Four letters written by Julius Plücker to Thomas Archer Hirst[1]
September 1866 –October 1867

September 1866

My dear Dr Hirst,

When at Nottingham you requested a selection of surface models to be sent to the secretary of R.S. I purposed leaving some of those I had with me and on my return to Bonn sending others thence. But some of the models having been injured by travelling I thought it better to order the whole set to be newly made at Bonn. Yesterday I returned to Bonn from a stay in the country and looked over the new made models. Being satisfied with them I have ordered Mr Epkens who made them, to send them to Mr White. The explanations he gives are his not mine.

Let me here repeat a few observations I made at Nottingham.

Each plane intersects a surface of the second degree along a conic; each point is the centre of a circumscribed cone. If the plane is the polar plane of the point, the cone touches the surface along the curve. With regard to a <u>complex</u> of the second degree to each right line corresponds another conjugated line. The surface formed by curves of the complex situated within planes passing through any one of two <u>conjugated</u> lines is at the same time enveloped by cones of the complex the centres of which are placed on the same line. The pole of that line with regard to any of the curves falls within the conjugated line, the polar plane of the same line with regard to any of the curves passes through the conjugated one. There are two conjugated surfaces belonging to every two conjugated lines. <u>Equatorial</u> surfaces belong to an infinitely distant line, its conjugated line passes through the centre of all curves, generating the surface. In a given complex there are three conjugated equatorial surfaces corresponding to any three conjugated diameters of a certain surface of the second degree.

I shall occasionally have models made of *three* such surfaces. The surfaces I send, all equatorial ones, have no relation one to another.

I shall be happy if the models contribute to give an idea of that marvellous relation that I have called a complex of the second degree.

Believe me, my dear Sir, most truly yours
Plücker

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[1] The letters, which were transcribed by Heather Ashby, belong to the London Mathematical Society. They are held in the Archives of University College London, together with the rest of Hirst's correspondence belonging to the Society. In 1866 Hirst was a Vice-President of the Society; he was President 1872–1874.

Bonn, *17th October 1866*

My dear Sir,

I can easily comply with your request concerning the models you have received. All belong to that class of surfaces I have called equatorials, formed by curves of the second class confined in parallel planes, and enveloped by cylinders of the second order having their axes parallel to the planes. The surfaces, both of the 4th class and the 4th order, depend generally upon 13 constants.

In admitting ordinary axes of coordinates OX, OY, OZ you may determine by x any plane parallel to $Y Z$, and within this plane by means of line coordinates u, v, w a curve by the equation

$$Aw^2 + (B - 2Rx + Fx^2)v^2 - 2(G + Ox)uv + (C - 2Ux + Ex^2)u^2 = 0$$

This equation─which admits one arbitrary constant corresponding with the arbitrary position on OX of the origin─may be said, by regarding x as variable, to represent the equatorial surface: OX being its diameter. By introducing, instead of u, v, w the coordinates y, z you will obtain in x, y, z the equation of the same surface. The same equation may be represented by the equation

$$(E\text{tang}^2\delta - 2C\text{tang}\delta + F)x^2 + Dz^2 + 2(U\text{tang}^2\delta - O\text{tang}\delta + R)x + C(\text{tang}^2\,\delta + B) = 0$$

Indeed in denoting by ± the angle formed by any plane XZ', passing through OX, with XZ (OZ' remaining within ZY) and by x and z' ordinary coordinates, the last equation represents, within any plane perpendicular to $Y Z$, the basis of a right cylinder <u>enveloping the surface</u>. The basis of the cylinder is replaced by its projection in the following equation

$$(Fv^2 - 2Guv + Eu^2)x^2 + Du^2z^2 + 2(Rv^2 - Ouv + Uu^2)x + (Bv^2 + Cu^2) = 0$$

where u and v denote plane coordinates. By replacing x and z by the line – (or plane –) coordinates t, v, w, the resulting equation between t, u, v, w again represents the same surface by its tangent-planes.

