. Muhiuddin, Ram Kripal Prasad, Margaret Readdy, Mansur Saburov, Onur Saglam, Subhrajyoti Saha and Jyoti Singh.

143 people were elected to Associate Membership for Teacher Training Scholars: Roba Abu Hantash, Aduragbemi Adebogun, Hasan Ahmad, Anees Ahmed, Kabir Akmal, Bayan Ali, Zahra Ali, Chris Allen, Henry Allen, Edward Antwi-Berchie, Joanna Aranowska, Dayana Arasarathnam, Thomas James Attrill, Jennifer Auger, Tayyib Azeem, Steven Barker, James Barlow, Ethan Barnes, Joseph Bashir, Rebecca Bedford, Jamina Begum, Jessica Bradbury, Thomas Brasier, Francis Brooks-Tyreman, Hannah Buckley, Christan Butters, Milan Chrastina, James Colton, Etienne Corish, Samuel Corrigan, Nicholas Cossins, Stephen Cox, Joshua Cunningham, Edward Curr, York Deavers, Francis Denton, Louise Devaney, Conrad Doggett, Thomas Donnelly, Ankita Dudani, Edet Efretuei, Christopher Ellis, David Errington, Mehmet Esen, Olivia Faggi, Ibrahim Farooq, Ciprian Faur, Peter Fitzpatrick, Katherine Frewer, Oliver Green,

Bethany Hall, Laura Hallmark, Mark Hamilton, Richard Hayes, Thomas Henshaw, Curtis Holmes, Chris Holmes, Lawrence Holmes, Nicole Houston, Tara Howard, Grant Hubbard, Stuart Huntley, Joshua Jenkins, Sharmin Joarder, Joshua Johnson, Sarah Johnson, Steve Joiner, Emma Jones, Meggie Jordan, Rajdeep Josan, James Keable, Dasina Kerai, Azrah Khan, Charlotte Kingdom, Emily Larkin, Robert Little, Yun Liu, Victoria Long, Ross Longden, Thomas Lowe, Muhammad Mamsa, Ethan Marks, Peter Mills, Elizabeth Morland, Simon Mullis, Angela Murphy, Mohammed Nasseri, Andrew Nolan, Alex Nuttall, Tim Oliver, Floriana Pacchiarini, Tilly Porthouse, Gareth Pugh, Katherine Purdy, Christopher Quickfall, Mohammed Habeeb Rabbani, Benjamin Raine, Zainab Razvi, Molly Reeve, Olivia Revans, Katy Rigby, Richard Robertson, Keith Rochfort, Erin Rodrigues, Connor Rollo, Lucille Rostron, George Savage, Lauren Sealey, Eloise Sear, Lucy Shelley, Amber Shiels, Daniel Shipp, Callum Shreeve, Matthew Simcock, Chris Simpson, Alexander Sinclair, Joe Skerman-Gray, Sophie Slade, Sarah Slater, Edward Slater, Susannah Smith, Nicholas Stancill, Shane Steele, Danny Sugrue, Megan Sullivan, James Taylor, Rachel Taylor, Benjamin Vickers, Beth Waghorn, Mark Walklin, Benjamin Walne, Kyle Ward, James Ward, Duncan Weaving, Joseph Webster, Toby Wehrle, Emma Wheeler, Francesca Williams, Luke Ryan Wilson, Crystal Wincy Wincent, Richard Winstanley, Chloe Wong and Carina Yew-Booth.

No members signed the Members' Book or were admitted to the Society. The President advised the audience that, while the Society was unable to offer the opportunity to members to sign it at this meeting, the Members' Book would once again be available for signing when face-to-face meetings could be resumed.

The President invited non-members within the audience to join the Society and advised that details about membership were on the Society's website.

The President informed the audience that donations to the Society were most welcome and that donations, including to the De Morgan Donations scheme, could be made online.

The President invited members of the audience to congratulate the 2020 Prize-winners:

Pólya Prize: Professor Martin Liebeck (Imperial College, London)
Senior Anne Bennett Prize: Professor Peter Clarkson (University of Kent)
Senior Berwick Prize: Professor Thomas Hales (University of Pittsburgh)
Shephard Prize: Regius Professor Kenneth Falconer (University of St. Andrews); Professor Des Higham (University of Edinburgh)
Fröhlich Prize: Professor Françoise Tisseur (University of Manchester)
Whitehead Prizes: Dr Maria Bruna (University of Cambridge), Dr Ben Davison (University of Edinburgh), Dr Adam Harper (University of Warwick), Dr Holly Krieger (University of Cambridge), Professor Andrea Mondino ((University of Oxford), Dr Henry Wilton (University of Cambridge)
LMS–IMA Christopher Zeeman Medal: Matt Parker

The certificates had been posted to the prize-winners.

The Scrutineer, Professor Chris Lance, announced the results of the ballot. The following Officers and Members of the Council were elected.

President: Professor Jon Keating
Vice-Presidents: Professor Iain Gordon, Professor Catherine Hobbs
Treasurer: Professor Simon Salamon
General Secretary: Professor Robb McDonald
Publications Secretary: Professor John Hunton
Programme Secretary: Professor Chris Parker
Education Secretary: Dr Kevin Houston

Members-at-Large elected for two-year terms: Professor Peter Ashwin, Professor Anne-Christine Davis, Professor Minhyong Kim, Professor Niall MacKay, Professor Anne Taormina, Dr Amanda Turner
Member-at-Large (Librarian): Dr Mark McCartney

Five Members-at-Large, who were elected for two years in 2019, have a year left to serve: Professor Elaine Crooks, Professor Andrew Dancer, Dr Tony Gardiner, Dr Frank Neumann and Professor Brita Nucinkis.

The following were elected to the Nominating Committee for three-year terms: Professor Chris Budd and Professor Gwyneth Stallard. The continuing members of the Nominating Committee are: Professor Kenneth Falconer (Chair), Professor I. David Abrahams, Professor Beatrice Pelloni, Professor Mary Rees and Professor Elizabeth Winstanley. One member of Council will also be nominated to the Nominating Committee.

Professor Nicholas J. Higham, University of Manchester, gave the Naylor Lecture 2020 on *The Mathematics of Today's Floating-Point Arithmetic*.

Before closing the meeting, Professor Keating thanked the retiring members of Council and welcomed the President Designate Professor Ulrike Tillmann, FRS.

Professor Keating also thanked the speaker at the Graduate Student Meeting on 16 November 2020 Theo Mary (Sorbonne), and congratulated the winners of the Graduate Student Talk Prizes: Carmen Cabrera-Arnau (UCL), Giulia Carigi (Reading), Carl-Fredrik Nyberg Brodda (UEA), Onirban Islam (Leeds), Raad Kohli (St. Andrews) and Gustavo Rodrigues Ferreira (Open University). The President thanked the other 12 graduate students who also gave talks.

The President thanked everyone who had worked to organise the online Annual General Meeting. The President closed the meeting. There was no reception or Annual Dinner.

Records of Proceedings at LMS Meetings: Society Meeting at the Joint Mathematics Meeting 2021

This meeting was held virtually on Zoom, at the Joint Mathematics Meeting co-hosted by the American Mathematical Society (AMS) and the Mathematical Association of America (MAA). Over 15 members and visitors were present for the LMS meeting session.

The Society meeting began at 5.00pm GMT on 7 January with the Publications Secretary, Professor John Hunton, in the Chair. Professor Hunton welcomed guests, thanked the organising parties, and then introduced Professor Tim Browning who spoke about the new *Proceedings of the London Mathematical Society*. Professor Browning then introduced a lecture given by Professor Sarah Zerbes (UCL) on *Special Values of L-Functions*.

Professor Hunton concluded the meeting by thanking Professor Zerbes, the organisers and the meeting attendees on behalf of the LMS.

Joining the De Morgan House Team — One Year On

CAROLINE WALLACE

LMS Executive Secretary Caroline Wallace reflects on her first eventful year in the role.

I became Executive Secretary of the London Mathematical Society in early April last year. Someone asked me recently what had attracted me to the role. There were several parts to my answer.

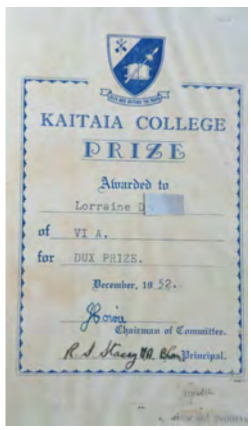
First and foremost, I love mathematics. In particular, I have a great interest in how it can be put to use to improve our lives. This interest intensified when, while still at school, I read the wonderful book *Mathematics and the Imagination* by Edward Kasner and James Newman. I am the proud inheritor of my mother's 1952 edition. It was given to her as a prize when she was a young woman, as she left her secondary school in the rural far north of New Zealand, and, unusually for a woman in that time and place, headed to university.

Pursuing my interest in the applications of mathematics, I gained a degree in engineering at the University of Cambridge and I spent the first part of my career working in industry. I can honestly say that the 'real world' power of mathematics was evident to me every day.

The second part of my answer to the question posed is that I was very impressed by the history and the reputation of the London Mathematical Society. It has a strong and clear mission: to advance, disseminate and promote mathematical knowledge. It is an organisation that has stood the test of time and that has always placed high value on the (still very topical) virtues of reasoning and research. Yet it also remains a relatively small Society where an individual can make a difference.

And this leads directly to the third part of my answer to the question posed. Not only does the Society have an incredible history, but it is also a charity. It seeks to do good in the world by supporting mathematical research and mathematicians and it promotes equality of opportunity. I find it highly motivating to work in an organisation that contributes in such a concrete way to the public good. For example, the Society's involvement in the Levelling Up Scheme, enabled by the extremely generous support of our donor, Dr Tony Hill, has the potential to increase the aspirations and the attainment of A-level maths students from under-represented groups across the country. You can read more about this Scheme on page 13.

As I approach the end of my first year as the Society's Executive Secretary, it is clear to me that I am very fortunate to have this role. As I noted earlier, the Society has a clear mission and it has a strong desire to achieve that mission. I lead a skilled and knowledgeable staff team. There are strong relationships amongst the staff and between the staff, the Council, the Membership and the wider volunteer community.



The book plate for my mother's copy of *Mathematics and the Imagination*

This is not to say that it has all been plain sailing. I took up my role at the Society two weeks after the first covid-19 lockdown began. While it was somewhat tempting to focus in this article on the impact of covid-19, I did not want the pandemic to dominate my reflections on my first year with the Society.

This is nonetheless a good opportunity to note that staff recognise and share the challenges that the covid-19 pandemic has created for the mathematics community. This includes amongst many other things managing greater caring responsibilities, limited workspace, altered work expectations and looking after our physical and mental wellbeing. I continue to be impressed by and deeply grateful for the flexibility and resilience of my colleagues as they cope with the sudden and disconcerting changes that the pandemic has imposed on all of us.

I am also impressed by — and very proud to be part of — the Society's response to the pandemic. The Society has demonstrated its willingness to listen to its Membership and to change rapidly what it does and how it does it. It has made additional Early Career Fellowships available and given additional funding to Research Groups to produce online lectures. It has pushed back its deadlines for LMS prize nominations, it has moved its meetings and events online and it has explored 'virtual' exhibition stands at conferences where face-to-face attendance is impossible. Recently, it has sought to improve the signposting not just to its own but to other organisations' funding opportunities. And I am pleased to say that there is more in the pipeline.

One of the many effects of the pandemic is that it has greatly reduced the opportunities for me to meet and get to the know the Society's Members. I have attended as many online Society meetings and events as possible. Unfortunately, it is just not the same as an 'in person' meeting, at which there would be a chance over coffee to introduce ourselves and talk about the latest developments at the Society. But if you do see me at an online meeting and would like to say hello, please just message me in the chat and hopefully we can arrange a separate call.

In the meantime, there are reasons to be hopeful as the vaccination programme is rolled out, as we continue to learn about the benefits of remote working, and as we plan for how we can retain the benefits of remote working (not least, reduced environmental impact) in our new post-covid world. I am looking forward to my second year with the Society!

Cover Image: Monte Carlo Simulation

IAN FLOOD

We are seeing unprecedented demand for radio spectrum with the advocates of emerging technologies pressing for access to frequency bands already populated by established services. This leads to studies of the radio interference environment with spectrum managers increasingly motivated towards consideration of mathematical models and spectrally efficient solutions.

The cover image is a snapshot of one step in a Monte Carlo simulation using *Visualyse Professional* software. The image shows a somewhat abstract model of interference sources in a mobile network deployment centred on St Louis, Missouri; local terrain features can be seen. The yellow markers represent sources of interference in the city, and the blue markers are sources of interference in the rural environment. There is one source for each mobile network cell and the black markers show the locations of cell centres.

At each step in the simulation, the location of an interference source is random within its cell allowing for an extensive search of possible interference geometries. The model calculates aggregate interference ΣI from all sources incident to a single victim receiver at the centre of the city. ΣI can be expressed in decibels relative to a unit of signal power in a specified bandwidth. The results from our simulation can be presented as a graph of the Complimentary Cumulative Distribution Function (CCDF).

In general, spectrum engineers are concerned with evaluating interference in relation to interference protection criteria which can take several forms. Typical examples are an aggregate interference-to-noise ratio $\Sigma I/N$ expressed in decibels, or a simple threshold for aggregate interference ΣI_T. Considering an aggregate interference threshold, if $\Sigma I = \Sigma I_T$ then our criterion is satisfied exactly, but if $\Sigma I > \Sigma I_T$ then excess interference is incident to the receiver. This may be acceptable if the criterion is associated with a constraint which allows the threshold to be exceeded for a specified percentage of time and this constraint is satisfied. We can easily use our graph of the CCDF to test such criteria when working in the time domain, but this has been the focus of some discussion recently as many simulations attempt to model large-scale network deployments and, because of uncertainties, include some variability in the deployment domain.

If the victim receiver and interferer are co-frequency in this simulation, with all potential sources of interference switched on, the modeller will not be surprised to find excess interference at the receiver.

When an interference protection criterion is exceeded, the modeller may consider mitigation. One method is to calculate the radius of a zone around the victim receiver where interference sources are excluded; this might be appropriate if the receiver is part of an important installation and at a fixed location.

Another approach is to introduce a frequency separation between interferer and victim receiver. Here, spectrum masks, characterising emissions from the interfering transmitters and the response of receiver filtering to incident signals, can be modelled. A convolution of these masks allows for the Net Filter Discrimination to be calculated at discrete frequency separations; this is the discrimination, expressed in decibels, available at the victim receiver when the interferer is offset in frequency.

FURTHER READING

[1] J. Pahl, Interference Analysis, Modelling Radio Systems for Spectrum Management, Wiley, Chichester, UK, 2016.

[2] NTIA, Interference Protection Criteria. Phase 1 - Compilation from Existing Sources, NTIA Report 05-432, U.S. Dept. of Commerce, Oct. 2005.

Ian Flood

Ian is a consultant with Transfinite Systems, London. His work involves modelling spectrum sharing problems. He is a Chartered Engineer and holds a PhD in graph-theoretic studies.

Notes of a Numerical Analyst

At the Edge of Infinity

NICK TREFETHEN FRS

2^n is bigger than n, and Cantor showed this is true even when n is infinite. The theory is beautiful, and most of us know the basics. But we are easily caught off guard by finite numbers when they are big enough.

Take this equation adapted from [2], with (!) as a warning that equality does not actually hold:

$$10^{10^{428{,}000}} = e^{10^{428{,}000}}. \qquad (!)$$

Of course the two numbers aren't really equal — their difference is enormous. Yet they are indistinguishable if you regard the top exponent as known to just three digits, for the number on the left is equal to $\exp(10^{428{,}000.36\ldots})$. Or consider this one:

$$\left(10^{10^{428{,}000}}\right)^2 = 10^{10^{428{,}000}}. \qquad (!)$$

This time the number on the left is equal to $10\hat{}(10^{428{,}000.30\ldots})$. Evidently the familiar rules of arithmetic break down, in a practical sense, when numbers are huge, giving us principles like

$$n^2 = n, \quad 2^n > n. \qquad (!)$$

Note how these formulae echo Cantor's results for true infinities, which we can write in shorthand as

$$\infty^2 = \infty, \quad 2^\infty > \infty.$$

For another curiosity at the edge of infinity, let A be an infinite "random Fibonacci matrix" with zero entries everywhere except ± 1 (independent coin tosses) on the first two superdiagonals, i.e., entries $a_{j,j+1}$ and $a_{j,j+2}$ [3]. The spectrum Σ of A as an operator on ℓ^2 is the closed disk $|z| \leq 2$ (with probability 1), which we can explain by noting that A contains arbitrarily large regions where all the signs are equal. Yet spectral theory is missing something essential about A if we view it as a limit of matrices A_n of finite dimension n. In an inner region $\Sigma_i \subset \Sigma$, roughly the disk $|z| < 1.3$, the resolvent norm $\|(z - A_n)^{-1}\|$ grows exponentially as $n \to \infty$, but in the remainder of Σ it grows only algebraically, as shown by the plot of $\log_{10}(\|(z - A_n)^{-1}\|)$ in Figure 1 for a matrix of dimension 200. The crown of this "witch hat" is very tall (truncated raggedly by floating-point arithmetic), but the brim is flat. If a physical system were governed by such matrices, the spectrum measured in the lab would probably be Σ_i, not Σ.

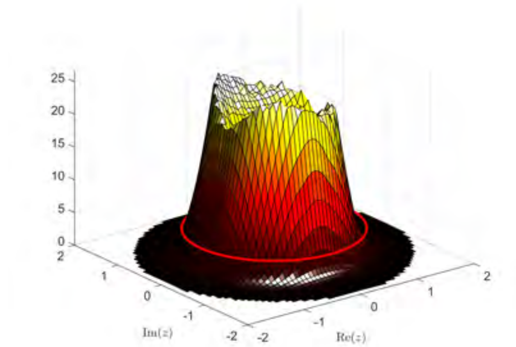


Figure 1. Random Fibonacci witch hat

Mathematics has a wonderful ability to reason rigorously about idealisations. Sometimes it is good to remember, however, that they are idealisations. In moral philosophy, the field of "infinite ethics" draws conclusions based on the supposition that there may be infinitely many worlds with infinitely many sentient beings, including a creature epsilon close to my own self down to the home address and the children's names [1]. Personally, I find it hard to believe that anything is quite that infinite.

FURTHER READING

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[2] S. J. Chapman, J. Lottes, and L. N. Trefethen, Four bugs on a rectangle, *Proc. Roy. Soc. A*, 467 (2010), 881–896.
[3] L. N. Trefethen and M. Embree, *Spectra and Pseudospectra*, Princeton, 2005.

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Mathematics News Flash

Jonathan Fraser reports on some recent breakthroughs in mathematics.

It is a pleasure to begin by thanking Aditi Kar for initiating the 'News Flash' section of the *Newsletter* and for writing many beautiful mathematical vignettes. I will do what I can to follow in her footsteps. The three papers discussed below establish deep results with easily understood statements. I hope the readership enjoys them as much as I did.

Flat Littlewood polynomials exist

AUTHORS: Paul Balister, Béla Bollobás, Robert Morris, Julian Sahasrabudhe and Marius Tiba
ACCESS: https://arxiv.org/abs/1907.09464

A *Littlewood polynomial* is a polynomial whose coefficients are all either $+1$ or -1. Littlewood conjectured in 1966 that there should exist constants $a, b > 0$ such that for every $n \geq 2$ there exists a Littlewood polynomial P of degree n such that

$$a\sqrt{n} \leq |P(z)| \leq b\sqrt{n}$$

for all $z \in \mathbb{C}$ with $|z| = 1$. This paper confirms Littlewood's conjecture and was published in *Annals of Mathematics* in 2020. As the authors explain, the most challenging part of the proof was establishing the lowering bound. Explicit polynomials satisfying the upper bound only had been constructed by Shapiro and Rudin over 60 years ago.

Littlewood's problem, and the solution described above, have implications in autocorrelation for binary sequences. Autocorrelation refers to the correlation of a signal with a delayed copy of the signal as a function of the delay.

The group of boundary fixing homeomorphisms of the disc is not left-orderable

AUTHORS: James Hyde
ACCESS: https://arxiv.org/abs/1810.12851

A group G is said to be *left-orderable* if it admits a total order $<$ such that for all $f, g, h \in G$,

$$f < g \Leftrightarrow hf < hg.$$

The integers under addition is the archetypal example of a left-orderable group, and a more sophisticated example is the group of homeomorphisms of the unit interval which fix the endpoints. The 2-dimensional analogue of this latter example became notorious: is the group of homeomorphisms of the closed disk which pointwise fix the boundary left-orderable? This question was posed in several esteemed circles (pun intended) including in a paper of Navas published in the *Proceedings* of the ICM in 2018 and in the famous *Kourovka Notebook*.

Hyde answered this question in the negative with an ingenious construction of a finitely generated subgroup which is itself not left-orderable. This remarkable paper is only five pages long and was published in the *Annals of Mathematics* in 2019.

On the Lebesgue measure of the Feigenbaum Julia set

AUTHORS: Artem Dudko and Scott Sutherland
ACCESS: https://arxiv.org/abs/1712.08638

The Julia set of a polynomial $P : \mathbb{C} \to \mathbb{C}$ is the boundary of the set of points whose orbit under P remains bounded. Julia sets are typically intricate fractal sets. Dudko and Sutherland consider the Julia set of the infamous *Feigenbaum polynomial* $z \mapsto z^2 + c_F$ where $c_F \approx -1.401155\ldots$. This polynomial is especially difficult to study due to the dynamics associated with the critical point: subtle behaviour which occurs only with delicate choice of c_F. The main result of this paper, published in *Inventiones Mathematicae* in 2020, is that the Lebesgue measure of the Feigenbaum Julia set is zero. This answers a famous open question in complex dynamics. In fact, the authors prove the stronger statement that the Hausdorff dimension of the Julia set is strictly less than 2. The proof uses a computer programme to rigorously establish that a certain condition is satisfied.