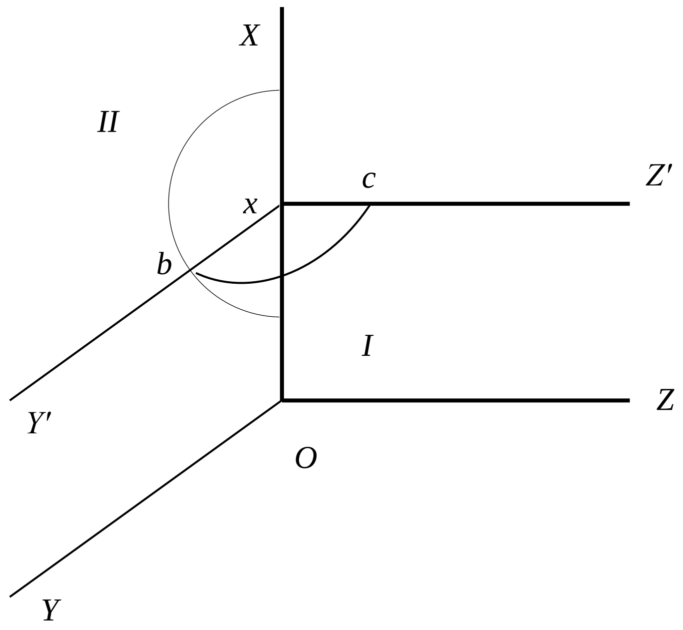
The surface has a double line, infinitely distant within $Y Z$.

If the term $(G + Ox)$ disappears, the resulting equation

$$Dw^2 + (B - 2Rx + Fx^2)v^2 + (C - 2Ux + Ex^2)u^2 = 0$$

represents curves, the axes of which are parallel to OY and OZ. In this case the curves are easily obtained by means of two curves, placed within XZ and XY and represented by the equations

$$z^2 + Fx^2 - 2Rx + B = 0$$
$$y^2 + Ex^2 - 2Ux + C = 0$$



Let $Y'Z'$ be any plane parallel to $Y\,Z$, determined by $Ox = x$. xc and xb will be the half-axes of the corresponding curve. If b and c are both real points the curve is an ellipsis, if one of both is imaginary the curve is an hyperbola. To imaginary points correspond an imaginary curve. To the four intersections of OX with the curves correspond four singular lines of the surface. When b and c fall within the same point of OX, the degree of the surface becomes the third.
When one or both curves degenerates into two right lines, to their intersection on OX corresponds a double line of the surface.

8 Models (their number refer to the bill only) belong to this case, containing 77 different surfaces.

If the term $(G + Ox)uv$ exists, the surfaces become twisted. I have distinguished 98 twisted surfaces, amongst which 77 correspond to the untwisted ones and 21 belong to a peculiar case indicated by the equation:

$$Dw^2 + (B - 2Rx + Fx^2)v^2 - 2(G + Ox)uv = 0$$

in which all curves are hyperbolae having an asymptote within $Y\,X$.

There is one model of the general case and one of the peculiar.

Finally an analogous number of equatorial surfaces is generated by a parabola in a plane moving parallel to itself. These surfaces, depending in the general case upon 12 constants, are represented by the equation

$$2(Q - Px)vw + 2(T + Mx)uw + (B - 2Rx + Fx^2)v^2 - 2(G + Ox + Kx^2)uv + (C - 2Ux + Ex^2)u^2 = 0$$

from which by a proper choice of axes OX, OY, OZ six constants disappear. Amongst the four parabolic equatorial equations, illustrated by models there is one untwisted. Three of a peculiar kind, (all parabolae having their foci on a right line) are twisted, one exhibiting no singular line, another one and the third two singular lines.

[Here I must signalise a mistake made in connecting the single parts of the two last models. The singular lines are not double lines. Each singular line is an infinite one, consisting of two parts joining in one point. Accordingly for instance, the extreme parts of the last model are to be turned through an angle of 180˚.]

I am, my dear Sir, highly indebted to you by the kind offer to have my researches printed by the R.S. I could transmit to you an Enumeration of about 800 different species of surfaces, both equatorial and meridional, of the fourth order and class─based on the same principles made use of in my Theory of algebraical curves. But that would disguise the true object of my inquiries, the nodal point of which is the conception of a complex, corresponding to four variables, while a surface corresponds to three and a plane curve to two variables only.

I think the following instance will explain my meaning. To any given line *A* corresponds an equatorial surface *I* generated by curves confined in planes (*a*) conjugated to *A* and enveloped by cylinders the axes of which are parallel to the planes (*a*), as well as a meridional surface *II* enveloped by cones the centre of which fall within *A* and generated by curves confined in planes passing through *A*. Having before you the equations of both surfaces *I* & *II*, you meet in these equations all constants of the complex, in a linear way except one. Hence it follows that the complex is determined in a linear way by two such combined surfaces and one of its lines *B*. Indeed, trace through *B* any plane (*b*) intersecting *A* in any point α. The point is the centre of an enveloping cone, two generatrices of which are confined within the plane (*b*). Besides the same plane (*b*) meets an enveloping cylinder, the axis of which is parallel to the intersection of (*b*) with the conjugated planes (*a*), along two generatrices. Thus are obtained within the plane (*b*) five lines belonging to the complex, therefore the curve itself, the tangents of which are lines of the complex.