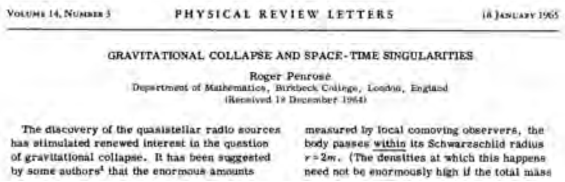
Jonathan Fraser is a Reader of Mathematics at the University of St Andrews. His research interests centre on fractal geometry. He is pictured with his son, Dylan, who makes his second appearance in the *Newsletter*.

Penrose's Incompleteness Theorem

MIHALIS DAFERMOS

On the occasion of Roger Penrose's 2020 Nobel Prize in Physics, I discuss his remarkable incompleteness theorem and its legacy for understanding black holes and singularities in general relativity.

This past year, Roger Penrose was awarded the 2020 Nobel Prize in Physics. The accompanying press release [21] cites his 1965 paper 'Gravitational collapse and space-time singularities' [13] below.


VOLUME 14, NUMBER 3 PHYSICAL REVIEW LETTERS 18 JANUARY 1965

GRAVITATIONAL COLLAPSE AND SPACE-TIME SINGULARITIES

Roger Penrose
Department of Mathematics, Birkbeck College, London, England
(Received 18 December 1964)

The discovery of the quasistellar radio sources has stimulated renewed interest in the question of gravitational collapse. It has been suggested by some authors[1] that the enormous amounts

measured by local comoving observers, the body passes within its Schwarzschild radius $r = 2m$. (The densities at which this happens need not be enormously high if the total mass

Penrose's [13] as published in Physical Review Letters

On the surface, this is a very unusual citation for a Physics Nobel: Penrose's [13] did not propose a new theory, formulate a new equation, or discover a new explicit solution, achievements which physicists more readily appreciate and celebrate. Despite appearing in *Physical Review Letters*, [13] is a quintessentially mathematical paper, sketching the proof of a theorem—indeed, a theorem of pure geometry. Yet it is hard to exaggerate how profoundly this theorem influenced the way all of us—mathematicans, physicists and even the wider public—today understand general relativity.

In this article, I will try to introduce Penrose's incompleteness theorem of [13] and its legacy to a broad mathematical audience. To set the stage, let me first describe briefly the mathematical structure of Einstein's celebrated theory of general relativity.

General relativity

General relativity postulates a unified structure, a *Lorentzian metric* g defined on a 4-dimensional manifold $\mathcal{M}$—*spacetime*—governing gravitation, inertia and what we perceive as time and geometry.

Lorentzian metrics g are the analogue of the more familiar Riemannian metrics, except that they have (in 4-dimensions) signature $(-,+,+,+)$. This just means that suitable local coordinates (x^0, x^1, x^2, x^3) around a spacetime point $p \in \mathcal{M}$ can be chosen such that the metric g_p at p may be written as

$$g_p = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

While for Riemannian metrics, $g_p(\mathbf{v}, \mathbf{v}) = 0$ would imply $\mathbf{v} = 0$, in Lorentzian geometry, the set

$$N_p = \{0 \neq \mathbf{v} \in T_p\mathcal{M} : g_p(\mathbf{v}, \mathbf{v}) = 0\}$$

forms a double cone in the tangent space $T_p\mathcal{M}$ which can be viewed as an infinitesimal version of Minkowski's light cone of special relativity (where units have been chosen such that the speed of light $c = 1$). We call such vectors $\mathbf{v} \in N_p$ *null vectors*.

The cone bounds the set of so-called *timelike* vectors

$$I_p = \{\mathbf{v} \in T_p\mathcal{M} : g_p(\mathbf{v}, \mathbf{v}) < 0\}.$$

We always assume that it is possible to select a distinguished connected component of N_p depending continuously on p. This defines the so-called *future null cone* N_p^+, which in turn bounds a connected component I_p^+ of I_p. We refer to vectors $\mathbf{v} \in N_p^+$ as *future null* and $\mathbf{v} \in I_p^+$ *future timelike*.

These concepts have immediate physical interpretation: test particles traverse curves $\gamma(\tau)$ in spacetime whose tangent $\gamma'(\tau)$ is future timelike, i.e. $\gamma'(\tau) \in I^+_{\gamma(\tau)}$. Such curves are called *worldlines*. The integral

$$\int_{\tau_1}^{\tau_2} \sqrt{-g_{\gamma(\tau)}(\gamma'(\tau), \gamma'(\tau))}\, d\tau$$

is known as *proper time*, the time of local physical processes, like a human observer's heartbeat. In the case of *proper time parametrisation*, where $g(\gamma', \gamma') = -1$, the vector γ' is known as the 4-*velocity*. If the test particles are 'freely falling', then these worldlines $\gamma(\tau)$ must in fact be *geodesics* of g (defined just as in Riemannian geometry). Light rays traverse future directed *null* geodesics of the metric, i.e. geodesics with $\gamma'(\tau) \in N^+_{\gamma(\tau)}$. Since such geodesics are in general curved, general relativity predicts the 'bending of light'.

The interaction of gravitational fields and macroscopic matter in the theory is provided by the so-called *Einstein equations*. These were formulated by Einstein [5] in November 1915 and take the form

$$\mathrm{Ric}(g) - \frac{1}{2}\mathrm{Scal}(g)g = 8\pi\mathrm{T}. \tag{1}$$

Here, $\mathrm{Ric}(g)$ denotes *Ricci curvature* of g and $\mathrm{Scal}(g)$ denotes *scalar curvature*, both defined just as in Riemannian geometry. (Recall that the Ricci curvature is itself a certain average of the full Riemann curvature tensor $\mathrm{Riem}(g)$, whereas the scalar curvature is simply the trace of the Ricci curvature. The precise formulae are of no particular relevance for this discussion.) The object T on the right hand side of (1) is the so-called *stress-energy-momentum* tensor of matter. We will see an example of such a tensor later on. (In writing (1), we note that we have chosen units such that in addition to $c = 1$ the Newtonian gravitational constant satisfies $G = 1$.)

One should view the Einstein equations (1) as the general relativistic analogue of the Poisson equation

$$\Delta\phi = 4\pi\rho \tag{2}$$

describing the Newtonian gravitational potential ϕ generated by the total mass density ρ of matter. We note already, however, some fundamental differences: equation (2), at fixed t, can be considered as a linear elliptic equation completely determining ϕ from ρ and appropriate boundary conditions at infinity. Thus, in Newtonian theory, gravity is only non-trivial in the presence of matter. In contrast, equation (1) is non-trivial already where $\mathrm{T} = 0$ globally, in which case it simplifies as:

$$\mathrm{Ric}(g) = 0. \tag{3}$$

These are the *Einstein vacuum equations*. Equations (3) in fact constitute a nonlinear hyperbolic system with a well-posed initial value problem.

Equations (1) must in general be supplemented with equations for matter fields, which are in turn coupled to (1) via the stress-energy-momentum tensor T. For all conventional matter however, T satisfies certain non-negativity properties, just as the mass density ρ in Newtonian theory satisfies $\rho \geq 0$. The most basic of these properties is the statement that $\mathrm{T}(\mathbf{v},\mathbf{v}) \geq 0$ for any null vector $\mathbf{v}$, in which case the Einstein equations (1) imply the following inequality:

$$\mathrm{Ric}(\mathbf{v},\mathbf{v}) \geq 0. \tag{4}$$

It is worth noting already that equation (3), and more generally inequality (4), encode geometric content, analogous to that encoded in positivity assumptions concerning Ricci curvature in Riemannian geometry.

Both the evolutionary pde point of view on (3) and the geometrical point of view on (4) will be essential to our story. We are already getting ahead of ourselves, however. Let us first return to 1915!

The Schwarzschild solution and the problem of 'singularity'

The problem of 'singularity' plagued Einstein's theory essentially from its inception. The issue arose already in connection with the first non-trivial solution of (3) to be discovered—only weeks after Einstein's formulation of the equations—namely that of Schwarzschild [19]. Let me briefly describe this metric and the issues it gave rise to.

In local coordinates (t, r, θ, ϕ) the Schwarzschild metric can be written as

$$\begin{aligned} g = -(1-2m/r)dt^2 + (1-2m/r)^{-1}dr^2 \\ + r^2(d\theta^2 + \sin^2\theta d\phi^2). \end{aligned} \tag{5}$$

With a little bit of computation, the keen reader can explicitly check that (5) indeed satisfies the vacuum equations (3) for all values of parameter $m \in \mathbb{R}$. Note that if $m = 0$, the expression (5) simply reduces to the flat Minkowski metric of special relativity, expressed in spherical polar coordinates.

The early well-known triumphs of general relativity (explaining anomalous precession of the perihelion of Mercury, predicting the bending of light [11]) can be easily deduced from (5), interpreting it as the vacuum metric outside a spherically symmetric star of mass m and radius R, measured in appropriate units.

As often happens, however, together with triumph came new problems! In particular, if $m > 0$, then the expression (5) defining the Schwarzschild metric seems to be inadmissible at $r = 2m$, where the metric coefficient g_{rr} manifestly blows up.

In the context of actual stars as understood at the time, the issue seemed academic: typical stars have $R \gg 2m$ in these units, so there is no problem if one only considers (5) for $r > R$. However, a prophetic 1939 paper [12] by Oppenheimer and Snyder—a paper little understood in its time—showed that the question was not as academic as first seemed! The point was that one should not restrict to static stars, but allow also collapsing ones.

We will discuss [12] in the next section. Briefly, [12] constructs a spherically symmetric spacetime $(\mathcal{M}, g)$ solving (1) coupled to the equations for a perfect fluid, where the fluid is pressureless, and thus

$$\mathrm{T} = \rho\, \mathbf{u}^\flat \otimes \mathbf{u}^\flat, \tag{6}$$

where ρ denotes the rest mass density and $\mathbf{u}^\flat$ the 1-form dual to the fluid 4-velocity. The precise form g takes in the support of (6) is not important here. We remark only that outside the support, the solution is both vacuum and spherically symmetric, and thus (by Birkhoff's theorem [11]) necessarily coincides with (5). The support of the matter has the interpretation of a collapsing star. If τ denotes the proper time of a freely falling observer $\gamma(\tau)$ on the boundary of the star, then its radius $R(\gamma(\tau)) \to 2m$ as $\tau \to \tau_{\text{critical}}$ for some $\tau_{\text{critical}} < \infty$. Thus, one does have to face the limit $r \to 2m$ in (5) after all!

The paper [12] did not quite make explicit what happens to freely falling observers γ when or after they reach $r = 2m$. It turns out, however, that this had in fact already been understood (at least in the vacuum region) by Lemaitre [9], who explicitly extended the metric (5) across $r = 2m$. It is in retrospect remarkable that this question caused as much confusion as it did, since it suffices to define

$$v = t + r + 2m \log|r - 2m|, \tag{7}$$

in which case the metric (5) transforms into

$$-(1-2m/r)(dv)^2+2dvdr+r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{8}$$

This metric is manifestly regular for all $r > 0$ and all $v \in \mathbb{R}$. We will discuss its behaviour at $r = 0$ later!

Even more astonishing than the initial confusion itself, however, was how long it took the correct solution to become common knowledge in the physics community! Indeed, as late as 1958, Lemaitre's work was being rediscovered, for instance by Finkelstein [6]. This turned out to be quite fortuitous for our story, as it was from a lecture of Finkelstein in London that Penrose was to learn about this extension [17].

Oppenheimer–Snyder à la Penrose

The starting point for Penrose's seminal [13] is precisely a lucid presentation of Oppenheimer–Snyder collapse, including Lemaitre's extension as he learned it from Finkelstein. The geometry of the spacetime is illustrated by a diagram reproduced here.

It is worth walking through this depiction: good visual aides are central to Penrose's work!

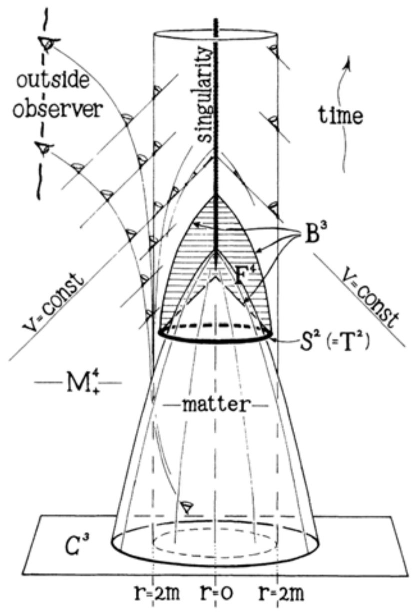


Diagram of Oppenheimer–Snyder spacetime from Penrose's paper [13]

The hypersurface C^3 is a 3-dimensional spacelike slice of spacetime $\mathcal{M}^4_+$, i.e. a hypersurface on which the induced metric is Riemannian; one may view C^3 as representing space at an instant. There is an initial ball of matter on C^3 of radius $R_0 > 2m$, and its world tube through spacetime is depicted (labelled 'matter'). This is where the fluid energy momentum (6) is supported. Outside the support, $\mathcal{M}^4_+$ is vacuum, and thus, described by the Lemaitre extension (8) of Schwarzschild. Note level surfaces $v = \text{const}$ depicted, with v increasing moving up.

As a Riemannian manifold, C^3 is as nice as can be; its metric is complete and asymptotically flat. (Asymptotic flatness, the condition that the metric approach Euclidean at large r, is the analogue of the boundary condition $\phi \to 0$ in Newtonian theory governed by (2).) Thus, we may view $\mathcal{M}^4_+$ as 'evolving' from a physically admissible initial state. In more technical terms, C^3 is in fact a *Cauchy hypersurface*, which, in pde language, means that its data determine uniquely the spacetime $(\mathcal{M}^4_+, g)$ as a solution of (1) with (6), and $(\mathcal{M}^4_+, g)$ is in fact the so-called *maximal Cauchy development* of these data. We will return to this issue later!

The cones depicted in the diagram are precisely the future light cones N^+_p at each spacetime point p. Drawing these cones allows us to pick out by sight worldlines and trajectories of light rays (such curves are collectively known as *causal*). Note that as depicted in the diagram, the N^+_p tilt inwards (with respect to the r coordinate) compared with flat Minkowski space, tilting more and more as $r \to 0$.

Considering the radius $R(v)$ of the support of the matter (6) as a function of v, one may moreover infer from the diagram that there is a V_{crit} (not labelled) for which $R(v) \to 0$ as $v \to V_{\text{crit}}$, while for $v > V_{\text{crit}}$, the spacetime is vacuum and thus completely described by (8). At least for $v > V_{\text{crit}}$, one sees easily that the future null cones N^+_p on $r = 2m$ are in fact tangent to $r = 2m$, with all other null directions pointing inside $r \le 2m$, i.e. $\mathbf{w}(r) \le 0$ for all $\mathbf{w} \in N^+_p$, where $\mathbf{w}(r)$ just denotes the action of vectors on functions by differentiation. For $r < 2m$, we have $\mathbf{w}(r) < 0$ for all $\mathbf{w} \in N^+_p$. Thus, if $\gamma(\tau)$ is a future causal curve and $r(\gamma(\tau_0)) < 2m$, $v(\gamma(\tau_0)) > V_{\text{crit}}$ then $r'(\gamma(\tau)) < 0$ for all $\tau \ge \tau_0$, and so $r(\gamma(\tau)) < 2m$. It follows in particular that there is a non-empty region of spacetime which cannot send signals to far away outside observers for which r is large. This region would later be named the *black hole region* [11].

Finally, we notice that all worldlines which enter the black hole region eventually reach $r = 0$. An easy computation with (8) reveals that this in fact happens in finite proper time, that is to say, given any worldline $\gamma(\tau)$ parametrised by proper time τ, then there exists a $\tau_{\max}$ such that $r(\gamma(\tau)) \to 0$ as $\tau \to \tau_{\max}$. Similarly, future null geodesics entering the interior of the black hole reach $r = 0$ in finite affine time, while there is the marginal case of those remaining for all affine time on the boundary, along which eventually $r = 2m$. This boundary came to be known as the *event horizon*. Examples of null geodesics lying on this horizon are depicted on the diagram.

We now finally turn to discuss $r = 0$. The spacetime $\mathcal{M}$ cannot be extended to include $r = 0$, at least not in a suitably regular fashion, as is clear by computing the Kretschmann scalar K, a contraction of the tensor $\mathrm{Riem} \otimes \mathrm{Riem}$, which equals $K = 48m^2r^{-6}$ and thus diverges as $r \to 0$. This thus represents a *singularity*, where physical quantities diverge, and presumably, general relativity itself breaks down.

Let us note already that there is another way of saying that something is 'wrong' with spacetime, without explicitly talking about the 'singularity' at $r = 0$. We say a future causal geodesic (i.e. one with $\gamma'(\tau) \in I^+_{\gamma(\tau)} \cup N^+_{\gamma(\tau)}$) is *future complete* if it can be extended to be defined on $[\tau_0, \infty)$, otherwise, *future incomplete*. Geodesics with $r(\gamma(\tau)) \to 0$ as $\tau \to \tau_{\max}$ are incomplete in view of the above, and since it contains such geodesics, $\mathcal{M}^4_+$ is itself said to be *future causally geodesically incomplete*.

In the above example, geodesic incompleteness seems intimately tied both with the black hole region and with the presence of the $r = 0$ 'singularity'. Future causal geodesics in Oppenheimer–Snyder turn out to be future incomplete if and only if they approach $r = 0$, in fact if and only if they enter the interior of the black hole region. In general, however, it is trivial to come up with spacetimes with future incomplete causal geodesics but which are in no reasonable sense 'singular' nor have black hole regions: one can just remove appropriate sets from Minkowski space. We shall return to this issue later! In the meantime, let us introduce Penrose's theorem.

Trapped surfaces and incompleteness

The Oppenheimer–Snyder spacetime depicted above is all well and good, but at the end of the day, it is just a single explicit solution of the equations (1)—and a very symmetric one at that. Moreover, the singular behaviour it exhibits corresponds to $r \to 0$, and it is natural to expect that behaviour there is very sensitive to perturbation away from symmetry. Indeed, on the eve of the appearance of [13], the general belief was that all this strange causal and singular behaviour—largely misunderstood in any case—was an artefact of symmetry [17].

The key to address, and finally frustrate, this expectation was a profound new concept, geometric in nature: that of a closed trapped surface.

Briefly, a *closed trapped surface* is a compact spacelike 2-surface T^2 (without boundary) such that its area element at every $p \in T^2$ is infinitesimally decreasing in both future null directions orthogonal to T^2. (Compare with usual spheres in Minkowski space decreasing in one and increasing in the other.) A more precise definition of this is given in the box.

In Oppenheimer–Snyder spacetime, any surface S^2 of constant (v, r) with $r < 2m$ lying in the vacuum region of the spacetime is in fact a closed trapped surface. (This is clear since $\mathbf{w}(r) < 0$ in this region for $\mathbf{w} \in N_p^+$.) An example of such a surface is labelled on the diagram. Let us already note, moreover, that the presence of such a surface is stable to perturbation of spacetime. This follows trivially from the compactness of S^2 and the fact that trappedness is defined by strict inequalities.

Trapped surfaces

If T^2 is a spacelike 2-surface (i.e. its induced metric is Riemannian), then for all $p \in T^2$ one can define two unique future null normals L and $\underline{L}$ to the tangent space of T^2 at p. We call T^2 *trapped* if $\mathrm{tr}\,(X, Y) \mapsto g(\nabla_X L, Y)$ and $\mathrm{tr}\,(X, Y) \mapsto g(\nabla_X \underline{L}, Y)$ are both negative, where X, Y are tangent to T^2. Note that the null geodesics generated by L and $\underline{L}$ span the two components of boundary of the causal future of T^2, and thus this definition can be interpreted as saying that the area element of T^2 is infinitesimally decreasing as it is flowed along either component of the boundary.

Given the notion of closed trapped surface, Penrose's theorem is incredibly simple to state, and, as it turns out, not so difficult to prove (see the Box at the end of the article), by methods of global geometry:

Theorem 1 (Penrose's incompleteness theorem [13]). *If $(\mathcal{M}, g)$ is sufficiently smooth, admits a non-compact Cauchy hypersurface, contains a closed trapped surface and satisfies the inequality* (4) *for all null* $\mathbf{v}$*, then it is future-causally geodesically incomplete.*

Closed trapped surfaces thus are 'surfaces of no return' which, once present, ensure incompleteness!

Note how the theorem's assumptions and conclusion are indeed exhibited for Oppenheimer–Snyder spacetime $(\mathcal{M}^4_+, g)$ itself. In particular, the curvature inequality (4) for null $\mathbf{v}$ follows from the Einstein equations (1) and the definition (6), in view of the fact that the fluid 4-velocity $\mathbf{u}$ satisfies $\mathbf{u} \in I_p^+$.

As remarked already, geodesic incompleteness in the special case of Oppenheimer–Snyder seems to be intimately connected to both black holes and singularities. It is thus tempting to interpret this theorem as predicting these. (Indeed, the traditional name for the above theorem, which we have avoided here for reasons we shall return to later, is the Penrose singularity theorem.) As we shall see, this interpretation is not in fact correct, and the true situation is far more interesting. Before trying to explain, let us introduce the evolutionary point of view, which will be essential for what follows.

The evolutionary point of view

The true significance of Penrose's theorem becomes apparent by interpreting it in an explicitly evolutionary context.

The precise language in order to do this was not in fact available in 1965, but was clarified a few years later in a paper [1] of Choquet-Bruhat and Geroch, which introduced the notion of the *maximal Cauchy development*. This important concept is explained further in the box. Briefly, given appropriate initial data for (1) on a 3-manifold C^3, this is the biggest spacetime $(\mathcal{M}, g)$ of (1), together with the matter equations, admitting C^3 as a Cauchy hypersurface. With this, one may now talk of a unique spacetime $(\mathcal{M}, g)$ which is 'predicted' by general relativity from initial data, resolving the ambiguity of domain, that, as discussed earlier, would allow for many trivial examples of geodesically incomplete spacetimes.

The maximal Cauchy development is the object to which one should apply Penrose's incompleteness theorem. We may in particular state the following:

Corollary 1. *For all initial data sufficiently close to data on C^3 in Oppenheimer–Snyder collapse, the resulting maximal Cauchy development $(\mathcal{M}, g)$ will still be future causally geodesically incomplete.*

In deducing the above, we have used also the fact that the presence of a closed trapped surface is stable not just to perturbation of spacetime but to perturbation of initial data, by general Cauchy stability arguments. Thus, geodesic incompleteness of the maximal Cauchy development is an inescapable prediction of the theory, following from assumptions expressible on initial data alone, robust to perturbation. Incompleteness cannot be avoided by perturbing the initial data.

Note that there are pure vacuum solutions like (8) which contain closed trapped surfaces. A more interesting question, however, is whether closed trapped surfaces can form in vacuum (3) from initial data which don't initially contain trapped surfaces, just like Oppenheimer–Snyder at the initial hypersurface $\mathcal{C}^3$. This was shown to be true in [3]. As in Corollary 1, it then follows that all initial data suitably close to those of [3] again lead to an incomplete maximal Cauchy development.

The initial value problem and the maximal Cauchy development

For convenience, let us restrict to the vacuum equations (3), although similar considerations apply for a wide class of Einstein matter systems. A *vacuum initial data set* is a triple $(\mathcal{C}^3, \bar{g}, K)$ with $\mathcal{C}^3$ a 3-manifold, $\bar{g}$ a Riemannian metric on $\mathcal{C}^3$ and K a symmetric 2-tensor, such that $\bar{g}$ and K satisfy the *vacuum constraint equations*. (These are the nontrivial relations arising for the first and second fundamental forms from differential geometry if $\mathcal{C}^3$ were a spacelike hypersurface of a Lorentzian manifold satisfying (3).) The theorem of Choquet-Bruhat and Geroch [1] states that given such a smooth $(\mathcal{C}^3, \bar{g}, K)$, there exists a unique smooth $(\mathcal{M}, g)$ which admits $\mathcal{C}^3$ as a Cauchy hypersurface, satisfies the Einstein vacuum equations (3), and is maximal in the sense that any other such spacetime isometrically embeds in $(\mathcal{M}, g)$ preserving $\mathcal{C}^3$. (The statement that $\mathcal{C}^3$ is a Cauchy hypersurface is simply the statement that all inextendible causal curves of $\mathcal{M}$ intersect $\mathcal{C}^3$ exactly once.)

The cosmic censorship conjectures

We have remarked already that Penrose's theorem is often misinterpreted as saying that black holes generically form or that 'singularities' (in the sense of local physics breaking down) generically arise. This is presumably because geodesic incompleteness in Oppenheimer–Snyder seems to be directly connected with both of these features.

What Penrose actually proved, however, is perhaps even more profound than what the press release [21] says. For it is fair to say that his theorem changed our very 'value system'. Whereas before, Oppenheimer–Snyder looked like the ultimate pathology, which would hopefully disappear once perturbed, Penrose showed us that we should not just tolerate black holes and singularities, but that we should in fact hope that black holes form, and what's more we should hope for singularities inside them—and the stronger those are, the better!

Why? Because, as we shall see, the alternative that his theorem allows is even worse.

Let us first understand the alternative to the presence of a black hole. We need look no further than negative mass Schwarzschild, i.e. the metric (5) with $m < 0$. Here, $r = 0$ can again be viewed as a singular boundary, but now one which is visible to outside observers. We say the spacetime possesses a 'naked singularity'. In contrast to the Oppenheimer–Snyder case, it would appear that one would need to go beyond general relativity to describe observations accessible to far-away observers.

Fortunately, the evolutionary point of view allows us to exclude the particular example of negative mass Schwarzschild outright, because it does not in fact arise as a maximal Cauchy development of complete asymptotically flat initial data $\mathcal{C}^3$. But who is to say that there do not exist spacetimes with naked singularities that do arise from such data, even perhaps from small perturbations of Oppenheimer–Snyder data as in Corollary 1?

The conjecture that naked singularities should not occur, or at least should generically not occur, is Penrose's original 'cosmic censorship' [16].

The conjecture can be nicely re-formulated [7] in the evolutionary context with the help of yet another fundamental concept introduced by Penrose, that of 'future null infinity' [14], typically denoted as $\mathcal{I}^+$, which under suitable circumstances can be attached as a conformal boundary of spacetime. Considering $\mathcal{I}^+$ is extremely useful for formulating the laws of gravitational radiation, but it can also serve as a stand-in for the role of far-away observers when defining black holes. For instance, one can define the black hole region as $\mathcal{M} \setminus J^-(\mathcal{I}^+)$, where J^- denotes causal past, although one should also impose that $\mathcal{I}^+$ itself is complete [7], loosely related to the statement that far-away freely falling observer worldlines be future complete. In this language, the defining feature of a 'naked singularity' is that information from there would arrive at $\mathcal{I}^+$ at finite affine time, rendering $\mathcal{I}^+$ incomplete. The modern formulation [2] of Penrose's conjecture, adapted to the evolutionary setting, is

Conjecture 1 (Weak cosmic censorship). For generic asymptotically flat initial data for (3) (or more generally (1) coupled to suitable matter), the maximal Cauchy development possesses a complete $\mathcal{I}^+$.

The word 'weak' is traditional, meant to distinguish this statement from the later 'strong' Conjecture 2. The genericity assumption turns out to be necessary even for vacuum (3) in view of recent examples [18] of naked singularities, related to the previous spherically symmetric [2]. A highly nontrivial symmetric toy version of Conjecture 1 has been proven [2]. The general problem, however, remains completely open!

The other 'good' feature of Oppenheimer–Snyder is perhaps even more difficult at first sight to recognise as good: as remarked earlier, all incomplete observers in Oppenheimer–Snyder spacetime fall into the singularity $r = 0$. Moreover, not only do they fall into the singularity, but an actual physical observer with arms and legs would be destroyed at $r = 0$, torn apart by infinite tidal deformations [11].

While this is hardly 'good' for that poor observer, one can argue that it is very 'good' for general relativity as a classical physics theory! For perversely, it gives the theory an attractive epistemological closure. Observers either live forever or are torn apart. Either way, their future as classical observers is completely described and determined by the theory—for as long as it makes sense to talk about them.

On the other hand, let us contemplate a very different situation. Imagine that there exist incomplete observers who encounter no singularity. The theory doesn't say what happens to them, but surely something must. What determines this?

Remarkably, this phenomenon is precisely what occurs in the celebrated Kerr solution, a 2-parameter family of vacuum metrics generalising (5). See [11]. Here the maximal Cauchy development of (2-ended) asymptotically flat initial data satisfies the assumptions of Theorem 1. It is thus incomplete, but it is extendible smoothly as a Lorentzian manifold, in fact as a vacuum solution, in fact it is extendible so that all incomplete observers may live another day in the extension. These extensions fail however to admit the initial hypersurface as a Cauchy hypersurface. The boundary of the maximal Cauchy development in such an extension is thus known as a *Cauchy horizon*. This notion is due to Hawking [8]. This strange situation can be understood better considering the solution's so called *Penrose diagram*, an influential way of representing the geometry of spacetimes which is beyond the scope of this article. See [14].

In a certain sense, Cauchy horizons can be viewed as 'worse' than singularities. The Kerr case is extremely pernicious in that not a single incomplete observer encounters anything that would even suggest that the regime of classical relativity has been exited. So it is a spectacular and seemingly inexplicable failure of the predictability of the theory, in no way accompanied by singularity. It was again Penrose who discovered a possible way out, noticing that Kerr's Cauchy horizon is subject to a blue-shift instability [15]. This led him to put forth his 'strong cosmic censorship', which in its evolutionary formulation [2] is the conjecture that, for generic initial data, Cauchy horizons should not occur, i.e.

Conjecture 2 (Strong cosmic censorship). For generic asymptotically flat initial data (3) (or more generally for (1) coupled to suitable matter), the maximal Cauchy development is inextendible as a suitably regular Lorentzian manifold.

(Note that as stated, Conjecture 2 is not in fact 'stronger' than Conjecture 1.) To make Conjecture 2 precise, one must specify how 'suitably regular' should be interpreted. Effectively, this corresponds to specifying how 'singular' the boundary of spacetime should be. The strongest formulation would have it that the boundary is so singular so that all incomplete observers are torn apart, just as in Schwarzschild, providing the definitive closure described above. This would correspond to the C^0 formulation of Conjecture 2, where 'suitably regular' just means 'continuous'. Unfortunately, this version is in fact false [4]. A weaker formulation is proposed in [3] and there are some positive non-trivial results [10] for a symmetric toy version. As with Conjecture 1, however, a positive resolution of a suitable version of Conjecture 2 remains completely open!

In conclusion, Penrose's theorem may not imply that black holes form or even true 'singularities' develop, but it very much taught us to live with black holes and singularities—indeed, to love them. Black holes are not themselves the singularity, but they are what protects us from singularity, and singularity in turn is what protects us from a much more dangerous kind of incompleteness associated with loss of predictability. The wide acceptance of black holes, now central both in astronomy and even popular culture, ultimately stems from this. It is difficult to imagine a more impactful contribution to general relativity—a more ironic reversal—arising from the proof of a mathematical theorem of pure geometry. With [13], our view of gravitational collapse irreversibly changed, and the resulting weak and strong cosmic censorship conjectures will undoubtedly remain the main source of inspiration for further progress in classical general relativity for many years to come.

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The proof of Penrose's incompleteness theorem

The proof of Theorem 1 can be thought of as an ingenious adaptation of ideas from the Bonnet-Myers theorem to the setting of Lorentzian geometry in the presence of a closed trapped surface. We sketch the proof here. One considers the so-called causal future $J^+(T)$ of the closed trapped surface T. If spacetime admits a Cauchy hypersurface C^3, then one can show that the boundary B of this set canonically projects to C^3. The variational theory of null geodesics, however, yields that B is covered by null geodesic segments none of which may extend beyond its first focal point to T. The curvature assumption (4) implies on the other hand that null geodesics emanating from T must develop focal points if they can be extended to arbitrary affine time τ. Thus, if these null geodesics are future complete it follows that every null geodesic emanating from T develops a focal point, whence one can show that B is compact without boundary, whence it cannot project to the non-compact C^3. This contradicts the null geodesic completeness. See also Penrose's Adams Prize essay [14]. There are many extensions of this result, starting from work of Hawking [8]. See the survey [20].

The Mathematics of Floating-Point Arithmetic

NICHOLAS J. HIGHAM

Floating-point arithmetic is ubiquitous in computing and its implementation in evolving computer hardware remains an active area of research. Its mathematical properties differ from those of arithmetic over the real numbers in important and sometimes surprising ways. We explain what a mathematician should know about floating-point arithmetic, and in particular we describe some of its not so well known algebraic properties.

Floating-point arithmetic has been in use for over seventy years, having been provided on some of the earliest digital computers. For the first half of that period there was tremendous variation in floating-point formats and in the ways the arithmetics were implemented. Some floating-point arithmetics could produce anomalous results and it was difficult or impossible to write programs that were portable, i.e., produce similar results on different computer systems with little or no change.

The 1985 ANSI/IEEE Standard for Binary Floating-Point Arithmetic [9] provided binary floating-point formats and precise rules for how to carry out arithmetic on them. Carefully designed over several years by a committee of experts, it brought much-needed order to computer arithmetic and within a few years virtually all computer manufacturers had adopted it.

From a mathematical perspective we can ask several questions about a floating-point arithmetic.

- What mathematical properties does it have compared with exact arithmetic?
- What sort of mathematical structure is it?
- How can we understand the accuracy of computations carried out in it?

First, we need to define the set of numbers under consideration. A floating-point number system is a finite subset $F = F(\beta, t, e_{\min}, e_{\max})$ of the real numbers $\mathbb{R}$ whose elements have the form

$$x = \pm m \times \beta^{e-t+1}. \tag{1}$$

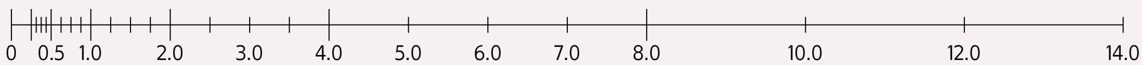
Here, β is the base, which is 2 on virtually all current computers. The integer t is the precision and the integer e is the exponent, which lies in the range $e_{\min} \le e \le e_{\max}$. The significand m is an integer satisfying $0 \le m \le \beta^t - 1$. To ensure a unique representation for each nonzero $x \in F$ it is assumed that $m \ge \beta^{t-1}$ if $x \ne 0$, so that the system is *normalised*.

The reason for the "+1" in the exponent of (1), which could be avoided by redefining $e_{\min}$ and $e_{\max}$, is for consistency with the IEEE standard. The standard also requires that $e_{\min} = 1 - e_{\max}$.

The largest and smallest positive numbers in the system are $x_{\max} = \beta^{e_{\max}}(\beta - \beta^{1-t})$ and $x_{\min} = \beta^{e_{\min}}$, respectively. Two other important quantities are $u = \frac{1}{2}\beta^{1-t}$, the *unit roundoff*, and $\epsilon = \beta^{1-t}$, the *machine epsilon*, which is the distance from 1 to the next larger floating-point number. See the box for a simple example of a floating-point number system.

A toy floating-point number system

This diagram shows the nonnegative (normalised) numbers in a binary floating-point number system with $t = 3$, $e_{\min} = -2$, and $e_{\max} = 3$. Note that the floating-point numbers are equally spaced between powers of 2 and the spacing increases by a factor of 2 at each power of 2. Here, the unit roundoff is $u = 0.125$ and the machine epsilon $\epsilon = 0.25$; these are the distances from 1 to the next smaller and next larger floating-point number, respectively.



The system F can be extended by including *subnormal numbers*, which have the minimum exponent and $m < \beta^{t-1}$, so that they are not normalised; they fill the gap between 0 and $x_{\min}$ with numbers having a constant spacing $\beta^{e_{\min}+1-t}$. The IEEE standard includes subnormal numbers.

We assume throughout the rest of this article that F is a binary system ($\beta = 2$), and that it follows the IEEE standard by including the special numbers $\pm\infty$ and NaN (Not a Number). The numbers $\pm\infty$ obey the usual mathematical conventions regarding infinity, such as $\infty + \infty = \infty$, $(-1) \times \infty = -\infty$, and $(\text{finite})/\infty = 0$. A NaN is generated by operations such as $0/0$, $0 \times \infty$, ∞/∞, $(+\infty) + (-\infty)$, and $\sqrt{-1}$.

We also assume, again following the IEEE standard, that the results of the elementary operations of addition, subtraction, multiplication, division, and square root are the same as if they were carried out to infinite precision and then rounded back to F, and that rounding of $x \in \mathbb{R}$ to F is done by mapping to the nearest floating-point number, with ties broken by rounding to the floating-point number with a zero last bit. We denote the operation of rounding by fl. With a standard abuse of notation, $\mathrm{fl}(\mathrm{expr})$, where expr is an arithmetic expression, is also used to denote the result of evaluating expr in floating-point arithmetic in some specified order.

With the inclusion of ∞ and NaN, F is a closed number system: every floating-point operation on numbers in F produces a result in F.

Algebraic properties

The real numbers form a field under addition and multiplication. It is natural to ask what sort of mathematical structure floating-point numbers form under the elementary (floating-point) arithmetic operations. To investigate this question we will explore some basic algebraic properties of floating-point arithmetic.

Let $a, b \in F$. By definition, $\mathrm{fl}(a+b)$ and $\mathrm{fl}(b+a)$ are equal, as are $\mathrm{fl}(a*b)$ and $\mathrm{fl}(b*a)$. However, with three numbers the usual rules of arithmetic break down: $\mathrm{fl}((a+b)+c))$ is not necessarily equal to $\mathrm{fl}(a+(b+c))$ and $\mathrm{fl}((a*b)*c)$ is not necessarily equal to $\mathrm{fl}(a*(b*c))$. In other words, floating-point addition and multiplication are not associative. For example, in our toy system $\mathrm{fl}(0.25+(8.0-7.0)) = 1.25$ but $\mathrm{fl}((0.25+8.0)-7.0) = \mathrm{fl}(8.0-7.0) = 1.0$. Similarly, $\mathrm{fl}(a*(b+c))$ is not necessarily equal to $\mathrm{fl}(a*b+a*c)$, so the distributive law does not hold.

If $a > b > 0$ then $\mathrm{fl}(a+b) > a$ need not hold. The reason is that b may be so small that a is unchanged after adding b and rounding. Indeed $\mathrm{fl}(1+x) = 1$ for any positive floating-point number $x < u$.

Does the equation $x*(1/x) = 1$ hold in floating-point arithmetic? The following result of Edelman says that it may just fail to do so [6, Prob. 2.12].[1]

Theorem 2. *For $1 < x < 2$, $\mathrm{fl}(x*(1/x))$ is either 1 or $1-\epsilon/2$*

A closely related question is which floating-point numbers are possible reciprocals of $x \in F$. Muller [10] showed that when $1/x \notin F$ there are two possibilities.

Theorem 3. *The only $z \in F$ that can satisfy $\mathrm{fl}(x*z) = 1$ are $\min\{y : y \ge 1/x, y \in F\}$ and $\max\{y : y \le 1/x, y \in F\}$.*

Perhaps surprisingly, these two possible z can simultaneously give equality, so a floating-point number can have two floating-point reciprocals. In fact, of the 24 positive numbers in the toy system, eight have two floating-point reciprocals; for example, $y = 0.625$ and $y = 0.75$ both satisfy $\mathrm{fl}(1.5*y) = 1$, and these are the two nearest floating-point numbers to $1/1.5 = 2/3$.

Now consider the computation $n*(m/n)$, where m and n are integers. If m/n is a floating-point number then $\mathrm{fl}(n*\mathrm{fl}(m/n)) = \mathrm{fl}(n*(m/n)) = \mathrm{fl}(m) = m$, as no rounding is needed. Kahan proved that the same identity holds for many other choices of m and n [4, Thm. 7].

Theorem 4. *Let m and n be integers such that $|m| < 2^{t-1}$ and $n = 2^i + 2^j$ for some i and j. Then $\mathrm{fl}(n*\mathrm{fl}(m/n)) = m$.*

The sequence of allowable n begins $2, 3, 4, 5, 6, 8, 9, 10, 12, 16, 17, 18, 20$ (and is A048645 in the On-Line Encyclopedia of Integer Sequences), so Theorem 3 covers many common cases. Nevertheless, the equality does not hold in general.

It can be shown that $\mathrm{fl}\left(\sqrt{x^2}\right) = |x|$ for $x \in F$, as long as x^2 does not underflow (round to zero) or overflow

[1] We give a minimal set of references in this article. Original sources can be found in the references cited.

(exceed the largest element of F), but $\mathrm{fl}((\sqrt{x})^2) = |x|$ is not always true (by the pigeonhole principle) [6, Prob. 2.20].

Rounding (to nearest) is monotonic in that for $x \in \mathbb{R}$ and $y \in \mathbb{R}$, the inequality $x \le y$ implies $\mathrm{fl}(x) \le \mathrm{fl}(y)$. As a result, it is easy to show that

$$x \le \mathrm{fl}\left(\frac{x+y}{2}\right) \le y.$$

While this result holds for base 2, the computed midpoint can be outside the interval for base 10.

The inequality $\mathrm{fl}\left(x/\sqrt{x^2+y^2}\right) \le 1$ always holds, barring overflow and underflow [6, Prob. 2.21]. Although this fact may seem unremarkable, in some pre-IEEE standard arithmetics this inequality could be violated, causing failure of an attempt to compute one of the angles in a right-angled triangle with shortest sides of lengths x and y as $\mathrm{acos}\left(x/\sqrt{x^2+y^2}\right)$.

The following result of Sterbenz guarantees that subtraction is exact for two numbers that are at most a factor 2 apart.

Theorem 5. *If x and y are floating-point numbers with $y/2 \le x \le 2y$ then $\mathrm{fl}(x-y) = x-y$ (assuming $x-y$ does not underflow).*

This result is notable because inaccurate results are often blamed on subtractive cancellation. It is not the subtraction itself that is dangerous but the way it brings into prominence errors already present in the numbers being subtracted, making these errors much larger relative to the result than they were to the arguments.

Finally, we note that a NaN is unique among elements of F in that it compares as unordered (including unequal to) everything, including itself. In particular, a statement "if x = x" returns false when x is a NaN. This is why some programming languages provide a function to test for a NaN (e.g., isnan in MATLAB).

We conclude that floating-point arithmetic is a rather strange mathematical object that does not correspond to any standard algebraic structure. These examples could make one pessimistic about our ability to carry out reliable numerical computations. Fortunately, these peculiar features of floating-point arithmetic are not a barrier to its successful use or to deriving satisfactory error bounds, as we now illustrate.

Experimenting with different floating-point arithmetics

It is instructive to run experiments in floating-point arithmetics based on different parameters t, $e_{\min}$, and $e_{\max}$.

We used the MATLAB function chop[a] [8] for this purpose. This function rounds single or double precision numbers to a specified target format (limited to $e_{\min} = 1 - e_{\max}$) and supports several rounding modes and other options. A library CPFloat offers similar functionality for C [3].