Again let (*B*) be any point of *B*. There is one conjugated plane (*a*), and another passing through the given line *A*, which meet in the point (*B*). Trace through this point two tangents to each curve confined in the two planes. Both couples of tangents, if joined to *B*, determine a cone, generated by lines of the complex, the centre of which is *B*.

You may give a great extent to construction of this kind, and, if you like, proceed in a pure geometrical way: starting from the general definition of a complex of the second degree.

Having now, after long preparatory attempt, a full geometrical oversight of complexes of the second degree and having met with an analytical method, surpassing by far, with regard to simplicity and symmetry whatever I before expected. I intend to publish a first volume next year, containing a complete classification and discussion of complexes, which involve the discussion of equatorial and meridional surfaces. The number of constants of a complex is 20 (reducible by division to 19) ranging into six groups of three and one group of two. The constants of two groups are the six constants of a surface of the second class, having an arbitrary centre. The sections of the asymptotic cone of the surface by infinitely distant plane belong also to the complex. The three axes of the complex are parallel to the axes of the surface. The planes conjugated to any diameter of the surface are, in the complex too, conjugated to lines parallel to the diameter. Among these parallel lines is one diameter of the complex. The three constants of a third group give the position in space of the three axes of the complex, the constants of the fourth the position of the axis of the three cylinders, generated by lines of the complex parallel to its axes. The six constants of the fifth and sixth group determine the cone formed by lines of the complex meeting in the original. By replacing the origin you may discuss the infinite part of the complex.

There is another Chapter treated by me with full success: The theory of linear polar complexes with regard to complexes of the second degree, as well as of tangent complexes.
Other Chapters has been reserved.

I shall be most happy to give any information whatever you request, but after all I think it desirable not to publish details now. I think a part of the developments which I have before me, if translated into French, would satisfy a question put by the Institute. I hesitated for a moment, by my indolence prevailed, the more as Mr. Chastles would recognise the author.

If I were twenty years younger I could hope to see the principles laid down in the two printed Papers, fully developed. Now, one of my former pupils will try to work out the last *x* of my last Paper into a treatise of Mechanics. I shall be glad if he succeeds and in this case assist him. A short elementary account of the analytical methods employed by myself in former Papers would be necessary or desirable as an introductory work. It would not be wise to overcrowd myself with heterogeneous work at present. When I have finished a geometrical volume I intend to concentrate my attention to the contents of the two first *xx* of my last Paper and its application to molecular Physics. Then I may at first not undertake to write a systematical work and I shall be happy to present elaborate Papers on especial points to the R.S. I wished to satisfy as much as possible your request, but do not know if I succeeded.

Believe me
My dear Sir
truly yours
Plücker

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Bonn, *11 November 1866*

My dear Sir!

Many thanks for your friendly communication of the decision in my favour of the Council of the Royal Society! I feel myself highly indebted to you for your personal interest and assistance. When my researches are published, I hope the Council will be justified in the honour it has now conferred.

I received the intelligence, first kindly noticed to me by you, soon after in a kind letter from the President.

Hoping to hear from you again.

I am, my dear Sir, most truly yours
Plücker

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Bonn, *21 October 1867*

My dear Sir,

In order to promote my work I went to the Lago de Como but was disappointed by the cold and miserable weather. When I returned I found your letter which I should have otherwise answered before.

Thanks for your expression of interest about the late appointment which I regard as a result of the honour done me the previous year in England and that not without your friendly cooperation.

With regard to the money for Herr Epkens, I shall ask you to send the amount to me for him in an ordinary Banker's cheque which I can get cashed at my Bankers here.

A new series of Models of complex surfaces of the general description—the locus of complex curves situated within a plane revolving round any fixed line—were sent by Mr Epkens to the Paris Exhibition, but were lost by fault of the Prussian Commissioners. These surfaces have one double line and besides 8 double points and 8 double planes (touching the surface along a conic). Other remarkable surfaces depend upon them. Prof. Cremona, which I met at Milano, has printed a Paper of mine, on the construction of surfaces reglées "by means of complex surfaces". If as I think, I shall get some copies of the Paper, I will send you one of them.

I have worked rigorously on the Complexes since I last wrote to you. The printing of the first volume will begin immediately and when finished I think of bringing it myself to England. I hope you will find it to contain many striking results, more so than I expected when I began to work. I will not enter into detail until I can present the volume to you.

Believe me, Sir, most truly
Yours
Plücker