[a] `https://github.com/higham/chop`

Error analysis

If we want to understand the effects of rounding errors on a floating-point computation then we need to analyse how the individual rounding errors interact and propagate. A natural way to try to do this is to define "circle operators" $\oplus$, $\ominus$, $\otimes$, and $\oslash$ by

$$\begin{aligned} x \oplus y &= \mathrm{fl}(x+y), \quad & x \ominus y &= \mathrm{fl}(x-y), \\ x \otimes y &= \mathrm{fl}(x*y), \quad & x \oslash y &= \mathrm{fl}(x/y), \end{aligned}$$

and then rewrite the expressions being evaluated in terms of these operators. For example, consider the evaluation of the cubic polynomial $p = ax^3 + bx^2 + cx + d$ by Horner's rule as $p = ((ax+b)x+c)x+d$ (using 6 operations instead of the 8 required if we explicitly form x^3 and x^2). We would then write the computed $\widehat{p}$ as

$$\widehat{p} = \big((a \otimes x \oplus b) \otimes x \oplus c\big) \otimes x \oplus d.$$

However, we cannot easily simplify this expression because the circle operators do not satisfy the associative or distributive laws.

The right way to do error analysis is to obtain equations in terms of the original operators and individual rounding errors. We need the result that [6, Thm. 2.2]

$$x \in \mathbb{R} \implies \mathrm{fl}(x) = x(1+\delta), \quad |\delta| \le u, \tag{2}$$

where u is the unit roundoff. Since $\mathrm{fl}(x \mathbin{\mathrm{op}} y)$ is defined to be the rounded exact value, it follows that for $\mathrm{op} = +, -, *, /$ we have

$$\mathrm{fl}(x \mathbin{\mathrm{op}} y) = (x \mathbin{\mathrm{op}} y)(1+\delta), \quad |\delta| \le u. \tag{3}$$

This is the standard model of floating-point arithmetic used for rounding error analysis. Note that it does not fully characterise floating-point arithmetic because (2) does not fully characterise rounding: for some x, two different floating-point numbers y satisfy $y = x(1+\delta)$ with $|\delta| \le u$. It is possible to use more refined models of floating-point arithmetic that more fully reflect the definition of rounding, which tends to give results with slightly smaller constants at the cost of a much more complicated analysis. The main purpose of a rounding error analysis, though, is to gain insight into accuracy and stability rather than to optimise constants.

For our cubic example, we can use the model to write

$$\begin{aligned}\widehat{p} &= \Big(\Big(\big(ax(1+\delta_1)+b\big)(1+\delta_2)x(1+\delta_3)+c\Big)\\ &\qquad \times (1+\delta_4)x(1+\delta_5)+d\Big)(1+\delta_6)\\ &= ax^3(1+\delta_1)(1+\delta_2)(1+\delta_3)(1+\delta_4)(1+\delta_5)(1+\delta_6)\\ &\quad + bx^2(1+\delta_2)(1+\delta_3)(1+\delta_4)(1+\delta_5)(1+\delta_6)\\ &\quad + cx(1+\delta_4)(1+\delta_5)(1+\delta_6)+d(1+\delta_6),\end{aligned}$$

where $|\delta_i| \le u$ for all i. This expression is rather messy, but we can rewrite it as

$$\begin{aligned}\widehat{p} &= ax^3(1+\theta_6)+bx^2(1+\theta_5)\\ &\quad + cx(1+\theta_3)+d(1+\theta_1),\end{aligned} \tag{4}$$

where the θ_i are bounded by the following lemma [6, Lem. 3.1].

Lemma 1. *If $|\delta_i| \le u$ and $\rho_i = \pm 1$ for $i = 1\colon n$, and $nu < 1$, then*

$$\prod_{i=1}^{n}(1+\delta_i)^{\rho_i} = 1+\theta_n,$$

where

$$|\theta_n| \le \frac{nu}{1-nu} =: \gamma_n.$$

Applying the lemma to (4), we obtain

$$|p-\widehat{p}| \le \gamma_6(|a||x|^3+|b||x|^2+|c||x|+|d|), \tag{5}$$

which is a concise and easily interpretable error bound, with constant $\gamma_6 = 6u + O(u^2)$.

With careful use of the lemma, the profusion of $1+\delta_i$ terms that arise in a rounding error analysis can be kept under control and manipulated, using the usual rules of arithmetic, into a useful bound.

Fused multiply-add operation

Since the 1990s some processors have provided a *fused multiply-add* (FMA) operation that computes $x + y * z$ with just one rounding error instead of two, so that

$$\mathrm{fl}(x+y*z) = (x+y*z)(1+\delta), \quad |\delta| \le u.$$

The motivation for an FMA is speed, as it is implemented in such a way as to take the same time as a single multiplication or addition.

When an FMA is used the number of rounding errors in a typical computation is halved. Our cubic polynomial can be evaluated with three FMAs, giving

$$\begin{aligned}\widehat{p} &= \Big(\Big(\big(ax+b\big)(1+\delta_1)x+c\Big)\\ &\qquad \times (1+\delta_2)x+d\Big)(1+\delta_3)\\ &= (ax^3+bx^2)(1+\theta_3)+cx(1+\theta_2)\\ &\qquad + d(1+\theta_1),\end{aligned}$$

which is more favourable than (4).

Although it generally brings improved accuracy, an FMA can also lead to some unexpected results.

If we compute the modulus squared of a complex number from the formula

$$(x+\mathrm{i}y)^*(x+\mathrm{i}y) = x^2+y^2+\mathrm{i}(xy-yx)$$

then the result is real, because $\mathrm{fl}(xy) = \mathrm{fl}(yx)$. But if an FMA is used in evaluating $xy-yx$ then the imaginary part may evaluate as nonzero.

Similarly, if the discriminant $b^2 - 4ac$ of a quadratic is nonnegative then the computed result is guaranteed to be nonnegative by the monotonicity of floating-point arithmetic, but with an FMA the result can be negative.

Error analysis strategy

Even with the use of Lemma 1, rounding error analysis can be tedious, and it is natural to ask whether it can be automated. Can we harness a computer to carry out the necessary manipulations? For focused classes of algorithms some progress has been made [1], but in general the task is difficult or impossible to automate. The reason is that the hardest part of an error analysis is deciding what one wants to prove. For the evaluation of the cubic we obtained a bound (5) on the error in the computed p, known as the forward error. Along the way we obtained (4), which is a backward error result: it shows that the computed $\widehat{p}$ is the exact result for a polynomial with perturbed coefficients $a(1+\theta_6)$, $b(1+\theta_5)$, $c(1+\theta_3)$, and $d(1+\theta_1)$, and it bounds the size of the relative perturbations by γ_6.

In general, for a computation $y = f(x)$, where x and y are vectors (say), we have three measures of error for the computed $\widehat{y}$:

- forward error: $\|y - \widehat{y}\|/\|y\|$,
- backward error:
$$\min\left\{\frac{\|\Delta x\|}{\|x\|} : \widehat{y} = f(x + \Delta x)\right\},$$
- mixed backward–forward error: the smallest ϵ for which there exist Δx and Δy such that
$$\widehat{y} + \Delta y = f(x + \Delta x), \qquad \frac{\|\Delta x\|}{\|x\|} \le \epsilon, \qquad \frac{\|\Delta y\|}{\|\widehat{y}\|} \le \epsilon.$$

Depending on the problem, any one of these errors may be the best one to bound in a rounding error analysis, or perhaps the only one that it is feasible to bound. Determining the right approach and working out how to achieve a result that is readable, understandable, and insightful can be difficult.

Backward error analysis was developed by J. H. Wilkinson in the 1950s and 1960s [5]. It has the attractive feature of decoupling the numerical stability properties of an algorithm from the conditioning of the underlying problem (its sensitivity to perturbations in the data).

What is quite remarkable is that despite the strange behaviour of floating-point arithmetic illustrated above, it is possible to carry out rounding error analysis of a wide variety of algorithms and obtain useful results.

Recovering the error

The sum $s = a + b$ of $a, b \in F$ is not in general in F, so the computed sum $\widehat{s} = \mathrm{fl}(a + b)$ may be inexact.

However, the error $e = s - \widehat{s}$ is in F, and for $|a| \ge |b|$ it can be computed (exactly) as [11, Sec. 4.3.1]
$$e = b - (\widehat{s} - a),$$
so that
$$a + b = \widehat{s} + e.$$

This computation is known as $\mathrm{Fast2Sum}$. Let us denote it by $[s, e] = \mathrm{Fast2Sum}(a, b)$. (There are other, more complicated, ways of computing e that do not require $|a| \ge |b|$.)

Of course, if we try to form $\mathrm{fl}(\widehat{s} + e)$ then we will just obtain $\widehat{s}$, because $\widehat{s}$ is the best floating-point representation of $a+b$. However, in a sequence of operations we can add the error from an earlier operation into a later operation, where it can potentially have an effect.

An important usage of Fast2Sum is in *compensated summation*, proposed by Kahan in 1965, which computes $\sum_{i=1}^{n} x_i$ by

```
s = x_1
e = 0
for i = 2: n
    t = x_i + e
    [s, e] = Fast2Sum(s, t)
end
```

For standard recursive summation the computed $\widehat{s}$ satisfies
$$|s - \widehat{s}| \le c_n u \sum_{i=1}^{n} |x_i| + O(u^2)$$
with $c_n = n - 1$, whereas the computed $\widehat{s}$ from compensated summation satisfies the same bound with $c_n = 2$ (even though compensated summation does not sort the arguments of $\mathrm{Fast2Sum}$). For large n, this reduction in the constant makes a significant difference.

The 2019 revision of the IEEE standard includes so-called augmented arithmetic operations for addition, subtraction, and multiplication, which (like $\mathrm{Fast2Sum}$) return both the computed result and the error in it.

Probabilistic analysis and stochastic rounding

A numerical computation with $n \times n$ matrices usually has a rounding error bound proportional to $c_n u$ with c_n growing at least linearly. Traditionally, numerical computations have been done in single precision arithmetic or double precision arithmetic, with unit roundoffs u of order 10^{-8} or 10^{-16}, respectively. Hence $nu \ll 1$ for practical problems.

However, half precision arithmetic is now increasingly available in hardware, with u of order 10^{-3} for the bfloat16 format and 10^{-4} for the IEEE half precision format. In these arithmetics, $nu = 1$ for quite modestly-sized problems, and in these cases an error bound proportional to nu provides no information.

Mixed-precision algorithms

It is becoming common for computer systems to offer half precision, single precision, and double precision floating-point arithmetics in hardware, possibly with quadruple precision arithmetic in software. In designing algorithms we wish to exploit the speed of execution of lower precision arithmetic while ensuring that enough higher precision is used to deliver a result of the desired accuracy. Rounding error analysis, parametrised by the unit roundoffs for the different precisions, helps to identify suitable algorithms.

Traditional rounding error bounds, such as those above for Horner's rule, are worst-case bounds. As Stewart observes [12], "To be realistic, we must prune away the unlikely. What is left is necessarily a probabilistic statement." The idea of obtaining probabilistic rounding error bounds by modelling rounding errors as random variables is not new, but a rigorous treatment producing bounds valid for any dimension has only recently been developed, by Connolly, Higham, and Mary [2], [7]. This analysis proves that under the assumption that the rounding errors are mean independent random variables of mean zero, error bounds with constants $\sqrt{f(n)}u$ hold with high probability in place of worst-case bounds $f(n)u$.

A form of rounding called *stochastic rounding* has recently been finding use in deep learning and other areas. It rounds a number lying between two adjacent floating-point numbers $a < b$ to a with a probability proportional to the distance to b, and conversely for b. Stochastic rounding is somewhat worse behaved than round to nearest vis-à-vis its algebraic properties for individual operations. However, the random nature of the rounding is beneficial. It can be shown [2] that the rounding errors from stochastic rounding are random variables satisfying both the mean independence and the mean zero assumptions, so that the $\sqrt{f(n)}u$ bounds hold unconditionally. This means that stochastic rounding can provide more accurate results than round to nearest for large problems.

As a simple example, we computed $\sum_{i=1}^{10^4} x_i$ in IEEE half precision arithmetic, where x_i is $1/i$ rounded to half precision with round to nearest. The sum computed with round to nearest had relative error 2.7×10^{-1}, whereas the minimum, mean, and maximum errors over ten sums computed with stochastic rounding were 2.2×10^{-3}, 1.2×10^{-2}, 3.0×10^{-2}, respectively. In this example, round to nearest suffers from stagnation, whereby the smallest terms cannot change the computed partial sum. By contrast, stochastic rounding gives all terms a nonzero probability of increasing the sum, and in fact it does so in just the right way to ensure that the expected value of the computed sum is the exact sum [2].

Outlook

The provision of half precision floating-point arithmetic in hardware is motivated by machine learning, where its greater speed is proving beneficial despite its lower accuracy. Half precision can also be exploited in general scientific computing, but rounding error analysis is needed to determine whether sufficiently accurate results are being computed.

An example of how half precision arithmetic can be harnessed to great effect is the HPL-AI Mixed Precision Benchmark[2], which is one of the benchmarks that the TOP500 project uses to rank the world's most powerful supercomputers. This benchmark solves a double precision nonsingular linear system $Ax = b$ of order n using an LU factorisation computed in half precision and it refines the solution using iterative refinement in double precision. As of November 2020, the world record

[2] `https://icl.bitbucket.io/hpl-ai/`

execution rate for the benchmark is 2.0 ExaFlop/s (2×10^{18} floating-point operations per second, where most of the operations are half precision ones) for a matrix of size $16{,}957{,}440$, which was achieved by the Fugaku supercomputer in Japan. For a successful benchmark run, the relative residual $\|A\widehat{x} - b\|/(\|A\|\|\widehat{x}\| + \|b\|)$ of the computed $\widehat{x}$ must be no larger than a threshold that is about 10^{-8} in this case. So after approximately $2n^3/3 \approx 3 \times 10^{21}$ floating-point operations, Fugaku's computed solution $\widehat{x}$ had a small residual, which is a testament to effectiveness of floating-point arithmetic given that each half precision operation has a relative error of order 10^{-4}.

Despite floating-point arithmetic having some strange mathematical properties, seventy years of experience show that it usually works well in practice, and it is supported by rigorous mathematical analysis—both worst-case and probabilistic. With hardware implementations of floating-point arithmetic evolving constantly and new algorithms regularly being developed, interesting mathematical questions will continue to arise over the coming years.

Acknowledgements

I thank Michael Connolly, Massimilano Fasi, Sven Hammarling, Theo Mary, Mantas Mikaitis, and Srikara Pranesh for suggesting improvements to a draft of this article. This work was supported by the Royal Society and Engineering and Physical Sciences Research Council grant EP/P020720/1.

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Nicholas J. Higham

Nick is Royal Society Research Professor and Richardson Professor of Applied Mathematics in the Department of Mathematics at the University of Manchester. His current research interests include mixed precision numerical linear algebra algorithms. He blogs about applied mathematics at `https://nhigham.com/`. Nick shudders to recall that in some pre-IEEE standard computer arithmetics one could have $\mathrm{fl}(1.0 * x) \neq x$ for a floating-point number x.

Random Lattices in the Wild: from Pólya's Orchard to Quantum Oscillators

JENS MARKLOF

Point processes are statistical models that describe the distribution of discrete events in space and time. Applications are everywhere, from galaxies to elementary particles. My aim here is to convince you that there is an exotic but interesting class of point processes — random lattices — that have fascinating connections with various branches of mathematics and some basic models in physics.

So what is a random lattice? First of all, a *lattice* in dimension one is any non-trivial discrete subgroup of the additive group of real numbers $\mathbb{R}$. (*Non-trivial* means anything but the group of one element.) The additive group of integers $\mathbb{Z}$ is an example and, up to rescaling by a constant factor, it is in fact the only example. Now in order to turn $\mathbb{Z}$ into a random object, let us translate $\mathbb{Z}$ by a real number α to obtain the set $\mathcal{S}(\alpha) = \mathbb{Z} + \alpha$, and then view α as a random variable uniformly distributed in the unit interval $[0,1]$. The choice of the unit interval is natural since α and $\alpha+1$ will lead to the same shifted lattice $\mathcal{S}(\alpha)$. With this, $\mathcal{S}(\alpha)$ becomes a random set, which we take (for the purposes of this discussion) to be synonymous with *random point process*. One can check that $\mathcal{S}(\alpha)$ is a translation-stationary random point process, i.e., $\mathcal{S}(\alpha)+t$ has the same distribution as $\mathcal{S}(\alpha)$ for every choice of $t \in \mathbb{R}$ — a simple consequence of the fact that α is assumed to be uniformly distributed in $[0,1]$. A random point process describes the probability of finding k points in a given set B. In the present setting, for B a bounded interval of length $|B|$ and integer $k \geq 0$, we have that

$$\mathbb{P}\Big(|\mathcal{S}(\alpha) \cap B| = k\Big) = \max\Big(1 - \big|k - |B|\big|\,,\ 0\Big).$$

It is not difficult to see that the expected number of points in B is $|B|$, which means that the process has *intensity one* — compare this with the corresponding probabilities for a Poisson process!

The above construction has produced a simple instance of a point process in $\mathbb{R}$. Independent superpositions of one-dimensional randomly shifted lattices explain for example the limiting gap distribution of the fractional parts of the sequence $\log n$, with $n = 1,2,3\ldots$ [14]. But the fun really starts in dimension two!

Poisson process

A homogeneous Poisson process with intensity one in $\mathbb{R}$ can be realised as a sequence of random points where the distances between consecutive points are independent random variables with an exponential distribution. That is, the probability that a gap is larger than s is e^{-s}. It follows that the probability of having k points in the interval B is given by the Poisson distribution

$$\frac{|B|^k}{k!}\,\mathrm{e}^{-|B|}.$$

Two-dimensional random lattices

To construct a two-dimensional random lattice, we begin with the integer lattice $\mathbb{Z}^2$. We could proceed as before and define a random point process in $\mathbb{R}^2$ by shifting $\mathbb{Z}^2$ randomly by a vector $\boldsymbol{\alpha}$, say, uniformly distributed in $[0,1]^2$. This is fine, but there is a more interesting avenue. Unlike in dimension one, we have a non-trivial group of linear volume-preserving transformations acting on $\mathbb{R}^2$. We can use this action, rather than the group of translations as above, to randomise $\mathbb{Z}^2$ and thus produce a two-dimensional random lattice with a fundamental cell of volume one. Here is how it works. We represent elements in $\mathbb{R}^2$ as row vectors $\boldsymbol{x} = (x_1, x_2)$. A linear transformation is then represented by real matrix multiplication from the right,

$$\boldsymbol{x} \mapsto \boldsymbol{x}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ax_1 + cx_2, bx_1 + dx_2).$$

Volume is preserved if and only if the determinant has modulus one, that is $|ad - bc| = 1$. We will only need to consider the case where also orientation is preserved, which means $ad - bc = 1$. Such matrices form a group, which we will label as $\mathrm{SL}(2,\mathbb{R})$. L stands for *linear* and S for *special* (referring to the unit determinant). To produce our first example of a random lattice in $\mathbb{R}^2$, consider the sheared lattice

$$\mathcal{P}_1(u) = \mathbb{Z}^2 \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}.$$

Note that, $\mathcal{P}(u+1) = \mathcal{P}(u)$, and it is therefore natural to consider u as a random variable uniformly distributed on $[0,1]$. This turns $\mathcal{P}(u)$ into a random set, a random point process. A similar construction is possible for the randomly rotated lattice

$$\mathcal{R}_1(\phi) = \mathbb{Z}^2 \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

where ϕ is uniformly distributed in $[-\frac{\pi}{2},\frac{\pi}{2}]$. It is a fact that any matrix in $M \in \mathrm{SL}(2,\mathbb{R})$ can be uniquely written as a product of a shear, stretch and rotation matrix

$$M = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

where u is real, v is real and positive, and $-\pi < \phi \le \pi$. This is known as the Iwasawa decomposition of $\mathrm{SL}(2,\mathbb{R})$, and provides a parametrisation of $\mathrm{SL}(2,\mathbb{R})$ in terms of (u,v,ϕ). It follows that any choice of random elements (u,v,ϕ) yields a random lattice $\mathbb{Z}^2 M$. The above examples of randomly sheared or rotated lattices are simply special cases! But is there a particular natural choice of probability measure for (u,v,ϕ) that plays the role of a uniform measure? One could start with u uniformly distributed in $[0,1]$ and ϕ uniformly distributed in $[-\frac{\pi}{2},\frac{\pi}{2}]$, as above — but what is a natural *uniform* probabilty measure on the positive axis for v? The answer is highly non-trivial, but has a beautiful geometric interpretation. The key to the solution is the modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, where now all matrix coefficients are restricted to integers. It is a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ and in fact precisely the subgroup of all $\gamma \in \mathrm{SL}(2,\mathbb{R})$ such that $\mathbb{Z}^2\gamma = \mathbb{Z}^2$. This means that M and γM lead to the same lattice $\mathbb{Z}^2 M$, and we can therefore restrict our attention to only one representative of the coset $\Gamma M = \{\gamma M \mid \gamma \in \Gamma\}$. A convenient set of such representatives is for example given by

$$\mathcal{F} = \Big\{ (u,v,\theta) \in \mathbb{R}^3 \mid -\tfrac{1}{2} < u < \tfrac{1}{2}, \\ u^2 + v^2 > 1,\ v > 0,\ -\tfrac{\pi}{2} < \phi < \tfrac{\pi}{2} \Big\}$$

(we should also include about half of the boundary). This set is called a *fundamental domain* of the Γ-action, just as the unit interval is a fundamental domain of the $\mathbb{Z}$-action on $\mathbb{R}$. The most natural *uniform* measure on $\mathcal{F}$ is obtained from the Haar measure of $\mathrm{SL}(2,\mathbb{R})$, restricted to $\mathcal{F}$ and normalised as a probability measure. Explicitly, this Haar probability measure is

$$\mu_{\mathcal{F}} = \frac{3}{\pi^2} \frac{du\, dv\, d\phi}{v^2}.$$

Geometers will have spotted the intriguing similarity with formulas from hyperbolic geometry: The group $\mathrm{SL}(2,\mathbb{R})$ acts on the upper complex halfplane $\mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$ by Möbius (fractional linear) transformations

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \qquad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The Möbius transformation for M as in the Iwasawa decomposition maps i to $u + \mathrm{i}v$, and thus the Möbius action really comes from group multiplication in $\mathrm{SL}(2,\mathbb{R})$. In fact, we can identify $\Gamma\backslash\mathrm{SL}(2,\mathbb{R})$ with the unit tangent bundle of the modular surface $\Gamma\backslash\mathbb{H}$, where the angle $\theta = -2\phi$ parametrises the direction of the tangent vector at the point $\tau = u + \mathrm{i}v$. With this identification, the Haar probability measure $\mu_{\mathcal{F}}$ becomes the natural invariant measure for the geodesic and horocycle flows for the modular surface.

Haar probability measure

If (x_1,x_2,x_3) is a uniformly distributed random vector in the unit cube $(-\frac{1}{2},\frac{1}{2})^3$, then

$$(u,v,\phi) = \left(\sin(\tfrac{\pi}{3}x_1), \frac{\cos(\frac{\pi}{3}x_1)}{\frac{1}{2} - x_2}, \pi x_3 \right).$$

is a random element in $\mathcal{F}$ distributed according to the Haar probability measure $\mu_{\mathcal{F}}$.

A key property of Haar measure on $\mathrm{SL}(2,\mathbb{R})$ is that it is invariant under left and right multiplication by its group elements. This implies that (using the invariance under right multiplication) for M distributed according to $\mu_{\mathcal{F}}$, the random lattices $\mathbb{Z}^2 M$ and $\mathbb{Z}^2 M g$ have the same distribution for every element $g \in \mathrm{SL}(2,\mathbb{R})$. In other words, the random point process $\mathbb{Z}^2 M$ is $\mathrm{SL}(2,\mathbb{R})$-stationary! The process is, however, not translation-stationary since the origin is always realised. But even with the origin removed, the random process $\mathbb{Z}^2 M \setminus \{\mathbf{0}\}$ is not translation-stationary (as the formulas below will show). Nevertheless, Siegel's famous mean value formula (published in 1945) shows that its intensity measure is the standard Lebesgue measure dy.

Siegel's mean value formula

Motivated by questions in the geometry of numbers, Siegel proved that for any measurable function $f : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$,

$$\int_{\mathcal{F}} \left(\sum_{x \in \mathbb{Z}^2 M \setminus \{\mathbf{0}\}} f(\boldsymbol{x}) \right) d\mu_{\mathcal{F}} = \int_{\mathbb{R}^2} f(\boldsymbol{y})\, dy.$$

Siegel's formula in fact works for lattices in arbitrary dimension d. In 1998 it was generalised by Veech to general $\mathrm{SL}(d, \mathbb{R})$-stationary point processes in $\mathbb{R}^d$. (Veech in fact proved it for a more general class of random locally finite Borel measures in $\mathbb{R}^d$).

One challenge is now to work out the probability

$$\mathbb{P}\left(|\mathbb{Z}^2 M \cap B| = k \right)$$

for a given Borel set B. This turns out to be more difficult than one would think, despite the explicit and simple form of the Haar probability measure. The problem is the domain of integration! Let us specialise to the case of lattice points in a strip.

Lattice points in a strip

Consider the lattice $\mathbb{Z}^2 M$ restricted to the vertical strip

$$\mathfrak{X}_{w,R} = (w - R, w + R) \times (0, \infty),$$

the green strip in Figure 1. For simplicity (and because it's all that is needed for our applications below) we assume that $-R < w < R$, so that the vertical axis interesects $\mathfrak{X}_{w,R}$. We can now look for the lattice point in the strip with the lowest height, i.e., with the smallest positive x_2-coordinate. For typical lattices this point will be unique, and we will denote it by $\boldsymbol{q}$.

It is remarkable that, for any given lattice $\mathbb{Z}^2 M$, there are at most three possible choices for $\boldsymbol{q}$: the two basis vectors $\boldsymbol{r}, \boldsymbol{s}$ of $\mathbb{Z}^2 M$ with minimal height in the larger vertical strip between $-2R$ and $2R$ (see Figure 1), and their sum $\boldsymbol{r} + \boldsymbol{s}$. This fact, and its link to the famous three gap theorem for circle rotations, is explained in [15]. This pretty observation enables us to calculate the distribution of the minimal height vector $\boldsymbol{q}$ [12].

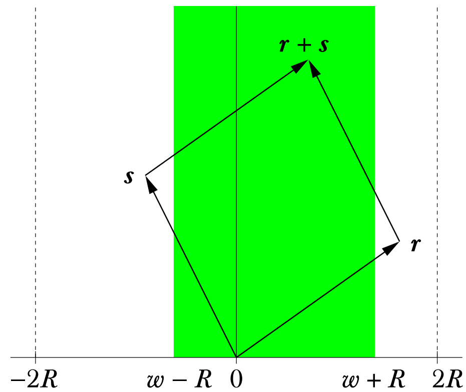

Figure 1. The two linearly independent lattice vectors with lowest and second-lowest heights in the vertical strip between $-2R$ and $2R$ form a basis. One can show that at any vertical strip of width one (in green) contains at least one of the three points, and hence the minimal height vector $\boldsymbol{q}$ is either $\boldsymbol{r}$, $\boldsymbol{s}$ or $\boldsymbol{r} + \boldsymbol{s}$.

Distribution of the lattice point with minimal height

If $\mathbb{Z}^2 M$ is a Haar random lattice, then the minimal height vector $\boldsymbol{q} = (q_1, q_2)$ in $\mathfrak{X}_{w,R}$ is distributed according to the probability measure $K_{w,R}(\boldsymbol{q}) dq$ with density $K_{w,R}(q_1, q_2)$ given by

$$\frac{6}{\pi^2} H\left(1 + \frac{q_2^{-1} - \max\left(|w|, |q_1 - w| \right) - R}{|q_1|} \right)$$

$$\text{where } H(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } 0 < x < 1 \\ 1 & \text{if } 1 \leq x. \end{cases}$$

The density $K_{w,R}(\boldsymbol{q})$ evidently depends on the choice of w, which proves that the random process $\mathbb{Z}^2 M \setminus \{\mathbf{0}\}$ is not translation-stationary. The $\mathrm{SL}(2, \mathbb{R})$-stationarity of our random lattice implies on the other hand that all distribution functions must be invariant under a simultaneously scaling of the horizontal and vertical directions by factors of $\lambda > 0$

and λ^{-1}, respectively. And indeed, the invariance

$$K_{\lambda w,\lambda R}(\lambda q_1, \lambda^{-1} q_2) = K_{w,R}(q_1, q_2)$$

is consistent with the explicit formula above.

If one is only interested in the height q_2 of $\boldsymbol{q}$ but not its direction, simply integrate over $q_1 \in [w-R, w+R]$. The result of this integration can be found in [12, Eq. (26)]. There is nothing to prevent us to further average over w, thus providing the distribution of the minimal height for a randomly shifted strip. The result of this second integration is as follows.

Distribution of minimal height on average

For a Haar random lattice $\mathbb{Z}^2 M$ the minimal height of a lattice point in the strip $\mathfrak{X}_{w,R}$, on average over w, is distributed according to the probability measure $P_R(q_2)\, dq_2 = 2R\, P(2Rq_2)\, dq_2$, with $P(s)$ given by (see also Figure 2)

$$\frac{6}{\pi^2} \times \begin{cases} 1 & (s \le 1) \\ \frac{1}{s} + 2\left(1-\frac{1}{s}\right)^2 \log\left(1-\frac{1}{s}\right) & \\ \qquad -\frac{1}{2}\left(1-\frac{2}{s}\right)^2 \log\left|1-\frac{2}{s}\right| & (s > 1). \end{cases}$$

The first moment is $\int_0^1 sP(s)ds = 1$. There is, however, a *heavy tail*: for s large, we have

$$P(s) \sim \frac{4}{\pi^2} s^{-3}.$$

So already the second moment diverges! Compare this with the exponential distribution in Figure 2, which we would have obtained for minimum height points from a Poisson point process with unit intensity, in a strip of unit width.

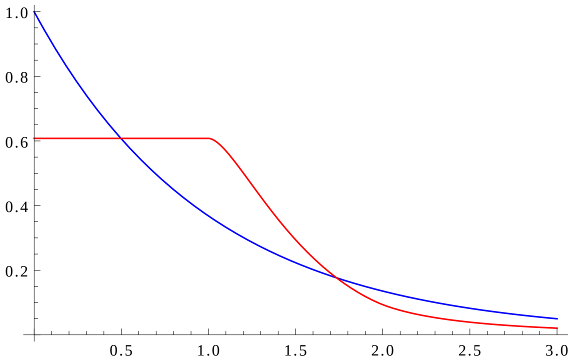

Figure 2. The exponential density e^{-s} (blue) vs. $P(s)$ (red).

Let us now discuss two natural examples where these distributions can be found in the 'wild'. The first describes visibility in Pólya's orchard or — equivalently — the free path length in the periodic Lorentz gas, and the second the energy level statistics for quantum harmonic oscillators.

Figure 3. The author in a perfectly periodic orchard: A poplar plantation near Pordenone, Italy.

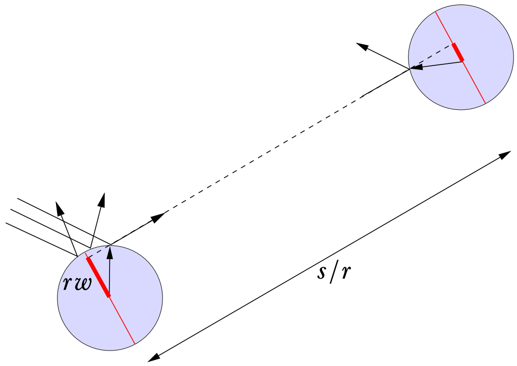

Figure 4. Intercollision flight of a particle in the Lorentz gas with scatterers of radius r. The free path length s is measured in units of $1/r$ and the the exit parameter w in units of r.

Pólya's orchard and the Lorentz gas

Pólya asked how far one could see in a forest, if all tree trunks had the same radius r and were either (a) randomly located or (b) planted on a perfect periodic grid. The same question arises in the study of the free path length for the two-dimensional Lorentz gas, where in the simplest setting a particle moves along straight lines in an array of spherical scatterers, see Figure 4. Let us here focus on the periodic setting, where the trees/scatterers are centered at points of $\mathbb{Z}^2$. What is the visibility, or free path length, with the observer at a given tree looking in direction $(-\sin\phi, \cos\phi)$? Is there a limit distribution when r is small and θ random?

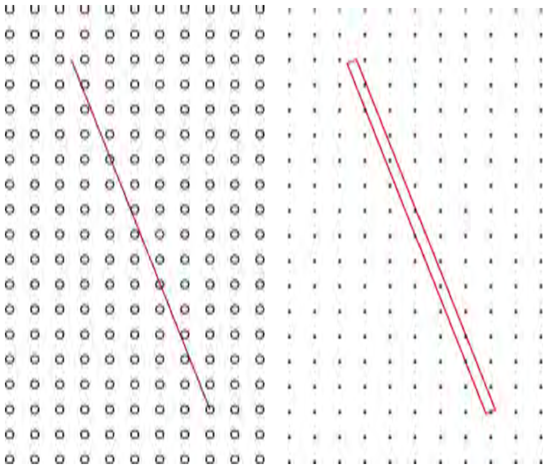

Figure 5. Left: A ray of length s/r in direction $(-\sin\phi, \cos\phi)$ intersecting k tree trunks of (small) radius r. Right: A rectangle containing k lattice points pointing the same direction, same length and width $2r$.

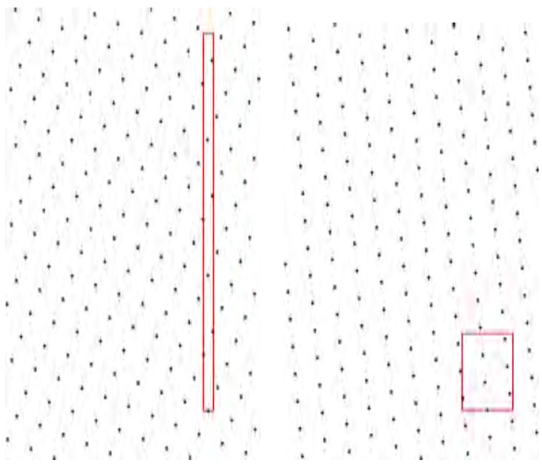

Figure 6. Left: The configuration in Figure 5 (right) rotated clockwise by ϕ. Right: The configuration on the left rescaled in the horizontal and vertical directions by factors of r^{-1} and r, respectively. The rectangle has now width 2 and height s.

The number of tree trunks of radius r intersecting a ray of length s/r is the same as the number of lattice points in a rectangle of width $2r$ and length s/r, see Figure 5. Now let's rotate the whole picture clockwise as in Figure 6 (left). The rectangle is now vertical, and instead of the lattice $\mathbb{Z}^2$ we have the rotated lattice

$$\mathcal{R}_1(\phi) = \mathbb{Z}^2 \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix},$$

which we have met before. Finally, we stretch the picture as described in Figure 6 (right), and obtain the rectangle of height s and width 2 — the r-dependence is gone! On the flipside, the underlying lattice has now transformed to the r-dependent lattice

$$\mathcal{R}_r(\phi) = \mathbb{Z}^2 \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} r^{-1} & 0 \\ 0 & r \end{pmatrix}.$$

The visibility, or free path length, can now be expressed as the minimal height of lattice points in the strip $\mathcal{Z}_{w,1}$, where w describes the offset of the ray relative to the center of the initial tree trunk; see Figure 6. (For example $w = 0$ means the ray emerges from its centre as in Figure 5.) The condition $|w| < 1$ ensures we are sitting somewhere on the tree trunk. In the context of the Lorentz gas, the fact that the minimal height can only take three values as w varies is known as Thom's problem, in turn a close variant of Slater's problem. The key fact we will now use is the following:

Randomly rotated lattices

If ϕ is a uniformly distributed random variable in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, then the random lattice $\mathcal{R}_r(\phi)$ converges in distribution to the Haar random lattice $\mathbb{Z}^2 M$ as $r \to 0$.

This statement is a consequence of the equidistribution of large circles in the homogeneous space $\Gamma\backslash \mathrm{SL}(2,\mathbb{R})$. The convergence implies that the limit distribution for the minimal height vector $\boldsymbol{q}$ in the lattice $\mathcal{R}_r(\phi)$ restricted to the strip $\mathcal{Z}_{w,1}$ is given by the density $K_{w,1}(\boldsymbol{q})$, and the corresponding distribution of the free path length is $P_1(s) = 2P(2s)$, see Figure 7. Note that if we had measured visibility in units of the diameter $2s$ rather than radius r, the limit distribution would be $P(s)$.

In the case of the Lorentz gas, $P_1(s)$ was in fact first found by the physicist Dahlqvist [3] in 1997, and only in 2007 established rigorously by number theorists Boca and Zaharescu [1], who employed analytic methods based on continued fractions and Farey sequences. The density $K_{w,1}(\boldsymbol{q})$ plays an important role in describing particles in transport in the periodic Lorentz gas, and in 2008 was calculated independently by Caglioti and Golse [2] by continued fraction techniques, and by Strömbergsson and the author [12] via random lattices. The principal advantage of the latter method is that it works in any dimension [13] and even extends to aperiodic, quasicrystalline point configurations! Now, on to the second 'real-world' appearance of random lattices.

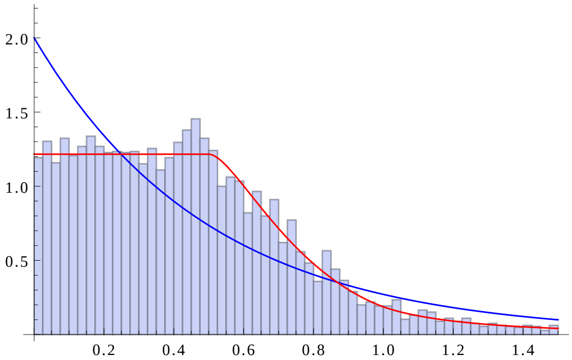

Figure 7. The distribution of free path for the periodic Lorentz gas with scatterers of radius $r = 10^{-8}$, sampled over 6000 initial conditions. Theoretical curves are the exponential density $2e^{-2s}$ (blue) vs. $P_1(s) = 2P(2s)$ (red). The data was computed using the algorithm in [9].

Quantum oscillators

In quantum mechanics, the energy levels of bound states can only take specific discrete ('quantized') values. One of the simplest and most fundamental quantum systems with a purely discrete spectrum is the harmonic oscillator. In two space dimensions, its energy levels are given by

$$E_{m,n} = (m + \tfrac{1}{2})\hbar\omega_1 + (n + \tfrac{1}{2})\hbar\omega_2$$

where $m, n = 0, 1, 2, \ldots$ run through the non-negative integers. The quantities ω_1, ω_2 are positive reals, the *oscillation frequencies* and $\hbar$ denotes Planck's constant. If we measure energy in units of $\hbar\omega_2$, we have the simpler expression

$$\epsilon_{m,n} = (m + \tfrac{1}{2})u + (n + \tfrac{1}{2}), \qquad u = \frac{\omega_1}{\omega_2}.$$

Of particular significance are the spacings between energy levels, as they determine the emission spectrum of the system. After a little thought, you can convince yourself that the spacings between consecutive levels $\epsilon_{m,n}$ in the interval $[E, E+1)$ are the same as the gaps between the fractional parts ξ_m of the sequence mu, where $m = 0, \ldots, N-1$ and N is number of $\epsilon_{m,n}$ in $[E, E+1)$. The three gap theorem mentioned earlier thus implies that we have the same phenomenon for the energy levels for a harmonic oscillator, at least for intervals of length one. A numerical illustration of this fact is given in Figure 8.

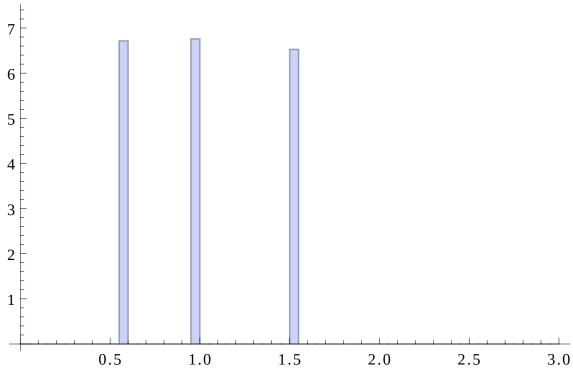

Figure 8. The gap distribution for the fractional parts of nu, with $n = 1, \ldots, 50000$ and $u = \pi$.

One can show, however, that the distribution in Figure 8 will not converge as N becomes large. The only hope to see a limit is to introduce a further average over u. Using the approach in [15], we can express the gap between ξ_m and its nearest neighbour to the right as the minimal height of all lattice points in the strip $\mathcal{Z}_{w,1/2}$ (of width one), with $w = \frac{m}{N} - \frac{1}{2}$ and the lattice

$$\mathcal{P}_N(u) = \mathbb{Z}^2 \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} N^{-1} & 0 \\ 0 & N \end{pmatrix}.$$

As in the case of randomly rotated lattices, also here we have a limit theorem.

Randomly sheared lattices

If u is a uniformly distributed random variable in $[0, 1]$, then the random lattice $\mathcal{P}_N(u)$ converges in distribution to the Haar random lattice $\mathbb{Z}^2 M$ as $N \to \infty$.

This fact is based on the equidistribution of long closed horocycles on $\Gamma\backslash \mathrm{SL}(2, \mathbb{R})$, which was proved by Zagier in 1979 in the case of the modular surface, and for more general discrete subgroups Γ by Sarnak in 1981. The most powerful extension of results of this type (as well as the rotational averages used for Pólya's orchard) is due to Ratner in the early 1990s [16]. It describes equidistribution of unipotent orbits on quotients $\Gamma\backslash G$ where G is now a general Lie group. (Horocycles are special examples of unipotent orbits.) Recent breakthroughs that build on Ratner's work include the deep measure classification and equidistribution theorems for moduli spaces by Eskin, Mirzakhani and Mohammadi. For an introduction to dynamics on homogeneous spaces and their relevance in number theory I recommend the excellent textbook by Einsiedler and Ward [4].

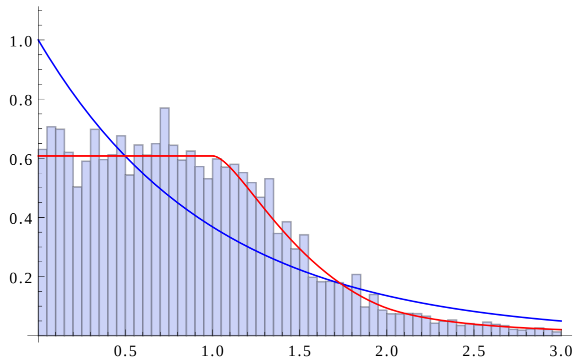

Figure 9. The gap distribution for the fractional parts of nu, with $n = 1, \ldots, 2000$ and u sampled over 2000 randomly chosen points in $[0,1]$. Theoretical curves are the exponential density e^{-s} (blue) vs. $P(s)$ (red).

By the same reasoning we used earlier for Pólya's orchard, the convergence of randomly sheared lattices to Haar distributed random lattices establishes the convergence of the gap distribution for the fractional parts of mu. The one difference is we now sum over $w = \frac{m}{N} - \frac{1}{2}$ $(m = 0, \ldots, N-1)$ rather than integrate — but this discrete average can be treated as a Riemann sum which approximates the Riemann integral for N large. We can conclude that the gaps between fractional parts on mu, and thus the energy level spacings for quantum oscillators, have the same limit distribution as the free path length in the periodic Lorentz gas! Figure 9 compares numerical data with the theoretical prediction.

The explicit form of the level spacing distribution for quantum oscillators (in Figure 9) was first established by Greenman [7] in 1996, following previous work on the problem by Berry and Tabor (1977), Bohigas, Giannoni and Pandey (1989), Bleher (1990-91), Pandey and Ramaswamy (1992), Mazel and Sinai (1992); see [10] for details and references. Greenman's paper predates Dahlqvist's and Boca and Zaharescu's work on the Lorentz gas; and perhaps more remarkably, the likeness of the two distributions seems to have been overlooked even in the recent literature [17]! That the two are the same is evident of course by simply staring at the explicit formulas, and perhaps no surprise given the similarity of their arithmetic setting. The beauty of using lattices is that we have a conceptual understanding of why the limit distributions must coincide: random rotations and random shears both converge to the same Haar probability measure — a non-trivial fact!

Other applications

We can construct random lattices that are not only $\mathrm{SL}(2,\mathbb{R})$-stationary but also translation-stationary as follows. Take the randomly shifted lattice $\mathbb{Z}^2 + \boldsymbol{\alpha}$ with $\boldsymbol{\alpha}$ uniformly distributed in the unit square $[0,1]^2$ (recall our construction in dimension one), then apply a linear transformation to obtain the random affine lattice $(\mathbb{Z}^2 + \boldsymbol{\alpha})M$ with M distributed in $\mathcal{F}$ with respect to Haar measure. This point process is now translation-stationary and it has intensity one. In fact, also its second moment coincides with that of a Poisson point process; again a consequence of Siegel's mean value formula [5, App. B]. In 2004, Elkies and McMullen [6] proved that the limiting gap distribution for the fractional parts of $\sqrt{n}$, $n = 1, 2, 3, \ldots$ can be derived via a random affine lattice. The proof uses equidistribution of certain nonlinear horocycles, which is a consequence of Ratner's measure classification theorem. The distribution found by Elkies and McMullen also describes the limiting distribution for directions in a fixed affine lattice [13].

Random lattices appeared in the probability literature in Kallenberg's disproof of the Davidson conjecture [8] on the classification of line processes which have (almost surely) no parallel lines. The counterexamples to the conjecture were constructed using two-dimensional random affine lattices restricted to a vertical strip, where each lattice point represents a line via the standard linear parametrisation. This is particularly impressive as Kallenberg was unaware of Siegel's classical construction in the geometry of numbers, as clarified by Kingman; see the quote at the end of Kallenberg's paper.

Other examples where random lattices play an important role are the value distribution of quadratic forms, such as in Margulis' proof of the Oppenheim conjecture, the Hall distribution describing the gaps between Farey fractions, random Diophantine approximation, diameters of random Caley graphs of abelian groups, the Frobenius problem, hitting times for integrable dynamical systems, deviations of ergodic averages of toral translations, etc. And how about random lattices in non-Euclidean settings?

But these are stories for another day!

Take home message

- Random lattices are important point processes with connections to ergodic theory, geometry, number theory, combinatorics, probability and physics.
- The level spacing distribution of a quantum oscillator equals the free path distribution of the periodic Lorentz gas.

Acknowledgements

Much of the material presented here is based on joint work with Andreas Strömbergsson, whom I thank for our longstanding collaboration. I am grateful to Atahualpa Kraemer and David Sanders for sending me the data for Figure 7. The author's research is currently supported by EPSRC grant EP/S024948/1.

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Marriages, Couples, and the Making of Mathematical Careers

DAVID E. DUNNING AND BRIGITTE STENHOUSE

By considering instances of mathematicians who have worked closely with a spouse or partner, we offer historical perspectives on gender and work-life balance in mathematical research. We aim to use history to open space for re-imagining how collaboration, home-life, and labour fit together in the mathematical community today.

The home life of mathematics

Though mathematicians are often imagined as the quintessential solitary researchers, many have managed the daily routines of a mathematical career through partnership with a spouse who was intimately involved in their working life. Whilst marriage is certainly not the only, nor even the most common form that collaboration can take, it does offer an especially clear window on the unstable boundaries dividing labour into the intellectual and the domestic, the masculinised and the feminised, or the credited and the unacknowledged. As historians of mathematics, we suggest that by looking at how such categories were made, sustained, and changed in the past, we can not only deepen our historical understanding but also support more equitable mathematical practice in the present.

A focus on collaboration is part of a broader trend in history of science scholarship which has sought to unravel the myth of the 'lone genius', that heroic, solitary — and usually white, male, European — individual who is celebrated as the sole mind behind innumerable discoveries. This is perhaps best encapsulated by Isaac Newton's so-called *annus mirabilis* or 'Year of Wonders', a period of intense productivity when he escaped from Cambridge to Woolsthorpe Manor during the Great Plague of 1665–6; it was here that he seemingly 'invented' calculus out of nothing, revolutionising physics and mathematics. However, this narrative sidelines and undervalues the work that had already been done by mathematicians such as Pierre de Fermat, René Descartes, or Isaac Barrow on the problems of finding tangents and quadratures. Futhermore it renders invisible the extensive network of mathematicians who corresponded with each other on such topics, and of which Newton was a part. These written exchanges could be facilitated by formal bodies, such as learned academies and societies, but just as often were part of personal correspondence.

Thus our need to understand individual achievements in their wider intellectual and social context should not end at the boundary of officially recognised scholarly activity. The importance of scientific knowledge production in the 'domestic sphere' — that is at home, in private, or through informal exchange — has been well treated in literature on women in science. Until very recently women were unable to access the 'public' institutions which have long been privileged as knowledge-making spaces: universities, scientific academies, or research laboratories. Only by looking beyond these spaces have historians recognised the many creative ways women found to participate in scientific endeavours. Ineligible to study at the École Polytechnique in 1794, Sophie Germain entered into correspondence with Joseph-Louis Lagrange under the pseudonym Antoine-Auguste Le Blanc in order to get a copy of his lecture notes to study. Germain subsequently situated herself within a wider network of mathematical correspondents, perhaps most notably Carl Friedrich Gauss, and although she never directly published her work on Fermat's Last Theorem it was certainly read by Adrien-Marie Legendre who explicitly attributed a result to her in a memoir he presented to the *Académie des Sciences* in 1823 [2].

To bring the collaboration that takes place *within a household* to the foreground is then to unite these two currents in historical research, viewing collaboration and domesticity together. Historians of science have studied collaboration between married couples and other domestic partners, but so far we lack a study dedicated to collaborative couples in the history of mathematics. Collaborative couples in mathematics, however, present a special case in

Figure 1: Participants at the 1950 ICM. HUPSF International Congress of Math (BP1), Harvard University Archives.

that so many kinds of mathematical practice are possible without any sort of specialised equipment or facilities; there need be no difference between domestic space and the space of mathematical research. At times this fact has made mathematical work more accessible to women than other forms of scientific contribution, though that access has not meant their work was regarded in equal or ungendered terms. Rather instances of mathematical collaborative couples provide us a window on the complex gendered terrain of collaboration within a marriage.

At home, the lines between the kinds of labour a couple divvied up among themselves and those which they delegated to servants, secretaries, or extended family, position mathematicians in a wider structure of class and familial relations. For Dorothy Vaughan, the transition from school teacher to professional mathematician was contingent on her wider family providing childcare when she moved 137 miles away from her children to take up a job at Langley Research Centre, part of the United States National Advisory Committee for Aeronautics, in 1943. Vaughan's life and career is treated in Margot Lee Shetterley's book *Hidden Figures*, and the 2016 film of the same name. Living through the global pandemic in 2020 has certainly underscored the relationship between gender, class, and caring responsibilities, with the greatest reduction in time available for research being felt by female scientists with young dependents [5].

Couples and careers

We have so far emphasised domestic settings, but a couple's collaborative activity is certainly not limited to the home. Many couples have worked together to construct a shared network of mathematical acquaintances via letter writing or, more recently, through attendance at international meetings, congresses, and conferences — sites at which it can be impossible to separate mathematical from purely social exchange. Officially, women often attended such conferences as spouses and therefore do not turn up on the list of participants, but nevertheless engaged with the mathematical community in a meaningful way. Indeed the Women's Committee of the 1950 International Congress of Mathematicians in Cambridge, Massachusetts was made up of the wives of the organisers, and oversaw some of the social activities at the conference which were vital to international exchange. Thus women, including many who were not mathematicians themselves, helped sustain the professional networks that made international mathematical research possible.

Exclusion from formal membership in such networks, however, was often one of the tactics used by elite scientists to contain the perceived threat to their professional status represented by rising gender, sexual, or racial diversity in science. Heterosexual couples who were also colleagues can serve as useful comparative illustrations of the differential obstacles women faced, even while their marriages also sometimes offered strategies for circumventing those obstacles. The mathematical logician, psychologist, and activist Christine Ladd-Franklin completed the requirements for a PhD in Mathematics at Johns Hopkins University in 1882. But the university — employing another increasingly common tactic for hindering women's scientific activity — drew the line at actually awarding degrees to the few women it grudgingly permitted to become students. Her husband Fabian Franklin's scientific career, however, offered them stability and even the opportunity for them both to spend a sabbatical year in Europe. Ladd-Franklin spent this time working in the labs of Georg Müller in Göttingen and Hermann von Helmholtz in Berlin. Franklin left academia for journalism in 1895, whereas Ladd-Franklin remained an active, highly regarded scholar into old age, but she never had access to the academic positions and resources he had had at his disposal. In 1926 she finally received the PhD she had earned 44 years earlier.

The so-called 'two-body problem', where both partners are early career researchers on the academic job market, continues to create tension for those hoping for work-life balance. The likelihood of both partners successfully finding work in the same geographic location is often decreased further when their research is in the same or very similar fields. According to the documentary film by George Csicsery, *Secrets of the Surface: The Mathematical Vision of Maryam Mirzakhani*, such considerations even influenced the career trajectory of Fields Medallist Mirzakhani, who was married to mathematician Jan Vondrák.

In the case of couples who have collaborated even more closely, working together on the finest details of their research, the distinction between cooperation and exploitation can be slippery. A challenge for historical interpretation arises in cases of joint work appearing under a single (usually male) name, an arrangement that may or may not have been mutually agreeable depending on each partner's interpretation of their own role. The most well-known case in mathematics is that of Grace Chisholm Young and William Henry Young. In 1895, aged 27, Chisholm Young was awarded her doctorate in mathematics at Göttingen University, and between then and 1929 the Youngs published over 200 mathematical papers. They collaborated closely throughout this time, however only 13 papers were published jointly, and only 18 were published under Chisholm Young's name alone. At a time when there were very few paid positions for women to teach or research mathematics (and even fewer for married women), it seems that it was more beneficial economically for them as a household to attribute the work solely to William Young.

The terms of a collaboration, however, do not always remain amiable. When Mileva Mari threatened her ex-husband Albert Einstein with revealing the extent of their collaboration on work published under his name, his chilling response was to point out that no one would believe her:

> "You made me laugh when you began to threaten me with your memories ... When a person is completely insignificant, there is nothing else to tell such a person but to remain modest and silent. This is what I advise you to do." [1, p. 241].

The exploitation of collaborators arising from an unequal power dynamic is still extremely relevant today and of course not confined to partnerships. PhD students and post-doctoral researchers face chronic job instability whilst being reliant on the support and collaboration of supervisors when preparing their work for publication. This is further complicated by the widespread sexual harassment which persists at universities in the UK. The 2018 NUS Report on staff-student sexual misconduct in higher education found that 41% of the 1535 students who responded to the survey had experienced sexual misconduct from staff, with postgraduates more likely to have experienced misconduct than undergraduates. Students were also more likely to have experienced sexual misconduct from university staff if they were women, and more again if they identified as gay, queer, or bisexual. [6, pp. 8–9].

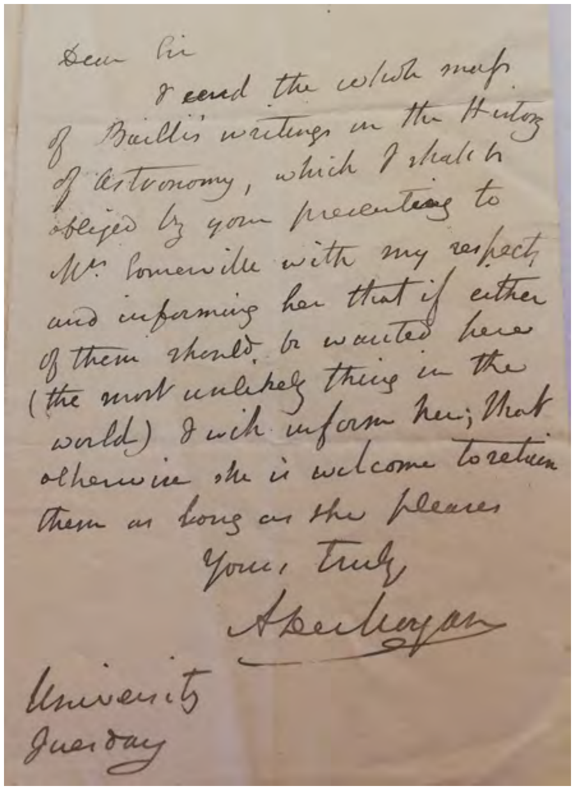
Given that the division of labour within a couple is so often governed by prevailing inequities in the society in which they live, it is no surprise that male mathematicians have tended to more easily receive credit, compensation, and prestige than their female partners. By favouring William Henry Young's name, the Youngs adopted a highly successful strategy in a publishing landscape that was not of their own design.

But we also find examples of cooperative efforts to prioritise a woman's mathematical career, such as the case of Mary Somerville (née Fairfax) and her husband Dr. William Somerville. Ineligible for a university education or for election to a learned society as a woman, Somerville's access to the mathematical knowledge circulating in these spaces was highly restricted. As a 'clubbable' gentlemen with interests in natural history and mineralogy, her husband, on the other hand, was elected a member of numerous learned societies including the prestigious Royal Society of London. He actively supported Somerville in her studies and scientific writing by borrowing books from libraries on her behalf, soliciting information from other society members, either in person at meetings, or via letter correspondence, and liaising with her publisher during the production of her books [7]. Dr. Somerville seems to have had no interest in mathematical research or in cultivating a reputation for himself as an eminent scientist. The importance of this disinterest was noted by geologist Charles Lyell in 1831 when he wrote the following:

> "had our friend Mrs. Somerville been married to La Place, or some mathematician, we should never have heard of her work. She would have merged it in her husband's, and passed it off as his." [3, p. 325]

While emphasising the role domestic partnerships have played in mathematical work, we should not neglect the converse influence that mathematical careers can exert on a given couple's way of building a life together. In his survey of collaboration of queer couples in the sciences, Opitz suggests that "the ethos of professional respectability claimed a significant role in shaping the dynamics of [queer] collaborative partnerships" [4]. That is to say, scientists curated an image of themselves and their relationships in order to conform with scientific practice of the time, whether that was as equal partners sharing expertise, or one partner being positioned as a researcher and the other as a domestic helpmate. This in turn affected the dynamics of the relationship itself, for example whether the partners desired or were able to achieve cohabitation. Moreover, the lived experience of a queer scientific couple was, and is, heavily influenced by social factors, such as the need to avoid harassment and discrimination in the workplace.

Paying attention to mathematicians' marriages also reveals ways that a mathematical career continues to be shaped and reimagined after an individual's death. After Bernhard Riemann's death, his widow Elise Riemann played an active role in the production of his *Collected Works*, while Emilie Weber helped buttress the friendship of Heinrich Weber and Richard Dedekind as they edited the publication. Similarly, Mary Everest Boole asserted quite an active voice in the commemoration of her husband, the logician George Boole, whom she survived by half a century. After his death she published prolifically on mathematical and logical pedagogy intertwined with religious issues, developing a mystical (and often mystifying) interpretation of George's work. In light of his well-documented reticence to speak publicly about his own religious beliefs, along with the temporal distance between his career as an author and hers, it is difficult to discern which of her ideas he shared. But whereas sexist dismissals of Mary's admittedly eccentric views were once common, scholarly consensus now rightly recognises her as a generally reliable witness to the more personal manifestations of George's thought. Today his contributions are better remembered through the lens of the information-theoretic interpretation developed by Claude Shannon in the mid-twentieth century. (Claude and his wife Betty Shannon, a computer at Bell Labs, offer another example of a mathematically collaborative marriage.) But Mary's efforts to shape the commemoration of George's legacy stand as an insightful body of work, offering a useful reminder that the meaning of a person's career is not fixed at the time of their death, and does not belong to the deceased alone.



Dear Sir
I send the whole mass of Bailli's writings in the Histoy of Astronomy, which I shall be obliged by your presenting to Mrs Somerville with my respects and informing her that if either of them should be wanted here (the most unlikely thing in the world) I will inform her; that otherwise she is welcome to retain them as long as she pleases

Yours truly
A De Morgan

University
Tuesday

Figure 2. A letter from Augustus De Morgan to Dr. Somerville, sending Bailly's *History of Astronomy* for "Mrs Somerville". Bodleian Library, Somerville Collection, Dep. c. 370, MSD-3 126, reproduced courtesy of the Principal and Fellows of Somerville College.

The work of mathematics, past and present

Mathematical research — as the readers of the *Newsletter* will hardly need reminding — is work. When we understand the history of mathematics as the history of a particular kind of work, it is clear that a full picture must include the related and interdependent kinds of labour that together form the context in which people make their lives as mathematicians. Such a historical perspective in turn compels us to recognise the seemingly mundane questions around various divisions of labour as meaningfully intrinsic to the work of mathematics in the present.

In suggesting marriages as a focal point, we certainly do not mean to overlook the many workers of diverse kinds who have not been part of a mathematical couple; this is just one line of historical inquiry among many. We call attention to it as a particularly illuminating one: given the feasibility of doing mathematics at home, and the paper-based practices so often constitutive of mathematical knowledge, studies of collaborative couples stand to offer much insight to the history of mathematics. Moreover, such studies naturally look beyond 'lone geniuses' and destabilise the history of mathematics as presented in university courses, namely as a body of knowledge steadily unearthed through the conjecturing and proving of theorems by the individuals after whom they are named.

To organise mathematical work in a particular way, to the advantage or disadvantage of particular people, has always been part of the making of mathematical careers. But the great diversity of ways this process has played out in the past illustrates the contingency of any given arrangement, and hence the possibility of re-imagining how collaboration, domesticity, and labour fit together in the mathematical community today.

To find out more...

We encourage readers to attend the forthcoming workshop *Marriages, Couples, and the Making of Mathematical Careers*, supported by the LMS and the British Society for the History of Mathematics, to be held online 29–30 April 2021.

For more details and free registration please visit mathmarriages.wordpress.com.

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David E. Dunning

David E. Dunning is a Postdoctoral Research Associate in the History of Mathematics research group at the Mathematical Institute of the University of Oxford. His current book project examines the rise of mathematical logic through the lens of notation, exploring both technical and social aspects of symbolic systems. He hails from Philadelphia and is an ardent dog person.

Brigitte Stenhouse

Brigitte Stenhouse is a PhD student in History of Mathematics at the Open University, UK. Her thesis looks at the work of Mary Somerville (1780–1872), and considers questions around translations; differential calculus in early-19th-century western Europe; and gendered access to knowledge. Her favourite way to unwind is to go splashing in the sea with her five-year-old nephew — the colder the better!

To Ithaca

MARINA ILIOPOULOU

“As you set out for Ithaka / hope your road is a long one, / full of adventure, full of discovery....”[1]. After ten years of postgraduate experience, Marina Iliopoulou reflects on the mathematician’s academic journey — bumpy, constant and exciting.

In February 2019, after four years of PhD and nearly six years of postdoctoral positions, I enthusiastically assumed my first permanent appointment, as Lecturer in Pure Mathematics at the University of Kent. Starting a new life in a new place, I was eager to bring my mathematical friends over, to discuss our work and show them around beautiful Canterbury. In February 2020, this became a reality: using my LMS Celebrating New Appointments grant, I organised a cosy one-day conference at the University of Kent.

The meeting featured four specialist talks on recent advances in harmonic analysis, and surpassed my expectations, attracting about 25 people from around the UK. I was particularly happy to see that there was a lot of mathematical interaction, even between participants who had not met before. The experience made me feel at home at Kent and renewed my connection to the UK harmonic analysis group, fuelling me with further excitement for upcoming collaborations. Now, after several months, the memory of the meeting is even more special, marking the last time our harmonic analysis group met, before coronavirus changed everything.

Having forgotten to take a photo of the meeting, I provide one of the venue: *SMSAS, University of Kent*

[1] The first lines of "To Ithaca" by C. P. Cavafy

As mathematicians, we primarily aim to create new mathematics. This goal largely shapes our lives. Before taking on permanent positions, we take on years of training (in my case, a decade) of PhD study and postdoctoral work, close to experts around the world. Solving a mathematical problem can take a lot of time — even years — and requires daily dedication and deep concentration. Being so focused without getting disheartened is not always easy. We like being productive, but unfortunately performing mathematical research offers no guarantee of results — at least not when the problem is worth it. For many of us, however, hunting down the truths behind difficult questions is reward in itself, motivating us to lead this, often uncertain, life.

My own mathematical journey started in my home town, with an undergraduate degree in mathematics at the University of Athens (Greece). My professors there were truly inspirational — and, even though I had not the remotest idea what academic life is like, I knew well enough that I loved puzzles and wanted advanced mathematics to stay in my life. My lecturers advised me to do a PhD abroad. I still remember my surprise when I was told that getting a PhD requires proving new theorems — somehow until then I had assumed that all maths had already been created, by people long dead. So, even though I had never thought of leaving Greece (or wanted to), I applied for postgraduate programmes abroad. I was exceptionally lucky to be accepted for a PhD at the University of Edinburgh, to work on harmonic analysis under the supervision of Tony Carbery.

My four PhD years in Edinburgh were the happiest of my life. Tony was a wonderful supervisor, who respected my personal taste in mathematics and gave me problems that I truly cared about — combinatorial at the time. He also granted me absolute intellectual freedom, trusting that I would ask for guidance if I needed to. This was exactly what I needed to be creative. Research became intertwined with a care-free life, and was constantly in my mind. In fact, I came up with the final piece of the solution of my first problem — a piece I was missing for months — after returning from a party at 4am. I mention this not to endorse heavy drinking, but to demonstrate that we always think our maths, and that a regular 9am–5pm office schedule doesn't necessarily get our ideas flowing.

At the start of my PhD, an unexpected breakthrough in harmonic analysis (not induced by myself!) made the field one of the most fertile in modern mathematics. In particular, harmonic analysis aims to understand the interaction of waves. Mathematicians had long been trying to understand this interaction via toy problems (including combinatorial questions, such as the ones I worked on during my PhD). As I was starting my PhD, such combinatorial problems were shown to have a deep algebraic nature. Since then, this algebraic behaviour has been systematically exploited, leading to major advances in geometric and analytic problems that in the recent past had been considered untouchable. For us who work in the field, these are exciting times to be alive. I quickly became eager to work on the original harmonic analytic problems that gave rise to the combinatorial ones I was focusing on, and to contribute a little to this wave of progress.

I achieved this during my postdoctoral positions over the next six years, in Birmingham, MSRI and UC Berkeley. Inspired by the vision of my mentors (such as Jon Bennett in Birmingham and Michael Christ in Berkeley), I started realising that being a successful researcher means much more than just solving problems. It also means developing a taste for what is interesting; seeking connections between different mathematical areas; and creating questions that matter. I started adopting this way of thinking, and creating research plans and proposals of my own.

All these years of effort and travelling had their good and bad moments, and naturally shaped me and my personal life. There are successes and disappointments: for every paper I have produced, I can provide a sizeable list of problems that I have failed to solve, despite trying very hard. Often work has been very hectic. For example, during my first semester at Berkeley, I somehow managed to impose upon myself the tightest travel restrictions short of house arrest: I was spending so much time on teaching preparations that I didn't get a chance to walk a single step west of my flat (I only had time to go to the university and the supermarket, which sadly were both east).

Not knowing where our next job will be, or even if we will manage to secure one, despite so many years of hard work, can be stressful, and can seriously hinder our personal life. However, it is also exciting, because, truly, anything can happen. It is a life full of travel and experiences, anticipation and strong excitement. Our love for research gives us energy and confidence, and can take us very far from where we started, to destinations that we never imagined.

Some destinations where mathematics has taken me

My long-anticipated permanent job gave me certainty and relief. Naturally, it comes with other responsibilities, apart from research and teaching, which I am still learning to balance. And while it means the end of care-free research-oriented years, the search for new questions and ideas never ends. This search has the power to make every moment interesting.

Marina Iliopoulou

Marina is a Lecturer in Pure Mathematics at the University of Kent. She is interested in the interface of harmonic analysis, incidence geometry and additive combinatorics. She also loves singing, but her neighbours prefer when she quietly does maths.

Microtheses and Nanotheses provide space in the Newsletter for current and recent research students to communicate their research findings with the community. We welcome submissions for this section from current and recent research students. See newsletter.lms.ac.uk for preparation and submission guidance.

Microthesis: A Novel Algorithm for Solving Fredholm Integral Equations

FRANCESCA ROMANA CRUCINIO

Fredholm integral equations of the first kind are the prototypical example of ill-posed linear inverse problems. They model, among other things, reconstruction from noisy or delayed observations and image reconstruction. My PhD project explores the use of Monte Carlo methods to solve these integral equations.

Fredholm Integral Equations

Fredholm integral equations of the first kind

$$h(y) = \int g(x,y) f(x)\, dx, \qquad (1)$$

are linear integral equations in which the function f is the unknown and g, h are given. They generalise linear systems of equations to the infinite-dimensional setting and describe the distortion caused by g on the function f.

Solving (1) corresponds to reconstructing f from its distorted version h. In the simplest case, the distortion g models addition of noise to the signal f, which has to be reconstructed from its noisy version h, a task known as deconvolution.

Fredholm equations find applications in medical imaging, where f corresponds to an image which is reconstructed from data provided by tomography scanners. In epidemiology, (1) links the incidence curve of a disease to the observed number of cases.

Regularisation

Fredholm integral equations (1) are generally ill-posed and stable solutions can be found minimising a distance between the h and the right-hand-side of (1). We consider regularised solutions f which minimise the Kullback-Leibler distance

$$\int h(y) \log\left(\frac{h(y)}{\int g(x,y) f(x)\, dx}\right) dy, \qquad (2)$$

with additional constraints to ensure smooth reconstructions of f.

To minimise (2), we resort to iterative techniques which, given an initial guess, reduce (2) sequentially until a fixed point is reached and the reconstruction of f stops improving.

Computational Considerations

Standard approaches to regularisation require discretisation of the domain of f, restricting their applications to low-dimensional scenarios, and make strong assumptions on the regularity of f. Often, knowledge of an analytic representation of h is required.

Monte Carlo methods are a class of simulation based techniques which approximate a (density) function f through a set of samples. These algorithms provide a stochastic discretisation of the domain of f which can be applied in high-dimensional scenarios and can be naturally implemented when only observations from h are available.

Interacting Particle Methods

Interacting particle methods are a class of Monte Carlo methods which approximate a probability density through a population of (weighted) samples evolving over time. My PhD project considers a particular family of interacting particle methods, sequential Monte Carlo (SMC).

In SMC, a population of weighted samples sequentially undergoes random mutations, which are weighted so that mutations that produce fitter individuals are more likely to survive (selection). A new population is originated by replicating fitter mutations, while the other individuals die out; see [2] for a more detailed account.

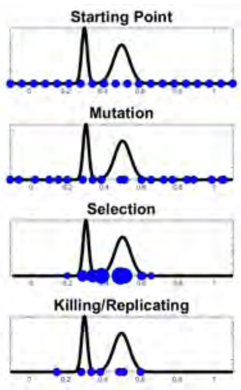

Figure 1. Sequential Monte Carlo

We use SMC to approximate the fixed point of the iterative scheme and show that the estimators we propose enjoy good asymptotic properties: as the discretisation gets finer they converge to a regularised solution of the integral equation. Currently, we are exploring the use of McKean-Vlasov stochastic differential equations to approximate the function f minimising a penalised version of (2).

Image Reconstruction

Given the blurred image in the first panel of Figure 2 we can reconstruct the corresponding clear image by solving a 2D Fredholm integral equation.

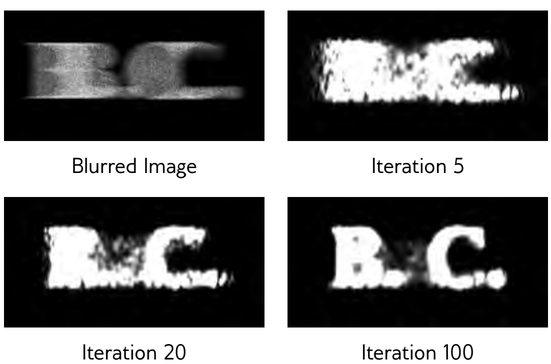

Figure 2. Given the blurred image, h, we iteratively reconstruct the original image, f. The function g describes the motion which caused the blur.

Reconstruction of cross-sections of the brain from the noisy measurements provided by positron emission tomography (PET) scanners is one of the most relevant applications of Fredholm integral equations. These reconstructions are used to analyse internal biological processes to detect medical conditions such as schizophrenia, cancer, Alzheimer's disease and coronary artery disease.

The algorithm reconstructs the reference image in the final panel of Figure 3 by refining the reconstruction until a fixed point is reached.

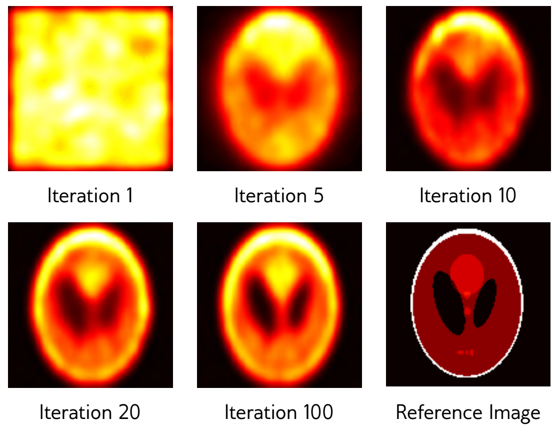

Figure 3. A cross-section of the brain is reconstructed from the data given by a PET scanner.

Acknowledgements

This work was supported by funding from the EPSRC and MRC OXWASP Centre for Doctoral Training (EP/L016710/1).

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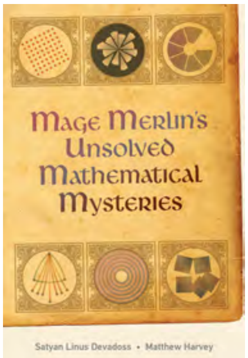
Francesca Romana Crucinio

Francesca is a PhD student at the Department of Statistics of the University of Warwick, supervised by Adam M. Johansen and Arnaud Doucet. Her main research interest is in Monte Carlo methods, with a particular focus on particle methods and their theoretical properties. Outside of research, she enjoys travelling, good food and board games.

Mage Merlin's Unsolved Mathematical Mysteries

by Satyan Linus Devadoss and Matthew Harvey, MIT Press, 2020, £20.00, US$25.00, ISBN: 978-0262044080

Review by Troy Kaighin Astarte



I was summoned to Camelot, where the great mage Merlin told me of sixteen mysteries...

This is a pleasant and satisfying little book, perfect for the aesthetically-inclined mathematician's coffee table. It is short (I read it cover-to-cover in forty-five minutes) and beautifully presented.

The book opens with a scan of the landscape of mathematics, as seen by the authors. They say that most people think of mathematics like a mountain: a solid base of well-known topics, tapering up through layers of increasing complexity, to the rarefied and mysterious peaks of unsolved problems. The authors propose that instead, mathematics should be seen like an ice-cream cone: a palatable, if mundane, conical base, growing steadily more tasty as one moves upwards towards the delicious, downwards-trickling frozen treat of mathematical mystery. The metaphor is perhaps a little odd as most people do not begin eating ice-cream cones from the point, but serves to illustrate that unsolved maths can be seen as reachable and desirable.

A new metaphor is swiftly employed after the first few pages, which sticks throughout the book: our narrator is Maryam, a young mathematician named after Fields Medallist Maryam Mirzakhani, who is a distant descendant of the legendary Merlin. A book of tales written by Merlin himself has been handed down to Maryam, and she has picked out sixteen mathematical puzzles from the book. While unsolved by Merlin or anyone else since, Maryam offers us a tantalising glint of hope that we might be able to solve them by reminding us of Andrew Wiles' famous solving of Fermat's Last Theorem.

It is these 'mysteries' which constitute the majority of the book. They cover a number of areas in mathematics. From geometry, we have a puzzle about arranging smaller squares to cover a larger one; in graph theory, we ponder the relationship between edges and vertices in thrackles; and in number theory, we wonder whether there are infinitely many twin primes. Many of the puzzles will be well-known to the mathematical reader; others may be less so.

It is in the presentation of the puzzles that the book shines. Each mystery is described by Merlin in a double-page spread with beautiful typography, lovely illustrations, and a short in-universe story in which characters from Arthurian lore display perverse attachments to particular mathematical concepts. (Guinevere insists that on her daughters' prime-numbered birthdays, red candles be lit.) Every tale ends with Merlin sighing that "even with his powers of magic and logic" he is never able to solve them.

Each puzzle is preceded by a short introduction to the mathematical concept handwritten in character by Maryam, and a longer discussion follows each, written (as far as I can tell) in the voice of Devadoss and Harvey. The discussions explain the problem in modern mathematical terminology and examine related concepts. They include some proven results related to the puzzle and explain who solved them; some pointers are given about the direction one might go to solve it. The puzzles are all clearly explained, and, if inspired, one could easily start work on them right away.

So, who is the book for? Despite its playful framing, it is probably not a book for young children to read themselves — the explanation sections are a little advanced for anyone younger than secondary-school age. The puzzles themselves are very accessible, though, and one can certainly imagine young children enjoying listening to the stories and talking about how to think about the puzzle. Personally, I have always struggled with puzzles, tending to feel stupid that I can't solve them; but at least here, knowing the best mathematical minds of generations hadn't solved them, there was no expectation that I should! I found the most fun in trying to work out the maths concepts behind each puzzle and beginning to develop a strategy for their solution.

I think this book would do its best work on a coffee table. I imagine most mathematically-minded people would enjoy reading quickly through it when first bought, and then dipping into it occasionally later. The experienced mathematician is unlikely to find anything new here, except perhaps the motivation to start thinking about one of the problems. It might also serve to interest a young relative caught by the illustrations and start a conversation about maths.

One thing that an older reader might be prompted to discuss is the role of computers in mathematics. Many explanations mention that computer-assisted projects have helped get some way towards solving puzzles but not provided complete proofs. It is a shame, then, that there is no discussion of the four-colour theorem, which was significant and contentious for being a long-standing conjecture whose computer-produced proof was too complex for a human to comprehend.[1]

Well then, should you buy the book? I think so! If you have the disposable income, twenty pounds on this is rather nice. Think of it as a nice art piece with a fun mathematical flavour that could prompt some good discussions.

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Troy Astarte

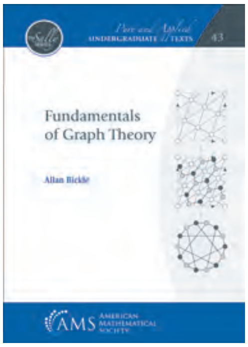
Troy K. Astarte is a researcher at Newcastle University. Their main research interest is history of computer science and the demilitarised zones between computing, mathematics, and logic. Troy comes from Lancaster (UK) and regularly leads a small team of enthusiastic problem-solvers through improvisational and creative challenges (we play Dungeons & Dragons).

[1]See [1] for the initial publication; a good discussion of the socio-philosophical implications of computers in proof is [2]. Chapter 4 of that book specifically deals with the four-colour theorem.

Fundamentals of Graph Theory

by Allan Bickle, American Mathematical Society, 2020, US$85.00,
ISBN: 978-1-4704-5342-8

Review by Claire Cornock



The author presents standard topics that you would expect to see within a graph theory book. These include Eulerian graphs, Hamiltonian graphs, trees, algorithms (e.g. to find a minimum spanning tree), planar and non-planar graphs, colour theorems and bipartite graphs. The contents extend far beyond this list, including more advanced topics such as generalised graph colourings.

This book is fairly expensive, but you certainly get a lot for your money. Amongst the 336 pages, there is a large number of definitions and theorems, over 1,200 exercises and a long list of references to other sources. There are nine main sections, each with an average of over 20 pages and an average of 125 exercises, with further information and questions in the appendices.

My background is within Pure Mathematics, with limited knowledge of graph theory. I am familiar with some of the basic methods and concepts without any of the depth. I found that parts of the book were very straightforward to follow, particularly when definitions and results were backed up with examples and/or diagrams. I had difficulty with some concepts that I had not encountered before, but all the information is there to persist with learning the material. The longer you spend with the book, the easier it is to follow.

This book aims to be appropriate for a range of audiences, which is ambitious for any publication. This includes undergraduates on Mathematics-related degrees (e.g., Computer Science), Mathematics undergraduates (with limited prior exposure to proof), more experienced Mathematics undergraduates and Mathematics postgraduate students. There is a very detailed guide for using the book which is confusing at first glance, but the detail includes handy information on which sections are needed before each part of the book, and contains recommendations for the content of lectures for each of the four student groups. There seems to be a lot of content for a series of lectures, but this is a useful guide for a lecturer to use.

I believe that the book is most suitable for lecturers and PhD students. It is a great reference book for anyone who wants to study the subject further to work towards some of the unsolved parts of graph theory. It can be used by more experienced undergraduate Mathematics students, but with caution. If they focus only on the parts that correspond to their studies, this is a great book for additional reading. It would be particularly good for those students studying graph theory within their final year project, under the guidance of a lecturer. I do not think this book is suitable for students who are not studying a Mathematics course. There is a lot of information that is not relevant and it would be difficult to pick out the parts that are, as the more complex results are alongside the more basic ideas.

The book is really well thought out. For example, the order of the topics has been carefully considered. At the end of each section, there is a list of topics that relate to the ideas that are presented. There is a very good section in the appendices on general proof. This includes techniques, examples and lots of exercises, with some linked to graph theory.

The best feature of this book is the extensive set of exercises. These are conveniently presented for each subsection, rather than listed together. Understandably there are no answers, but the questions are such a valuable resource regardless of this.

My favourite section was the one on Hamiltonian graphs. Really interesting facts are presented, such as the connection with puzzles. There is a particularly nice example of a Hamiltonian graph, which contains a very detailed description of how a Hamiltonian cycle was found. The applications include the Knight's Tour and visiting different cities. I was unfamiliar with the use of tournaments in voting theory and found this part especially interesting. I also liked the sections where the historical background is provided. For example, a good account of the historical developments is presented within the section on the Four Colour Theorem.

Real-life examples are used to motivate some of the topics, which include road networks, data storage and sporting fixtures. I especially liked how graph theory is introduced at the start with the consideration of social networks. Results move far beyond the practical, motivating examples and are studied mostly from an abstract perspective and regarding specific graphs. There is extensive consideration of when certain conditions hold. Proofs are presented for most results, and references are generally provided when they are not. The book was especially good at highlighting areas that had unsolved or partially solved problems. It is made clear when results are only known for certain cases. This makes the book especially useful for the more advanced students.

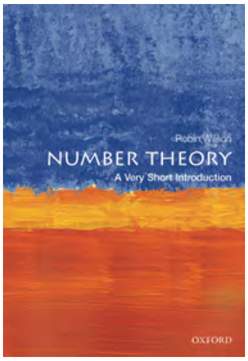
Claire Cornock

Claire is a Principal Lecturer at Sheffield Hallam University. She studied Semigroup Theory for her PhD and now researches teaching and learning pedagogy. Claire is known for teaching with Rubik's cubes to help her students' understanding of abstract ideas within group theory.

Number Theory: A Very Short Introduction

by Robin Wilson, Oxford University Press, 2020, £8.99, US$11.95, ISBN: 978-0198798095

Review by Zachary Walker



Robin Wilson's *Number Theory: A Very Short Introduction* is a concise book but very informative, aimed at readers unfamiliar with the subject. It is part of a series of 'Very Short Introductions' known for being of a very high quality, this book certainly lives up to that reputation. It is only 150 pages and split into nine chapters. Despite its relatively in-depth content given the target audience, it is a digestible short read.

The introductory chapter provides a fantastic list of questions to provide motivation for the investigation of the topic. These questions cover both real world and abstract ideas, which one would expect to capture the attention of the wide audience the book is intended for. The majority of the book is spent going over many standard results of the field, but this does not seem academic or dry. Particularly in the second half, Wilson goes on to include applications of some of the ideas covered, a few of these appear more contrived than others but succeed in keeping momentum throughout the book. The crescendo of the chapter is a look at some of the most well-known unsolved problems and recent results which can be explained well with the material introduced.

As former president of the British Society for the History of Mathematics, Wilson unsurprisingly takes the opportunity to include some fascinating references to the timeline of number theory. An impressive span of Eratosthenes through to Andrew Wiles. The context of how number theory was developed helps to justify interest in the subject. Even if one was familiar with the maths in this book then I think simply seeing how ideas were developed could make it an interesting read.

Wilson starts with explanations that seem to presume little prior knowledge, going over congruences, factors and Euclid's algorithm. These first ideas are explained very well but as the ideas become more complicated the fine details of the proofs are generally omitted, which could be frustrating or confusing to someone reading into the subject for the first time. I do not think this is a significant issue but it does leave some ambiguity as to whom the book is intended for. While there are a set of official questions set out, Wilson is not constrained to these and is constantly using questions to point to theorems as a solution to them. Knowledge of which results were likely to be shown did not spoil the anticipation of resolving the problems.

So many of the examples could be picked out but I found the short section about Charles Dodgson's method for determining the day of the week of any date especially satisfying. On one level this is an amusing party trick, but I think it is more than that. Wilson demonstrates that the simple tools that have been explained can be used to provide a solution to something, such that there is a sense of an underlying order, although it is not clear what order that is. This reveals something of the beauty of numbers and mathematics in general.

The structure of call and response between the problems and solutions being developed by revealing more results from number theory sets up expectation for the reader, which is broken in the final chapters. After hearing about the Goldbach conjecture one almost expects Wilson to introduce a new idea, building on the rest of the book to solve the conjecture, but the lack of an answer is far from dissatisfying. The cliffhanger of open problems is not only exciting but gives relevance to the subject by showing how far away it is from being a complete field. Unrealistic as it may be, I think leaving the reader in a place where they are so desperate for resolution that they attempt to find the answer themselves is testament to this book being an inspiration.

In case the reader missed it, the final short chapter summarises all the questions covered and gives succinct answers to all of them. It is in reading the solutions to these problems that one realises that the maths developed goes beyond the questions themselves to provide a powerful framework.

Overall, this book is a good introduction to the number theory but does an even better job of getting a reader excited enough about the subject that I think they would want to pursue it further. Wilson has impressively captured the essence of the topic.

Zachary Walker

Zachary Walker is an undergraduate in his third year at The Queen's College, Oxford. This year he has chosen to study algebra and the history of maths. When he is not trying to finish problem sheets, he enjoys playing the cello.

Obituaries of Members

Peter M. Neumann: 1940–2020

Photograph by Veronika Vernier (2007). © Mathematical Institute

Peter Neumann, who was elected a member of the London Mathematical Society on 17 December 1964, died on 18 December 2020, aged 79.

Cheryl Praeger and Martin Liebeck write: Peter Neumann was the first son of the well-known mathematicians Bernhard and Hanna Neumann, who came to the UK from Germany in the 1930s. Peter was born in Oxford on 28 December 1940, where Hanna was working on her DPhil while Bernhard served with the Pioneer Corps in the British Army. After the war, Bernhard and Hanna obtained university appointments in Hull, where Peter grew up before going to Queen's College Oxford in 1959. During 1961–62, while still an undergraduate, Peter joined his parents on their sabbatical year at the Courant Institute in New York, and began mathematical research on varieties of groups. This resulted in his first research paper in 1962, the '3N' paper written jointly with his parents, and in 1964, the 'B+3N' paper published jointly also with Gilbert Baumslag. Peter started his DPhil at Oxford in 1963 under the supervision of Graham Higman, and by the time he finished in 1966, he had published five more research articles, and had been awarded a Tutorial Fellowship at Queen's, to be followed a year later by a university lectureship. He remained at Oxford for his entire career, retiring in 2008.

Peter's lifelong contribution to mathematics in the UK and worldwide was monumental and wide-ranging: through his research in algebra and the history of algebra; his supervision of over 40 doctoral students, many of whom went on to have distinguished academic careers; his extensive service to the London Mathematical Society; and his enormous contribution to mathematics education.

Let us first briefly discuss Peter's research. Peter was a leading figure in algebra for over 50 years, publishing around 100 papers and books on a wide range of topics: varieties of groups, soluble groups, group enumeration, permutation groups, and algorithms in computational algebra. Each of his publications is beautifully crafted, and its place in mathematics carefully thought out and explained, together with insightful comments on where further work might lead. His work was highly influential, and we are just two of his many beneficiaries. Peter was also a great collaborator, publishing with 38 different co-authors, and holding visiting positions at many places around the world. Peter described himself as a 'mathematician historian' and wrote extensively on the history of algebra, including his comprehensive book on the mathematical writings of Évariste Galois, published in 2011. Peter's contributions to research and scholarship were recognised by the London Mathematical Society with the award of the Senior Whitehead Prize in 2003, and by the British Society for the History of Mathematics which established the Neumann Prize in 2009 in his honour.

Peter's service to the London Mathematical Society was very extensive: he was Publications Secretary (1967–72); Journal Editor (1976–79); *Bulletin* Book Review Editor (1979–81); *Bulletin* Editor (1979–84) and *Monographs* Editor (1999–2003). He was also an Officer of the Society, holding the position of Vice-President from 1990–92. The LMS honoured Peter not just with the Senior Whitehead Prize, but also with the joint LMS-IMA David Crighton Medal in 2012.

Peter also made an enormous contribution to mathematics education in the UK. He was the founding Chairman of UK Mathematics Trust (UKMT), serving from 1996 to 2005. The Trust works with hundreds of volunteers across the UK to organise competitions promoting problem solving and team work and other mathematical enrichment activities for schoolchildren. During his period as chairman, Peter led UKMT in taking on the staging of the 2002 International Mathematical Olympiad. For his services to mathematics education, Peter was awarded an OBE in 2008.

Peter loved music and was a fine violin and viola player. Before his stroke in early 2018 he would frequently cycle long distances to meetings. In July 2018, Peter moved to a care home on Cumnor

Hill. He continued to solve The Guardian cryptic crosswords regularly — with numerous Facetime discussions with his wife Sylvia throughout the pandemic lockdown when visitors were not allowed. Peter was a wonderfully generous, warm and wise person, and is deeply missed by his many friends, colleagues and students.

Peter is survived by his wife of 58 years, Sylvia, their sons David and James and daughter Jenny, their ten grandchildren, and their first great-grandchild Isaac, born 4 October 2020.

Gordon D. James: 1945–2020

Professor Gordon James, who was elected a member of the London Mathematical Society on 10 May 1985, died on 5 December 2020, aged 74. Professor James was LMS *Journal* Editor 1989–93.

Rob Curtis writes: Gordon's natural talent for mathematics first became apparent at Eastbourne College where he was taught by Eric Laming, an inspirational teacher who became a firm friend of the family for many years. From Eastbourne, Gordon won a scholarship to Sidney Sussex College, Cambridge, where he was tutored by John Conway, and where he obtained a First Class degree in 1967 followed by Distinction in Part III of the Tripos the following year. He was then taken on as a research student by John Thompson, the pre-eminent finite group theorist of the day, and wrote a PhD thesis on the modular representations of the Mathieu group M_{24} for which he was awarded a Smith Prize for his first year research.

It was during Gordon's Part III year whilst we shared a house in Cherry Hinton that he met Mary, his wife-to-be, and they married in 1971. Shortly after receiving the PhD in 1972, he was elected to a Fellowship at Sidney Sussex, a post he held until 1985 when he moved to Imperial College, London. He was very soon promoted to a Readership in 1986 and then to a Professorship in 1989, when he delivered an inaugural lecture entitled 'What the Hecke Algebras?', being unable to resist the pun on the area of mathematics in which he had become an international expert. Indeed, from the sporadic groups, Gordon's consuming interest had shifted to the representation theory of the symmetric groups. In 1975 he had spent his sabbatical leave in Canada and visited G. de B. Robinson, himself famous for his contributions to the representations of the symmetric groups, and Gordon proceeded to extend the delightful and highly combinatorial classical theory to modular representations. He produced two books on this work, one joint with Adalbert Kerber, putting the whole theory on a rigorous foundation. He then became interested in developing an analogous theory for the general linear groups and, together with Richard Dipper, introduced the concept of q-Schur algebras. His collaboration with Dipper, Andrew Mathas and others produced a body of significant results during this period and posed tantalising conjectures which have led to further important developments in the area. Gordon's ground-breaking book on unipotent representations of the finite general linear groups was awarded the Adams Prize in 1981.

Besides these advanced research monographs, Gordon, together with Martin Liebeck, produced a highly regarded and popular undergraduate text on the representation theory of finite groups.

During his time at Imperial, Gordon served as Head of Pure Mathematics from 1991–97 and supervised 8 PhD students. He was highly respected as a dedicated, unselfish and sympathetic member of the department.

Sadly, in 2002 Gordon was diagnosed with Parkinson's disease and a few years later had to take early retirement through ill health. He and Mary retired to the Yorkshire Dales and Gordon determined to keep the disease at bay by walking miles over the wonderful moorland. I myself have struggled to keep up with him as he went up over those hills like a gazelle, although he was already less sure-footed going downhill. On one occasion I recall we were both winched down into the remarkable Gaping Gill cavern, 300 feet below ground, halfway to the Ingleborough Peak. Gordon fought the disease with passion and fortitude but inevitably it caught up with him, and by the end he struggled to keep his balance. Throughout this ordeal Mary was a constant and indefatigable support to him.

Gordon had many interests. He was a fine chess and bridge player, although his fondness for a 'psych' one spade opening bid could mislead his partner as much as his opponents! It was also not unknown for him to play a game in which hands traditionally contain five cards, the standard Cambridge ante being one tenth of a penny. After retirement Gordon threw himself energetically into Yorkshire village life and soon became a hugely valued member of the local community.

Gordon was a fine mathematician, a superb colleague and a loyal friend; he is survived by his wife Mary, their two children Elizabeth and William, and five grandchildren.

A.E.L. Davis: 1928–2020

A.E.L. Davis with Antonin Švejda (left) and Igor Janovský (right). Photograph from ntm.cz

Ann Elizabeth Leighton Davis, who was elected a member of the London Mathematical Society on 15 January 1988, died on 23 November 2020, aged 92.

Snezana Lawrence writes: A.E.L. Davis (who always preferred this form of address), a mathematical historian, began her academic career with a thesis on Kepler, which she completed in 1981 at Imperial College, University of London. Her thesis, 'A Mathematical Elucidation of the Bases of Kepler's Laws', established her as a foremost scholar on Kepler. This took her on to be an active member of the International Astronomical Union (IAU) and the British Society of the History of Mathematics in the years to come.

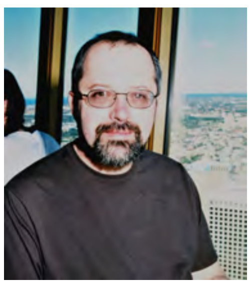
Davis became a Vice-Chair of IAU's Commission on Johannes Kepler twice in her lifetime, and was the leading scholar on Kepler until her passing. She was a productive and energetic historian of mathematics, in more general terms, too. Her greatest output in the history of mathematics was certainly her compilation of the online archive named after her, *The Davis Historical Archive: Mathematical Women in the British Isles, 1878–1940*, part of the larger MacTutor History of Mathematics archive at St Andrews University (bit.ly/39yhQ8h). The archive lists the names of all women graduates in mathematics, around 2500 in total, from the twenty-one colleges and universities that educated women in the given period. As she went about her work on compiling the archive, Davis collected about two hundred books written by or about the women whose lives and careers she investigated. This collection is diverse, including academic but also school-books, discourses or biographies. She donated the collection to the London Mathematical Society, under the name of 'Philippa Fawcett Collection' (bit.ly/3cyTtJx).

The Philippa Fawcett Collection is now one of the LMS's Special Collections, and is housed in De Morgan House. Both the collection and the insistence on calling it after Fawcett testifies to the generosity of spirit as well as her unwavering efforts to promote the work of women in mathematics and to record and inspire future female mathematicians. It also tells something about Davis' own regard for the rights of women: Fawcett was 'above the Senior Wrangler' at Cambridge in 1890, and a daughter of a noted suffragist Millicent Fawcett.

I met A.E.L. Davis many times at the BSHM meetings; her approach to life and to the history of mathematics was refreshing, piercing, and inspiring. A fearlessly independent woman, she was always genuinely interested in others' work and stories, and rarely spoke about herself — I wish I had had more time to ask her many more questions about her own life.

What I know is scarce and does not do justice to such an important and productive historian of mathematics. For many years she worked as an Associate Lecturer for the Open University (1989–2004), and towards the end of her life became an Honorary Research Associate of University College London and an Honorary Visiting Fellow at the Mathematical Sciences Institute, Australian National University. It is in Australia that Davis died last year; she will be sorely missed amidst the historians of mathematics of UK, and in our global community.

Robin J. Chapman: 1963–2020



Dr Robin Chapman, who was elected a member of the London Mathematical Society on 19 June 1987, died on 18 October 2020, aged 57.

Peter Cameron writes: Robin was born in May 1963 in Swansea. He attended Dynevor Comprehensive, where he won a Postmastership to Merton College, Oxford. After taking one of the top Firsts in his year, he went to Cambridge to do Part III and was accepted as a PhD student to work with Martin Taylor. He followed Martin to Manchester, completing his PhD in 1987. After a Junior Research Fellowship at Merton College, he took a position at Exeter University where he remained for the rest of his career, though he retained great affection for Oxford.

Robin was a very able undergraduate. When the Galois theory lecturer listed the subgroups of the symmetric group of degree 4 and asked the students to find polynomials realising each as a Galois group, Robin's comment was "You missed one". His tutorial partner Peter Kronheimer took pride in finding the shortest and most elegant answer to any problem; at first Robin simply fought the problem into submission, but as his stature as a mathematician grew he found he was capable of shorter and more beautiful arguments. Later in his career, his co-author Patrick Solé praised the elegance of his work, when (for example) he proved by hand the equivalence of two constructions of the Leech lattice.

I greatly valued the information he provided on his web page, which included short and efficient proofs of various 'folklore' results such as Bertrand's postulate and the characterisation of orders for which every group is abelian. Others shared this opinion, and he was often cited on MathOverflow and StackExchange.

Robin was a mathematician first and foremost, but his interests were very wide indeed. Peter Kronheimer played French horn in a wind-quartet; his quartet perfomed Ligeti's Six Bagatelles, and Peter was surprised to find that not only did Robin come to the performance, but he could expound on the work and its place in Ligeti's oeuvre. This knowledge stood him well in Mastermind, where he reached the final in 2005: his special subjects in the heats and final were The Life and Music of Igor Stravinsky, One Foot in the Grave and The Science Fiction Novels of Philip K. Dick. It is said that the pub quiz machine in the students' bar at Manchester helped fund his studies there.

Robin's mathematical interests lay in discrete mathematics and number theory. One thing he is remembered for is his "evil determinant problem", subsequently solved by Maxim Vsemirnov. He published 50 papers, was on the editorial board of two journals, and organised the British Combinatorial Conference in 2011.

After the opening of the Heilbronn Institute for Mathematical Research in 2005, Robin split his time between there and Exeter, doing collaborative mathematical research supporting the work of Government Communication Headquarters. He worked with the UK Olympiad team, and both Tony Gardiner and Imre Leader write warmly of him.

Robin took great joy from mathematics and brought joy to many friends. He is survived by his brother and family.

Death Notices

We regret to announce the following deaths:

- Patrick D. Barry, Professor Emeritus of University College Cork, who died on 2 January 2021.
- Colin J. Bushnell, Emeritus Professor at King's College London, who died on 1 January 2021.
- Walter Forster, formerly of University of Southampton, who died on 17 January 2021.
- Robin L. Hudson, formerly of Loughborough University, who died on 12 January 2021.
- Brian H. Murdoch, formerly Erasmus Smith Professor at Trinity College, Dublin, who died on 9 December 2020.
- Stephen Pride, formerly of University of Glasgow, died on 21 October 2020.
- Tommy A. Whitelaw, formerly of the University of Glasgow, who died on 21 January 2021.

Biographical Memoirs

Memoirs of Michael Atiyah (bit.ly/39CeJMP), Christopher Hooley (bit.ly/3ap16ja), Frank Bonsall (bit.ly/36xSYvr) and Edward Fraenkel (bit.ly/2MM0yLP) have recently appeared in *Biographical Memoirs of Fellows of the Royal Society*.

Early Career Mathematicians' Spring Conference 2021

Location: Online
Date: 13 March 2021
Website: tinyurl.com/y2povzt3

This IMA conference will interest mathematicians early in their career, in academia and industry, students of mathematical sciences, as well as those with an interest in the subject. It will feature plenary talks from distinguished speakers covering a wide range of subjects, as well as networking activities. The Invited Speakers include Mihaela Rosca (DeepMind and UCL) and Nick Higham (University of Manchester).

LMS Meeting at the Joint BMC–BAMC 2021

Location: Online
Date: 8 April 2021
Website: tinyurl.com/yarpowdo

This event was originally scheduled for 2020 and was postponed owing to covid-19. The meeting will begin with Society business, followed by an LMS lecture by Ciprian Manolescu (Stanford). Further details and updates on the meeting can be found on the website.

Korteweg-de Vries Equation, Toda Lattice and their Relevance to the FPUT Problem

Location: University of Lincoln
Date: 26 May 2021
Website: https://wp.me/PcBUF5-6

This meeting aims to highlight aspects of integrable systems theory applied to near-integrable many-body dynamical systems. Postgraduate and final year undergraduate students are particularly encouraged to apply. Participation is open to final year students, early career researchers and academics.

Modelling in Industrial Maintenance and Reliability

Location: Online
Date: 28 June–2 July 2021
Website: tinyurl.com/IMAMIMAR

This conference is the premier maintenance and reliability modelling conference in the UK and builds upon a very successful series of previous conferences. It is an excellent international forum for disseminating information on the state-of-the-art research, theories and practices in maintenance and reliability modelling.

LMS Women in Mathematics Day 2021

Location: Online
Date: 24 March 2021
Website: tinyurl.com/y5uwol5f

This event, open to mathematicians of all genders and from all backgrounds, aims to promote interest and careers in mathematics for women. In addition to talks, the event will include a panel discussion and a poster competition open to women mathematicians at undergraduate, postgraduate and early career levels. The deadline for registration is 21 March 2021, 16:00. Register your attendance at tinyurl.com/y6bvdfg8.

LMS Spitalfields History of Mathematics Meeting

Location: Online
Date: 14 May 2021
Website: tinyurl.com/y3kpv6ye

This event, held by the LMS and UCL Special Collections, will celebrate the launch of the Educational Times Digital Archive. Talks will have a mathematical historical focus, and will include a presentation from UCL about their work on the collections.

Dynamics and Geometry Summer School

Location: University of Bristol
Date: 21 June–2 July 2021
Website: tinyurl.com/s5hn592j

Dynamics and Geometry are two intertwined areas of mathematics that have seen revolutionary breakthroughs in recent years. In this summer school world-leading experts will speak about some of these developments, alongside problem sessions and other opportunities for discussion and interaction.

Research Students' Conference in Population Genetics

Location: University of Warwick
Date: 21–23 July 2021
Website: tinyurl.com/y9fjp36b

This conference is aimed at young researchers interested in mathematical and statistical aspects of population genetics, including coalescent theory, stochastic processes in population genetics, computational statistics and machine learning for genomics.

Young Geometric Group Theory X

Location: Newcastle University
Date: 26–30 July 2021
Website: conferences.ncl.ac.uk/yggt2021

The aim of this YGGT conference is to bring together young researchers in geometric group theory, post-docs and graduate students. It will allow them to learn from one another and from senior mathematicians invited to give tutorial courses and lectures in several branches of geometric group theory. Supported by an LMS Conference grant.

Scaling Limits: From Statistical Mechanics to Manifolds

Location: Cambridge
Date: 1–3 September 2021
Website: statslab.cam.ac.uk/james60

This workshop, postponed from 2020, is in honour of James Norris' 60th birthday. There will be 16 invited talks covering: Random growth processes and SPDEs; Yang-Mills measure; Limits of random graphs, random planar maps, and fragmentation processes; Markov chains, interacting particle systems and fluid limits; Diffusion processes and heat kernels. A workshop dinner will be held at Churchill College.

Young Functional Analysts' Workshop

Location: Lancaster University
Date: 12–14 August 2021
Website: tinyurl.com/yce6j3gy

This is an event aimed at early-stage researchers (PhD students and postdocs) in functional analysis and related areas. It is a great opportunity to bring researchers with shared interests together and provides the opportunity for participants to present their own work in front of a supportive and interested audience.

Heilbronn Annual Conference 2021

Location: Heilbronn Institute
Date: 9–10 September 2021
Website: tinyurl.com/y63vvoar

The Annual Conference of the Heilbronn Institute for Mathematical Research is the Institute's flagship event. The eight invited speakers are: Caucher Birkar, Jon Brundan, Ana Caraiani, Heather Harrington, Gil Kalai, Peter Keevash, Jeremy Quastel and Tatiana Smirnova-Nagnibeda. They will deliver lectures intended to be accessible to a general audience of mathematicians.

Covid-19: Owing to the coronavirus pandemic, many events may be cancelled, postponed or moved online. Members are advised to check event details with organisers.

Society Meetings and Events

March 2021

24 LMS Women in Mathematics Day (online) (493)

April 2021

8 Society Meeting at the joint BMC–BAMC 2021 (online) (493)

May 2021

14 LMS Spitalfields History of Mathematics Meeting: Educational Times Digital Archive Launch, London (493)

June 2021

2-4 Midlands Regional Meeting and Workshop, Lincoln

22 Society Meeting at the 8ECM, Portorož, Slovenia

July 2021

2 General Meeting of the Society, London

September 2021

6-10 Northern Regional Meeting, Conference in Celebration of the 60th Birthday of Bill Crawley-Boevey, University of Manchester

January 2022

4-6 South West & South Wales Regional Meeting, Swansea

Calendar of Events

This calendar lists Society meetings and other mathematical events. Further information may be obtained from the appropriate LMS Newsletter whose number is given in brackets. A fuller list is given on the Society's website (www.lms.ac.uk/content/calendar). Please send updates and corrections to calendar@lms.ac.uk.

March 2021

13 Early Career Mathematicians' Spring Conference 2021 (online) (493)

14 International Day of Mathematics (491)

30-31 Mathematics in Defence and Security IMA Conference (online) (492)

April 2021

6-9 British Mathematical Colloquium and British Applied Mathematics Colloquium 2021 (online) (492)

20-23 Mathematics of Operational Research (online) (492)

29-30 Marriages, Couples, and the Making of Mathematical Careers (online) (492)

May 2021

26 Korteweg–de Vries Equation, Toda Lattice and their Relevance to the FPUT Problem, University of Lincoln (493)

June 2021

20-26 8th European Congress of Mathematics, Portorož, Slovenia (492)

21-2 Jul Dynamics and Geometry Summer School, University of Bristol include (493)

28-2 July Modelling in Industrial Maintenance and Reliability (online) (493)

July 2021

7-9 Nonlinearity and Coherent Structures, Loughborough University (492)

12-16 New Challenges in Operator Semigroups, St John's College, Oxford (490)

19-23 Rigidity, Flexibility and Applications, Lancaster University (492)

21-23 Research Students' Conference in Population Genetics, University of Warwick (493)

26-30 Young Geometric Group Theory X, Newcastle University (493)

August 2021

12-14 Young Functional Analysts' Workshop, Lancaster University (493)

16-20 IWOTA, Lancaster University (481)

18-20 Young Researchers in Algebraic Number Theory III, University of Bristol (492)

September 2021

1-3 Scaling Limits: From Statistical Mechanics to Manifolds, Cambridge (493)

9-10 Heilbronn Annual Conference 2021, Heilbronn Institute (493)

16-17 Statistics at Bristol: Future Results and You 2021, Heilbronn Institute

19-24 8th Heidelberg Laureate Forum, Heidelberg, Germany

21-23 Conference in Honour of Sir Michael Atiyah, Isaac Newton Institute, Cambridge (493)

July 2022

24-26 7th IMA Conference on Numerical Linear Algebra and Optimization, Birmingham (487)